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Comparing trimaximal mixing and its variants with deviations from tri-bimaximal mixing

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Abstract We analyze in detail the predictions of "trimaximal" neutrino mixing, which is defined by a mixing matrix with identical second column elements. This column is therefore identical to the second column in the case of tri-bimaximal mixing. We also generalize trimaximal mixing by assuming that the other rows and columns of the mixing matrix individually can have the same forms as for tri-bimaximal mixing. The phenomenology of these alternative scenarios and their mixing angle and CP phase correlations are studied. We emphasize how trimaximal mixing scenarios can be distinguished experimentally from broken tri-bimaximal mixing.

1 Introduction

Ten years after observation of the depletion of atmospheric muon-neutrinos was established by the SuperKamiokande collaboration [1], our knowledge of the neutrino oscillation parameters has been noticeably sharpened by the ensuing atmospheric [2–4], solar [5–12], reactor [13–16], and long-baseline [17, 18] neutrino experiments. Recent global analyses [19–22] limit the oscillation parameters to the 1σ and 3σ ranges determined by Fogli et al., for example, to be [21, 22]

$$\begin{aligned}
&\Delta m^2_{32} = \left(2.39^{+0.11,0.42}_{-0.08,0.33}\right) \times 10^{-3}\ \text{eV}^2,\\
&\Delta m^2_{21} = \left(7.67^{+0.16,0.52}_{-0.19,0.53}\right) \times 10^{-5}\ \text{eV}^2,\\
&\sin^2\theta_{23} = 0.466^{+0.073,0.178}_{-0.058,0.135},\\
&\sin^2\theta_{12} = 0.312^{+0.019,0.063}_{-0.018,0.049},
\end{aligned} \tag{1}$$

while the reactor neutrino oscillation angle remains more uncertain. In a recent analysis [21, 22] it is weakly (1.6σ) constrained to be non-zero according to (see also Refs. [23, 24])

$$\sin^2\theta_{13} = 0.016 \pm 0.010\ (\leq 0.046). \tag{2}$$

These mixing angles help to specify the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) mixing matrix defined in the standard convention [25] by

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{3}$$

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$ with δ the unknown CP-violating Dirac phase. Harrison, Perkins, and Scott first emphasized that the experimentally obtained mixing matrix is close to the simple tri-bimaximal mixing (TBM) form where [26–31]

$$U_{\text{TBM}} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & -\sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{6}} & \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{2}} \end{pmatrix}. \tag{4}$$

The column vectors are just the eigenvectors of the three diagonal $U(3)$ operators. For exact tri-bimaximal mixing,

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the mixing angles are[1]

$$\sin^2\theta_{12} = \frac{1}{3}, \qquad \sin^2\theta_{23} = \frac{1}{2}, \qquad \sin^2\theta_{13} = 0, \tag{5}$$

which are seen to be close to the present data quoted above in (1).

In a top-down approach, the charged lepton mass matrix, m_ℓ, and light Majorana neutrino mass matrix, m_ν, are specified in some model with particular family and flavor symmetries, often by invoking the seesaw mechanism. The two mass matrices are diagonalized by two unitary transformations such that

$$\begin{aligned} &U_\ell^\dagger m_\ell^\dagger m_\ell U_\ell = \mathrm{diag}(m_e^2, m_\mu^2, m_\tau^2), \\ &U_\nu^T m_\nu U_\nu = m_\nu^{\mathrm{diag}} = \mathrm{diag}(m_1, m_2, m_3). \end{aligned} \tag{6}$$

The PMNS mixing matrix then follows from $U_{\mathrm{PMNS}} P = U_\ell^\dagger U_\nu$, where the diagonal Majorana phase matrix is given by $P = \mathrm{diag}(1, e^{i\alpha}, e^{i\beta})$. Note that without this matrix the phase transformation required on the right side of U_{PMNS} (and hence on U_ν itself) to bring it into the conventional phase structure of (3) is not possible, when one demands real positive diagonal neutrino mass entries in (6). The presence of P allows one to compensate for the phase transformation on the right side of U_{PMNS} without altering U_ν. In this top-down approach one can then compare the U_{PMNS} obtained with U_{TBM}.

Instead, one might employ a bottom-up procedure to identify the μ–τ symmetric neutrino mass matrix in the flavor basis as the most general one giving rise to tri-bimaximal mixing,

$$\begin{aligned} (m_\nu)_{\mathrm{TBM}} &= U_{\mathrm{TBM}}^* P^* m_\nu^{\mathrm{diag}} P^\dagger U_{\mathrm{TBM}}^\dagger \\ &= \begin{pmatrix} A & B & B \\ \cdot & \frac{1}{2}(A+B+D) & \frac{1}{2}(A+B-D) \\ \cdot & \cdot & \frac{1}{2}(A+B+D) \end{pmatrix}. \end{aligned} \tag{7}$$

The parameters A, B, D are in general complex and functions of the neutrino masses and Majorana phases:

$$\begin{aligned} &A = \frac{1}{3}\left(2m_1 + m_2 e^{-2i\alpha}\right), \qquad B = \frac{1}{3}\left(m_2 e^{-2i\alpha} - m_1\right), \\ &D = m_3 e^{-2i\beta}. \end{aligned} \tag{8}$$

[1]The experimental results are so close to TBM that parameterizations of the PMNS matrix with TBM as the starting point have been proposed [32–35].

Another way to write the mass matrix is to decompose it in terms of the three individual masses.

$$\begin{aligned} (m_\nu)_{\mathrm{TBM}} = \frac{m_1}{6}\begin{pmatrix} 4 & -2 & -2 \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix} + \frac{m_2 e^{-2i\alpha}}{3}\begin{pmatrix} 1 & 1 & 1 \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix} \\ + \frac{m_3 e^{-2i\beta}}{2}\begin{pmatrix} 0 & 0 & 0 \\ \cdot & 1 & -1 \\ \cdot & \cdot & 1 \end{pmatrix}. \end{aligned} \tag{9}$$

Since observable departures from exact tri-bimaximal mixing are expected, it is of interest to study how deviations may arise. In a recent paper the authors linearly perturbed the matrix elements in (7) in order to observe how large the departures of the mixing angles from their tri-bimaximal values in (5) can be [36]. Allowing for up to 20% deviations in the matrix elements in the normal hierarchy case results in non-zero values for $\sin^2\theta_{13}$ up to 0.001, for example. We argued that larger deviations in connection with a normal mass hierarchy would signal that the apparent nearly tri-bimaximal mixing should be considered accidental in nature, rather than the result of a softly-broken symmetry.

