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COMPARISON GEOMETRY REFERRED TO WARPED PRODUCT MODELS

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Abstract. We generalize the Alexandrov-Toponogov comparison theorem to the case of complete Riemannian manifolds referred to warped product models. We prove the maximal diameter theorem and the rigidity theorem. In particular, we discuss collapsing phenomena where the curvature explosion may occur.

1. Introduction. Comparison theorems play an important role in investigating the curvature and topology of Riemannian manifolds. It was Klingenberg [10] who first introduced the notion of radial curvature of a pointed manifold (M, o). The restriction on the range of sectional or Ricci curvature of a certain class of manifolds is required to have a uniform lower bound when the comparison theorems are used. However, our comparison geometry developed in this note does not require any restriction on the range of curvature. Pointed Hadamard surfaces $(\tilde{M}, \tilde{o})$ with rotationally symmetric metric around $\tilde{o}$ are discussed by Greene and Wu [6] and Abresch [1, 2], for example, as reference spaces of complete noncompact pointed Riemannian (Kähler) manifolds. Here the radial curvature with respect to a base point $o \in M$ is bounded below by that of a Hadamard model surface. Further investigations of the radial curvature and topology of pointed manifolds can be seen in, for example, [13, 14, 15, 17, 20, 27].

Recently, the Alexandrov-Toponogov comparison theorem was established in [8, 9] for pointed manifolds whose reference surfaces admit rotationally symmetric metrics. The topology of pointed manifolds referred to such model surfaces has been discussed in [9, 12, 25], for example.

It is our purpose to develop comparison geometry whose reference spaces are warped product models of the form $\tilde{M} = (-l_-, l_+) \times_f N$, where N is a connected compact $(n-1)$-manifold, f is the warping function, and $0 < l_\pm \leq \infty$ are constants. The models previously discussed are contained as special cases of our warped product models.

Let (M, N) be a pair of a connected complete Riemannian n-manifold M and a connected compact Riemannian $(n-1)$-manifold N, where N is isometrically immersed into M. Throughout this note, when the normal bundle $\perp N$ over N is trivial, the distance function

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$t : M \to \boldsymbol{R}$ to N may take negative values. Hence, t is smooth on $M \setminus C(N)$, where $C(N)$ is the cut locus to N, and defines the oriented distance to N. It defines the usual distance to N when $\perp N$ is non-trivial. A *warped product model* is defined to be a pair (M, N) for which the sectional curvature of M on the plane sections containing ∇t depends only on the (oriented) distance to N, and t is constant on each component of $C(N)$. We first establish the characterization of the warped product models. This is obtained in a manner similar to that developed in [11].

Some notation is needed for the precise statement of our characterization: on a pair (M, N), a geodesic $\gamma : [0, a) \to M$ is called a minimizing geodesic from N if $\dot\gamma(0) \in \perp N$ and if $d(N, \gamma(t)) = t$ for every $t \in [0, a)$. *A radial curvature of* (M, N) is by definition the sectional curvature $K_M(\Pi)$ of the plane Π containing a unit vector tangent to a minimizing geodesic from N. If the normal bundle $\perp N$ is trivial, then $M \setminus C(N)$ is expressed as the union $M_- \cup M_+$, where M_+ (resp. M_-) is the set of points taking non-negative values (respectively non-positive values) of the oriented distance to N. Thus, we define

$$\rho_\pm := \inf_{x \in C(N) \cap M_\pm} \pm d(N, x), \quad l_\pm := \sup_{x \in M_\pm} \pm d(N, x) \le \infty \tag{1.1}$$

if $\perp N$ is trivial, and

$$\rho := \inf_{x \in C(N)} d(N, x), \quad l := \sup_{x \in M} d(N, x) \le \infty \tag{1.2}$$

if $\perp N$ is non-trivial. Clearly, $\rho_\pm \le l_\pm$, $\rho \le l$, and equality holds if and only if (M, N) is isometric to a warped product model.

With these understandings, we state our characterization of the warped product models as follows. We only state the case where $\perp N$ is trivial and omit the other case.

THEOREM 1.1. *Assume that the radial curvature of (M, N) depends only on the oriented* (*resp. usual*) *distance to N and that the radial curvature function is non-constant near $-l_-$ and l_+* (*resp. l*). *We then have $\rho_\pm = l_\pm$* (*resp. $\rho = l$*) *and $M \setminus C(N)$ is isometric to one of the following warped product models.*

(1) $(-l_-, \infty) \times_f \boldsymbol{S}^{n-1}$, $l_- < \infty$, $f(-l_-) = 0$, $l_+ = \infty$ ($\boldsymbol{R}^n$*-model*).