In this paper we study other mixing scenarios which deviate from tri-bimaximal mixing by leaving only one of the columns or one of the rows invariant. We shall refer to such scenarios as "generalized trimaximal mixing." The term "trimaximal mixing" [37] was originally introduced to describe a mixing matrix in which only the second column of (4) remains invariant with the absolute value of every element of that column equal to $1/\sqrt{3}$. To distinguish the different versions, we label the original version of trimaximal mixing considered in [37–41] as TM_2, i.e.,

$$\mathrm{TM}_2: \quad |U_{\alpha 2}|^2 = \frac{1}{3}, \quad \forall\alpha = e, \mu, \tau. \tag{10}$$

We shall also study the effects of allowing the other two columns to remain independently invariant under tri-bimaximal mixing perturbations and label them TM_1 and TM_3, respectively. It is also of interest to allow one of the rows to remain invariant for which we adopt the labeling TM^i, with $i = 1, 2,$ or 3. In total there are six possibilities. By examining these variations we shall learn that the three mixing angles and the CP-violating Dirac phase are not all independent and that restricted ranges of the deviations are imposed by the present mixing data. In fact, keeping a row or a column of U_{PMNS} fixed leads to a 2-parameter scenario, i.e., two of the four mixing observables in U_{PMNS} are determined. In four of the six trimaximal variants we find an interesting, characteristic and testable correlation between the mixing parameters. Future more precise data will allow one to test some of the trimaximal variations considered here. In particular, we stress that while TM_2 allows naturally for non-zero θ_{13}, non-maximal θ_{23} and for $\sin^2\theta_{12} \neq \frac{1}{3}$, it predicts that $\sin^2\theta_{12} \geq \frac{1}{3}$, while the current best-fit points all lie at $\sin^2\theta_{12} \le \frac{1}{3}$. Some of the alternative trimaximal scenarios can accommodate this, while still allowing for non-zero θ_{13}.

We would like to stress that in the present paper not merely deviations from tri-bimaximal mixing are discussed, but also novel and testable mixing scenarios are presented. Though tri-bimaximal mixing dominates the current phenomenological and theoretical discussion in neutrino physics, alternative proposals are surely of interest, and here we study one possible avenue.

In Sect. 2 we summarize the results obtained previously for broken tri-bimaximal mixing in [36]. In Sect. 3 we derive the results for the originally proposed trimaximal mixing and for the variant forms of trimaximal mixing in Sect. 4. A comparison of the results and conclusions are presented in Sect. 5.

2 Deviations from tri-bimaximal mixing

In [36] we raised the issue of deviations in the mixing observables from tri-bimaximal mixing when the mass matrix is perturbed. Our strategy was to perturb every entry of the mass matrix with a small complex parameter ϵ_i, i.e.,

$$m_\nu = \begin{pmatrix} A(1+\epsilon_1) & B(1+\epsilon_2) & B(1+\epsilon_3) \\ \cdot & \frac{1}{2}(A+B+D)(1+\epsilon_4) & \frac{1}{2}(A+B-D)(1+\epsilon_5) \\ \cdot & \cdot & \frac{1}{2}(A+B+D)(1+\epsilon_6) \end{pmatrix}, \tag{11}$$

where the complex perturbation parameters were taken to be $|\epsilon_i| \le 0.2$ for $i = 1-6$ with their phases ϕ_i allowed to lie between zero and 2π. The results obtained for the oscillation parameters depend on the neutrino mass values and ordering. In case of a normal hierarchy ($m_3 \gg m_2 > m_1$), the maximal expected values are

$$\begin{aligned} \text{NH:} \quad & |U_{e3}| \lesssim \frac{2}{3}\sqrt{\frac{\Delta m_\odot^2}{\Delta m_A^2}}|\epsilon| \simeq 0.027, \\ & \left|\frac{1}{2} - \sin^2\theta_{23}\right| \lesssim \frac{1}{2}|\epsilon| \simeq 0.1, \end{aligned} \tag{12}$$

where $|\epsilon|$ denotes the absolute value of one of the six independent breaking parameters in (11). Figure 1 shows for a maximal 20% deviation the parameter $|U_{e3}|^2$ as a function of the smallest neutrino mass m_1. Note that the maximal value of $|U_{e3}|$ depends linearly on ϵ. Hence, if the deviation is allowed to range up to 50% for the normal hierarchy case, one finds that $|U_{e3}|^2$ can reach a maximum value of roughly 0.005 for $m_1 \lesssim 4$ meV. Even for a normal hierarchy it turns out that $\sin^2\theta_{12}$ can take values anywhere in its allowed range. The latter is also true in case of an inverted hierarchy ($m_2 \simeq m_1 \gg m_3$), for which one furthermore finds

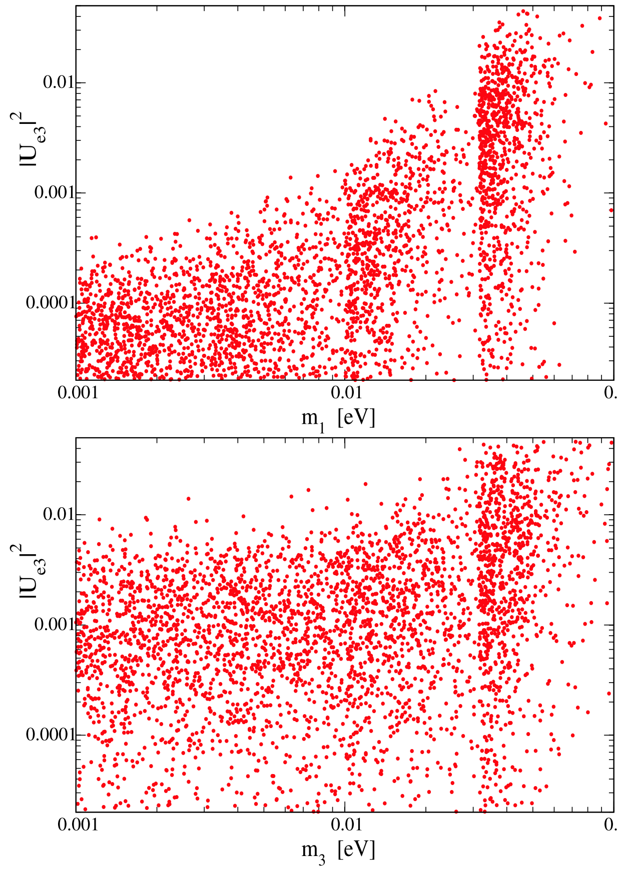

Fig. 1 Scatter plot of the predictions for $|U_{e3}|^2$ versus the smallest neutrino mass for softly-broken tri-bimaximal mixing. Shown are the normal mass ordering (*upper plot*) and the inverted one (*lower plot*)

that the other mixing observables are bounded from above by

$$\begin{aligned} \text{IH:} \quad & |U_{e3}| \lesssim \frac{1}{3}|\epsilon|\sqrt{\frac{8}{9} + \frac{16}{3}\frac{m_3}{\sqrt{\Delta m_A^2}}} \simeq 0.12, \\ & \left|\frac{1}{2} - \sin^2\theta_{23}\right| \lesssim \frac{8}{9}|\epsilon| \simeq 0.18. \end{aligned} \tag{13}$$

Almost all of the 3σ range can be covered. The interesting accessible range for $|U_{e3}|^2$ as a function of the smallest neutrino mass m_3 is also plotted in Fig. 1. If neutrinos are quasi-degenerate, then the fully allowed parameter space can be covered.

In Ref. [36] we also presented the results for some predictive $SO(10)$ symmetric Grand Unified models exhibiting a normal mass hierarchy. There we found that $|U_{e3}|^2 \gtrsim 2 \times 10^{-3}$, while $|\frac{1}{2} - \sin^2\theta_{23}| \gtrsim 0.07$ with $\sin^2\theta_{12}$ typically $< \frac{1}{3}$. Hence one can distinguish the results of these models from softly-broken tri-bimaximal mixing once the value of $|U_{e3}|^2$ and the neutrino mass hierarchy is known.

3 Trimaximal mixing deviations from TBM

We now turn to study the deviations from tri-bimaximal mixing which can arise with the less restrictive symmetry referred to as trimaximal mixing, defined by (10):

$$\mathrm{TM}_2\colon \quad \begin{pmatrix} |U_{e2}|^2 \\ |U_{\mu 2}|^2 \\ |U_{\tau 2}|^2 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix}. \tag{14}$$

This is the original version of trimaximality, for which also a model based on the $\Delta(27)$ flavor symmetry was proposed [37, 41]. From the condition $|U_{e2}|^2 = \frac{1}{3}$ it follows that

$$\sin^2\theta_{12} = \frac{1}{3}\frac{1}{1-|U_{e3}|^2} \geq \frac{1}{3}. \tag{15}$$

Note that TM_2 predicts $\sin^2\theta_{12} \geq \frac{1}{3}$, to be compared with the current best-fit values and 1σ ranges of Ref. [21, 22], $\sin^2\theta_{12} = 0.312^{+0.019}_{-0.018}$, and Ref. [20]: $\sin^2\theta_{12} = 0.304^{+0.022}_{-0.016}$. Inserting the range $|U_{e3}|^2 = 0.016 \pm 0.010$ would give $\sin^2\theta_{12} = 0.339 \pm 0.003$.

The second prediction of trimaximal mixing can be obtained from $|U_{\mu 2}|^2 = \frac{1}{3}$ and is

$$\begin{aligned} &\cos\delta \tan 2\theta_{23} \\ &\quad = \frac{2\cos\theta_{13}\cot 2\theta_{13}}{\sqrt{2-3\sin^2\theta_{13}}} = \frac{1-2|U_{e3}|^2}{|U_{e3}|\sqrt{2-3|U_{e3}|^2}} \\ &\quad \simeq \frac{1}{\sqrt{2}}\frac{1}{|U_{e3}|}\left(1-\frac{5}{4}|U_{e3}|^2 + \mathcal{O}(|U_{e3}|^4)\right). \end{aligned} \tag{16}$$

We observe that the three mixing angles are not independent but related as above. Because $|U_{\tau 2}|^2$ is related by unitarity with $|U_{e2}|^2$ and $|U_{\mu 2}|^2$, there is no third independent condition.

For $\delta \neq \pi/2$ (or $\delta \neq 3\pi/2$) and θ_{13} non-zero, θ_{23} is non-maximal. On the other hand, for maximal CP violation ($\delta = \pi/2$ or $3\pi/2$) it follows that $\sin^2\theta_{23} = \frac{1}{2}$, independent of $|U_{e3}|$. We can use the expressions for $\sin^2\theta_{12}$ and $\tan 2\theta_{23}$ to evaluate the Jarlskog invariant for leptonic CP violation, which in general reads $J_{\mathrm{CP}} = \mathrm{Im}(U_{e1}^* U_{\mu 3}^* U_{e3} U_{\mu 1}) = \frac{1}{8}\sin 2\theta_{13}\cos\theta_{13}\sin 2\theta_{23}\sin 2\theta_{12}\sin\delta$. We find

$$J_{\mathrm{CP}} \simeq \frac{\sin 2\delta}{6\sqrt{2}\sqrt{\cos^2\delta}}|U_{e3}|, \tag{17}$$

where we have expanded the lengthy exact equation, which is an odd function of θ_{13}.

In Fig. 2 we show plots of $\sin^2\theta_{12}$ and $\sin^2\theta_{23}$ as functions of $|U_{e3}|$ and δ, respectively. As can be seen, solar neutrino mixing is well within the allowed 2σ range, and the possible deviation from maximal atmospheric neutrino mixing is largest for CP conserving values of δ, grows with $|U_{e3}|$, and can exceed the 3σ range. Also displayed in Fig. 2 is a plot of $\sin^2\theta_{12}$ vs. $\sin^2\theta_{23}$, when $|U_{e3}|$ is allowed to vary. It is evident that the deviation from maximal atmospheric mixing can be larger than the deviation from $\sin^2\theta_{12} = \frac{1}{3}$. Indeed, from $\theta_{23} = \pi/4 - \epsilon$, one obtains the leading order expressions $\tan 2\theta_{23} \simeq \frac{1}{2\epsilon}$ and $\sin^2\theta_{23} \simeq \frac{1}{2} - \epsilon$. By taking only the first order term on the RHS of (16)

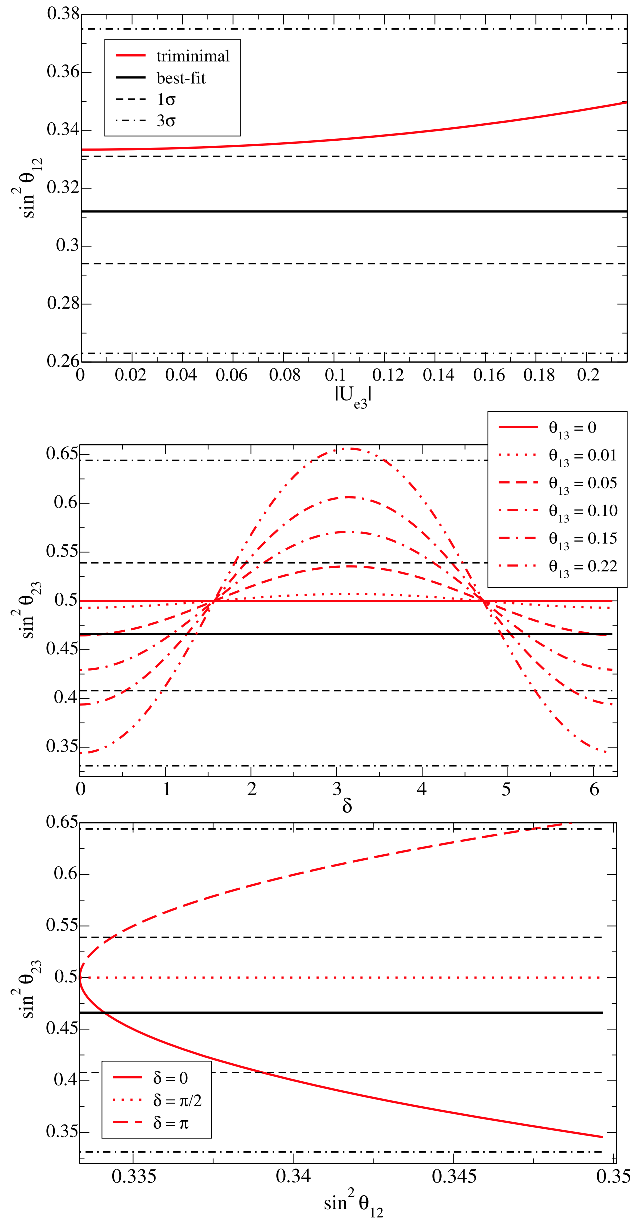

Fig. 2 Phenomenology of exact TM_2 mixing. Shown are the solar neutrino parameter $\sin^2\theta_{12}$ against $|U_{e3}|$, the atmospheric neutrino parameter $\sin^2\theta_{23}$ against δ for different values of $|U_{e3}|$ and $\sin^2\theta_{12}$ against $\sin^2\theta_{23}$. Also given are the current best-fit value and the 1σ as well as 3σ ranges from a global fit [21]

one obtains the relation

$$\left(\frac{1}{2}-\sin^2\theta_{23}\right)^2 \simeq \frac{\cos^2\delta}{2}|U_{e3}|^2 \simeq \frac{3\cos^2\delta}{2}\left(\sin^2\theta_{12}-\frac{1}{3}\right). \quad (18)$$

This expression shows that the deviation from maximal atmospheric neutrino mixing can be stronger than the deviation from $\sin^2\theta_{12} = \frac{1}{3}$. In general, the deviation in $|U_{e3}|^2$ obtained here can be considerably larger than that entertained in Sect. 2.

The trimaximal mixing matrix can be parametrized by the application of a general 13-rotation from the right [40]

$$U_{\mathrm{TM}_2} = U_{\mathrm{TBM}} R_{13}(\theta;\psi), \quad \text{where}$$
$$R_{13}(\theta;\psi) = \begin{pmatrix} \cos\theta & 0 & \sin\theta e^{-i\psi} \\ 0 & 1 & 0 \\ -\sin\theta e^{i\psi} & 0 & \cos\theta \end{pmatrix}. \quad (19)$$

It is easy to see that $|U_{e3}|^2 = \frac{2}{3}\sin^2\theta$ and $\sin^2\theta_{12} = 1/(3 - 2\sin^2\theta)$, which is equal to $\frac{1}{3}/(1-|U_{e3}|^2)$ as before. We find furthermore that

$$\sin^2\theta_{23} = \frac{1}{2} + \frac{1}{2\sqrt{3}}\frac{\sin 2\theta\cos\psi}{1-|U_{e3}|^2} \quad \text{and}$$
$$J_{\mathrm{CP}} = \frac{\sin 2\theta \sin\psi}{6\sqrt{3}}, \quad (20)$$

where $\sin 2\theta = \sqrt{6}|U_{e3}|\sqrt{1-\frac{3}{2}|U_{e3}|^2}$. The CP phase δ in this parameterization is related to the phase ψ.

By using the above form for U_{TM_2} in (19), we can obtain the corresponding neutrino mass matrix in the lepton flavor basis from

$$(m_\nu)_{\mathrm{TM}_2} = U^*_{\mathrm{TM}_2} P^* m_\nu^{\mathrm{diag}} P^\dagger U^\dagger_{\mathrm{TM}_2} = \begin{pmatrix} A & B+C & B-C \\ \cdot & \frac{1}{2}(A+B+D-2C) & \frac{1}{2}(A+B-D) \\ \cdot & \cdot & \frac{1}{2}(A+B+D+2C) \end{pmatrix}$$

where we identify (with $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$)

$$A = \frac{1}{3}\left(2m_1c_\theta^2 + 2m_3s_\theta^2 e^{2i(\psi-\beta)} + m_2e^{-2i\alpha}\right),$$
$$B = \frac{1}{3}\left(m_2e^{-2i\alpha} - m_1c_\theta^2 - m_3s_\theta^2e^{2i(\psi-\beta)}\right),$$
$$C = \frac{1}{\sqrt{3}}\left(m_1e^{-i\psi} - m_3e^{i(\psi-2\beta)}\right)s_\theta c_\theta,$$
$$D = m_3e^{-2i\beta}c_\theta^2 + m_1e^{-2i\psi}s_\theta^2.$$

It is clear from this mass matrix that the additional C terms break the original μ–τ symmetry present with tri-bimaximal mixing in a well-defined way. Note that C vanishes for $\theta = 0$ and that in this case $(m_\nu)_{\mathrm{TBM}}$ from (7) is recovered. Interestingly, if we decompose the mass matrix for TM_2 in terms of the individual neutrino masses, as done for TBM in (9), we find that m_2 is multiplied with the same flavor-democratic matrix as in (9). The other two masses are multiplied now with more complicated matrices, having entries depending on the angle θ.

Since we can now trade θ_{12} for θ_{13} with the use of (15), it is possible to obtain a simple value for the effective mass $\langle m_{ee}\rangle$ governing neutrinoless double beta decay ($0\nu\beta\beta$). With inverted hierarchy ($m_2 \simeq m_1 \gg m_3$) the general result is $\langle m\rangle = c_{13}^2\sqrt{\Delta m_{\mathrm{A}}^2}\sqrt{1-\sin^2 2\theta_{12}\sin^2\alpha}$. In case of TM_2 mixing we then find

$$\langle m\rangle \simeq \sqrt{\Delta m_{\mathrm{A}}^2}\left[\sqrt{1-\frac{8}{9}\sin^2\alpha} - \frac{1-\frac{2}{3}\sin^2\alpha}{\sqrt{1-\frac{8}{9}\sin^2\alpha}}|U_{e3}|^2\right]. \quad (21)$$

The $\frac{8}{9}$ in the first term is of course the value of $\sin^2 2\theta_{12}$ in case of tri-bimaximal mixing.

4 Variant trimaximal mixing scenarios

We have seen that the prediction of trimaximality implies $\sin^2\theta_{12} \geq \frac{1}{3}$, while the best-fit points are all below $\frac{1}{3}$. If this trend continues, then one may introduce variants of trimaximal mixing, such as

$$\mathrm{TM}_1\colon \quad \begin{pmatrix} |U_{e1}|^2 \\ |U_{\mu1}|^2 \\ |U_{\tau1}|^2 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/6 \\ 1/6 \end{pmatrix}. \quad (22)$$

Here we have fixed the first column of U_{PMNS} to have the same form as in the case of tri-bimaximal mixing. A consequence of (22) is that $\sin^2\theta_{12} \leq \frac{1}{3}$. Indeed, from $|U_{e1}|^2 = \frac{2}{3}$ one finds

$$\sin^2\theta_{12} = \frac{1}{3}\frac{1-3|U_{e3}|^2}{1-|U_{e3}|^2} \simeq \frac{1}{3}\left(1-2|U_{e3}|^2\right). \quad (23)$$

Using the range $|U_{e3}|^2 = 0.016 \pm 0.010$ gives $\sin^2\theta_{12} = 0.322 \pm 0.007$. Figure 3 shows $\sin^2\theta_{12}$ as a function of $|U_{e3}|$. In contrast to the original trimaximal mixing scheme, TM_2, the best-fit value of $\sin^2\theta_{12}$ can be obtained (for $|U_{e3}| \simeq 0.179$). Note that $\sin^2\theta_{12}$ decreases with $|U_{e3}|$. As long as $|U_{e3}| \gtrsim 0.06$, $\sin^2\theta_{12}$ is within its allowed 1σ range.

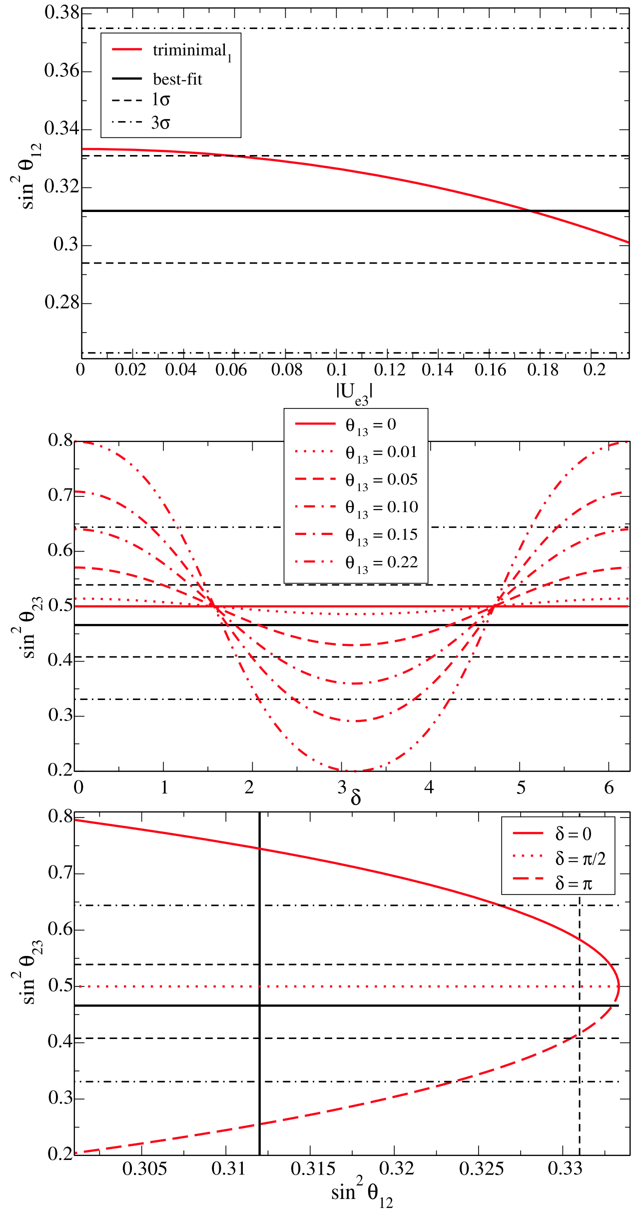

Fig. 3 Phenomenology of exact TM_1 mixing. Shown are the solar neutrino parameter $\sin^2\theta_{12}$ against $|U_{e3}|$, the atmospheric neutrino parameter $\sin^2\theta_{23}$ against δ for different values of $|U_{e3}|$ and $\sin^2\theta_{12}$ against $\sin^2\theta_{23}$. Also given are the current best-fit value and the 1σ as well as 3σ ranges from a global fit [21]

The second independent condition in (22) involving $|U_{\mu 1}|^2 = 1/6$ gives

$$\cos\delta\tan 2\theta_{23} = -\frac{1-5|U_{e3}|^2}{2\sqrt{2}|U_{e3}|\sqrt{1-3|U_{e3}|^2}} \simeq \frac{-1}{2\sqrt{2}|U_{e3}|}\left(1-\frac{7}{2}|U_{e3}|^2\right). \tag{24}$$

Here the results are qualitatively similar to the ones for tri-maximal mixing treated in Sect. 3. In Fig. 3 we also show $\sin^2\theta_{23}$ as a function of δ. As for TM_2 mixing, CP conserving values of δ maximize the deviations, but here the 3σ range can easily be overshot. Finally, in Fig. 3, $\sin^2\theta_{23}$ is plotted against $\sin^2\theta_{12}$. Again, the possible departure from $\sin^2\theta_{23} = \frac{1}{2}$ can be larger than the one from $\sin^2\theta_{12} = \frac{1}{3}$.

A mixing matrix with the TM_1 property can be obtained by multiplying U_{TBM} with a 23-rotation of angle θ from the right, $U_{\mathrm{TM}_1} = U_{\mathrm{TBM}} R_{23}(\theta;\psi)$. The observables are in this case

$$\begin{aligned} |U_{e3}|^2 &= \frac{1}{3}\sin^2\theta, \\ \sin^2\theta_{23} &= \frac{1}{2} - \frac{\sqrt{\frac{3}{2}}\sin 2\theta\cos\psi}{3-\sin^2\theta}, \\ \sin^2\theta_{12} &= 1 - \frac{2}{3-\sin^2\theta}, \\ J_{\mathrm{CP}} &= \frac{1}{6\sqrt{6}}\sin 2\theta\sin\psi, \end{aligned} \tag{25}$$

where the CP phase δ is again related to ψ. For the mass matrix we find

$$\begin{aligned} (m_\nu)_{\mathrm{TM}_1} &= U^*_{\mathrm{TM}_1} P^* m_\nu^{\mathrm{diag}} P^\dagger U^\dagger_{\mathrm{TM}_1} \\ &= \begin{pmatrix} A & B+C & B-C \\ \cdot & \frac{1}{2}(A+B+D+4C) & \frac{1}{2}(A+B-D) \\ \cdot & \cdot & \frac{1}{2}(A+B+D-4C) \end{pmatrix}, \end{aligned}$$

where we identify

$$\begin{aligned} A &= \frac{1}{3}\left(2m_1 + m_2 c_\theta^2 e^{-2i\alpha} + m_3 s_\theta^2 e^{2i(\psi-\beta)}\right), \\ B &= \frac{1}{3}\left(-m_1 + m_2 c_\theta^2 e^{-2i\alpha} + m_3 s_\theta^2 e^{2i(\psi-\beta)}\right), \\ C &= \frac{1}{\sqrt{6}}\left(m_2 e^{-i(\psi+2\alpha)} - m_3 e^{i(\psi-2\beta)}\right) s_\theta c_\theta, \\ D &= m_2 e^{-2i(\psi+\alpha)} s_\theta^2 + m_3 c_\theta^2 e^{-2i\beta}. \end{aligned}$$

Again we see that the original μ–τ symmetry is broken by the extra terms involving C. We can decompose the mass matrix for TM_1 in terms of the individual neutrino masses and find that m_1 is multiplied with the same matrix as in (9).

If we would insist that the third column of U_{TBM} remain invariant instead, i.e., $|U_{e3}|^2 = 0$, $|U_{\mu 3}|^2 = |U_{\tau 3}|^2 = \frac{1}{2}$, then $\theta_{13} = \delta = 0$, $\theta_{23} = \pi/4$, while θ_{12} is a free parameter. This case (TM_3 in our notation) is nothing other than the well-known μ–τ symmetry.

It is also of interest to consider the case where one of the rows of the tri-bimaximal mixing matrix remains invariant. Again such a result can be obtained by multiplying with a suitable two dimensional rotation matrix, but now from the left. In fact, this class of deviations from tri-bimaximal mixing corresponds to the charged lepton flavor matrix differing from its diagonal mass matrix, thus introducing a $U_\ell^\dagger$ factor in the PMNS mixing matrix, i.e., $U_{\rm PMNS} = U_\ell^\dagger U_{\rm TBM}$ [42–46].

Let us start with the case of the first row in $U_{\rm TBM}$ remaining invariant. We denote this with a superscript as

$$\text{TM}^1: \quad \left(|U_{e1}|^2, |U_{e2}|^2, |U_{e3}|^2\right) = \left(\frac{2}{3}, \frac{1}{3}, 0\right). \tag{26}$$

As a result, θ_{23} is a free parameter, while $\sin^2\theta_{12} = \frac{1}{3}$, as well as $\theta_{13} = \delta = 0$. With a rotation of $U_{\rm TBM}$ by the matrix $R_{23}(\theta;\psi)$ from the left, the light Majorana neutrino mass matrix becomes

$$(m_\nu)_{\text{TM}^1} = R_{23}^*(m_\nu)_{\rm TBM} R_{23}^\dagger. \tag{27}$$

If we consider the second or third row we can correlate all four mixing parameters. Starting with the second row, i.e.,

$$\text{TM}^2: \quad \left(|U_{\mu1}|^2, |U_{\mu2}|^2, |U_{\mu3}|^2\right) = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right), \tag{28}$$

one immediately finds from $|U_{\mu3}|^2 = \frac{1}{2}$:

$$\sin^2\theta_{23} = \frac{1}{2(1-|U_{e3}|^2)} \simeq \frac{1}{2}\left(1+|U_{e3}|^2\right) \geq \frac{1}{2}, \tag{29}$$

i.e., atmospheric neutrino mixing on the "dark side," with a maximal value of $\sin^2\theta_{23} \simeq 0.524$ for $|U_{e3}|^2 = 0.046$. Inserting (29) in $|U_{\mu2}|^2 = \frac{1}{3}$ gives a complicated and lengthy expression including $\cos\delta$, $\sin^2\theta_{12}$ and $|U_{e3}|$, which can be approximated as

$$\sin^2\theta_{12} \simeq \frac{1}{3} - \frac{2\sqrt{2}}{3}|U_{e3}|\cos\delta + \frac{1}{3}|U_{e3}|^2\cos 2\delta. \tag{30}$$

Figure 4 shows the result for the TM^2 scenario. The parameter dependence is similar to the one for the TM_1 and TM_2 scenarios, the main difference being the exchanged roles of θ_{12} and θ_{23}.

On the other hand with the third row remaining invariant under the transformation

$$\text{TM}^3: \quad \left(|U_{\tau1}|^2, |U_{\tau2}|^2, |U_{\tau3}|^2\right) = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right), \tag{31}$$

the same approximate formula with a relative sign for the term of order $|U_{e3}|$ is found. Now, however, one finds atmospheric neutrino mixing on the "bright side":

$$\sin^2\theta_{23} = \frac{1-2|U_{e3}|^2}{2(1-|U_{e3}|^2)} \simeq \frac{1}{2}\left(1-|U_{e3}|^2\right) \leq \frac{1}{2}. \tag{32}$$

The maximal deviation occurs for the largest possible $|U_{e3}|^2 = 0.046$, in which case $\sin^2\theta_{23} \simeq 0.476$. In Fig. 5 the resulting correlations for the TM^3 scenario are given.

As in the case of TM^1, the U_{TM^2} mixing matrix can be obtained by a rotation of $U_{\rm TBM}$ from the left by R_{13}. For TM^3, the rotation matrix is R_{12}. For all three cases with the first, second, or third row of $U_{\rm TBM}$ remaining invariant, solar neutrino mixing can occur in the complete 3σ range. The deviations from tri-bimaximal mixing can be considerably larger for θ_{13} in the normal hierarchical case than were observed in Sect. 2 and Ref. [36].

Finally, it is amusing to compare the trimaximal mixing schemes with the recently proposed tetramaximal one [47]. Its name stems from the fact that it can be obtained by four consecutive rotations each having a maximal an-

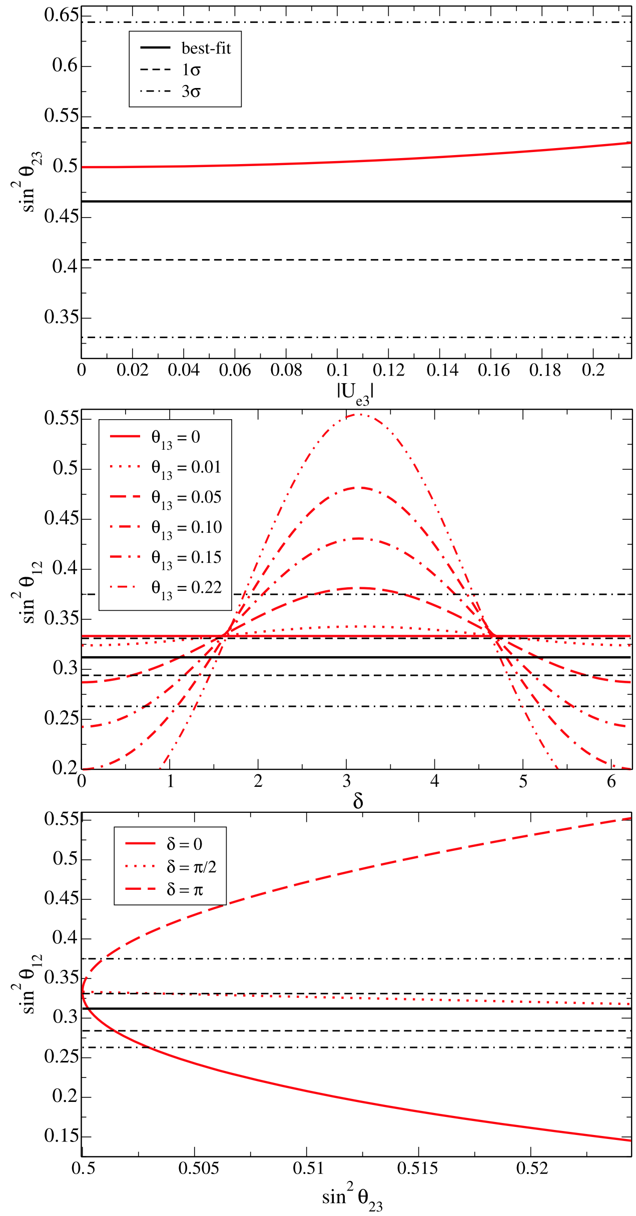

Fig. 4 Phenomenology of exact TM^2 mixing. Shown are the atmospheric neutrino parameter $\sin^2\theta_{23}$ against $|U_{e3}|$, the solar neutrino parameter $\sin^2\theta_{12}$ against δ for different values of $|U_{e3}|$ and $\sin^2\theta_{12}$ against $\sin^2\theta_{23}$. Also given are the current best-fit value and the 1σ as well as 3σ ranges from global fit [21]

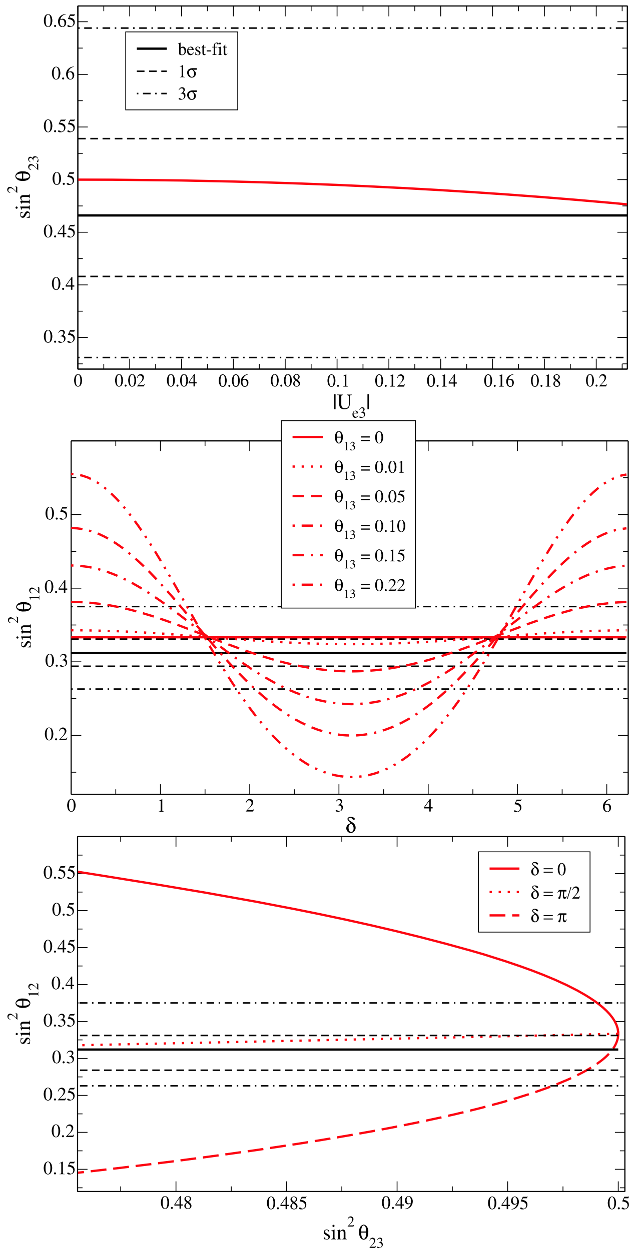

Fig. 5 Phenomenology of exact TM^3 mixing. Shown are the atmospheric neutrino parameter $\sin^2\theta_{23}$ against $|U_{e3}|$, the solar neutrino parameter $\sin^2\theta_{12}$ against δ for different values of $|U_{e3}|$ and $\sin^2\theta_{12}$ against $\sin^2\theta_{23}$. Also given are the current best-fit value and the 1σ as well as 3σ ranges from a global fit [21]

gle of $\pi/4$, and with appropriately chosen phases: $U_{\text{tetra}} = R_{23}(\pi/4;\pi/2)R_{13}(\pi/4;0)R_{12}(\pi/4;0)R_{13}(\pi/4;\pi)$. The predictions are $\delta=\pi/2$, $\sin^2\theta_{23}=\frac{1}{2}$, $|U_{e3}|^2=\frac{1}{4}(\frac{3}{2}-\sqrt{2})\simeq 0.021$ and $\sin^2\theta_{12}=1/(\frac{5}{2}+\sqrt{2})\simeq 0.255$. None of the trimaximal variants discussed in this paper could be confused with tetramaximal mixing.

5 Summary and conclusions

Although the present PMNS lepton mixing matrix deduced from experiment is consistent with tri-bimaximal mixing, it is of interest to study possible deviations which may arise in the future. In a previous paper the authors considered linear complex perturbations of the neutrino matrix away from the most general μ–τ symmetric texture which yields tri-bimaximal mixing. Here we have considered variations of trimaximal mixing which can arise, e.g., with a simple complex rotation of the U_{TBM} matrix from the right or the left. The original trimaximal mixing matrix U_{TM_2} preserves the second column of U_{TBM} for which each element has absolute value of $1/\sqrt{3}$. We have generalized this mixing scenario here and refer to the other variations as TM_k or TM^k according to which $k=1,2,3$ column or row remains invariant, respectively. Independent of our comparisons with tri-bimaximal mixing, the "trimaximal" mixing scenarios considered here are alternative, novel and testable mixing schemes.

We summarize our findings in Table 1 for the squares of the sines of the three mixing angles. For broken TBM where up to 20% deviations are allowed in every neutrino mass matrix element, we had found that the allowed variations in sine squared of the mixing angles are uncorrelated to a large extent. The perturbed solar and atmospheric neutrino mixing angles cover the entire mixing ranges presently allowed, while $|U_{e3}|^2=\sin^2\theta_{13}$ can range from zero up to 0.001 (0.014) for the case of normal (inverted) hierarchy. These upper bounds depend sensitively on the lightest neutrino mass as shown in Fig. 1, especially for a normal ordering. The largest upper bounds presently allowed are reached in the case of three-fold neutrino mass degeneracy.

For the six generalized trimaximal mixing cases considered, on the other hand, the mixing angles and Dirac CP phase are characteristically correlated for four of the cases. The exceptional cases arise for TM_3 and TM^1. For TM_3 both $|U_{e3}|^2$ and $\sin^2\theta_{23}$ remain fixed at their TBM values, while θ_{12} is a free variable and limited only by the present experimental bounds. This case corresponds to the well-known μ–τ symmetry. For TM^1, $|U_{e3}|^2$ and $\sin^2\theta_{12}$ remain fixed, while $\sin^2\theta_{23}$ remains bounded only by experiment. For TM_1 and TM_2 the solar mixing is tightly limited, with the former ranging just below the TBM value and the latter just above the TBM value of $\frac{1}{3}$. The atmospheric neutrino mixing in these two cases can cover the full presently allowed region with the former (latter) peaking at $\delta=0(\pi)$ and bottoming at $\delta=\pi(0)$. For TM^2 and TM^3 the opposite situation holds, where the atmospheric neutrino mixing range is tightly limited close to the TBM value in the bright side or dark side, respectively, while the full range for solar neutrino mixing can be realized.

When more refined ranges for the mixing angles are known, one will be able to rule out or confirm the new

Table 1 Summary of the mixing angles obtained in the exact and broken tri-bimaximal, and generalized trimaximal mixing schemes. We have for the sake of illustration expanded the exact correlations (see Figs. 2–5). For the ranges given, the present 3σ data bounds on the three mixing angles have been imposed

	$\|U_{e3}\|^2$	$\sin^2\theta_{12}$	$\sin^2\theta_{23}$
TBM	0	$\frac{1}{3}$	$\frac{1}{2}$
Broken TBM			
NH	$\lesssim$0.001	0.26 – 0.38	0.37 – 0.63
IH	$\lesssim$0.014	0.26 – 0.38	0.35 – 0.64
TM_1	$\leq$0.046	$\frac{1}{3}(1-2\|U_{e3}\|^2)$	$\frac{1}{2}-\sqrt{2}\|U_{e3}\|(1-\frac{1}{2}\|U_{e3}\|^2)\cos\delta$
		0.30 – 0.33	0.34 – 0.64
TM_2	$\leq$0.046	$\frac{1}{3}(1+\|U_{e3}\|^2)$	$\frac{1}{2}+\frac{1}{\sqrt{2}}\|U_{e3}\|(1+\frac{1}{4}\|U_{e3}\|^2)\cos\delta$
		0.33 – 0.35	0.34 – 0.64
TM_3	0	0.26 – 0.38	$\frac{1}{2}$
TM^1	0	$\frac{1}{3}$	0.34 – 0.64
TM^2	$\leq$0.046	$\frac{1}{3}-\frac{2\sqrt{2}}{3}\|U_{e3}\|\cos\delta+\frac{1}{3}\|U_{e3}\|^2\cos 2\delta$	$\frac{1}{2}(1+\|U_{e3}\|^2)$
		0.26 – 0.38	0.50 – 0.52
TM^3	$\leq$0.046	$\frac{1}{3}+\frac{2\sqrt{2}}{3}\|U_{e3}\|\cos\delta+\frac{1}{3}\|U_{e3}\|^2\cos 2\delta$	$\frac{1}{2}(1-\|U_{e3}^2\|)$
		0.26 – 0.38	0.49 – 0.50

mixing scenarios discussed here and narrow down the acceptable deviations from tri-bimaximal mixing we have discussed in this paper.

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