(2) $\boldsymbol{R} \times_f N$, $l_- = l_+ = \infty$ (*cylinder model*).

(3) $(-\infty, l_+) \times_f N$, $l_+ < l_- = \infty$, $f(l_+) > 0$, *and M is isometric to the quotient space of $\boldsymbol{R} \times_{\hat f} N$ by the fixed point free isometric involution* (*open Möbius strip model*).

(4) $(-l_-, l_+) \times_f \boldsymbol{S}^{n-1}$, $l_-, l_+ < \infty$, $f(-l_-) = f(l_+) = 0$ ($\boldsymbol{S}^n$*-model*).

(5) $(-l_-, l_+) \times_f \boldsymbol{S}^{n-1}$, $l_\pm < \infty$, $f(l_+) \cdot f(l_-) = 0$, $f(l_+) + f(l_-) > 0$, *and M is diffeomorphic to $\boldsymbol{R}P^n$* (*real projective model*).

(6) $(-l_-, l_+) \times_f N$, $l_- = l_+ < \infty$, $f(-l_-) = f(l_+) > 0$ (*torus model*).

(7) $(-l_-, l_+) \times_f N$, $l_-, l_+ < \infty$, $f(l_+) \cdot f(l_-) > 0$, *and M is isometric to the quotient space of $\boldsymbol{S}^1(l_- + l_+) \times_f \hat N$ by the fixed point free isometric involution* (*Klein-bottle model*). *Here $\boldsymbol{S}^1(r)$ denotes the circle of circumference r and $\hat N$ is the orientable double cover of N.*

Here the warping functions f (on M) and $\hat{f}$ (on the oriented double cover of M) have the following properties: under the assumption of Theorem 1.1, if $\perp N$ is trivial, we have the radial curvature function $K : (-l_-, l_+) \to \boldsymbol{R}$ of (M, N). Hence, $f : (-l_-, l_+) \to \boldsymbol{R}$ is the solution of the Jacobi equation

$$f'' + Kf = 0\,, \quad f(0) = 1\,, \quad f'(0) =: \mu\,, \tag{1.3}$$

where μ is the principal curvature of N with respect to $\nabla t|_N$. When $\perp N$ is non-trivial, we also have the radial curvature function $K : [0, l) \to \boldsymbol{R}$ of (M, N) and the solution $f : [0, l) \to \boldsymbol{R}$ of (1.3). We show, in the proof of Theorem 1.1 in Section 2, that $\hat{f}$ is symmetric and determined by f.

The proof of Theorem 1.1 requires fundamental properties of the cut locus $C(N)$ (Lemma 2.1) and the relation between the radial curvature function and the cut locus (Lemma 2.2). On each model as stated in Theorem 1.1, $C(N)$ takes the following special form: if $\perp N$ is trivial and $C(N)$ is disconnected, then $C(N)$ has exactly two connected components $\{x \in M\,;\, d(N, x) = \pm l_\pm\}$. Each component of $C(N)$ is isometric to $f(\pm l_\pm)N$, where aN for $a \geq 0$ denotes the scaling of N by a. In particular, if $f(\pm l_\pm) = 0$ for $l_\pm < \infty$, $\{x \in M\,;\, d(N, x) = \pm l_\pm\}$ consists of a single point which is the first focal point to N with multiplicity $\dim M - 1$. If $\perp N$ is non-trivial, then $C(N)$ is connected, $C(N) = \{x \in M\,;\, d(N, x) = l\}$, and $C(N) = f(l)N$. Clearly, $C(N) = \emptyset$ if $l_- = l_+ = \infty$.

Comparison geometry will be discussed for each warped product model as obtained in Theorem 1.1. We say that *the reference space of* (M, N) *is* (M^*, N) (or (M, N) *is referred to* (M^*, N)) if and only if the following properties are satisfied.

(1) (M^*, N) is a warped product model.

(2) $M \setminus N$ and $M^* \setminus N$ have the same number of connected components.

(3) Denote the radial curvature function of (M^*, N) by $K : (-l^*_-, l^*_+) \to \boldsymbol{R}$ when $\perp N$ is trivial, and by $K : [0, l^*) \to \boldsymbol{R}$ when $\perp N$ is non-trivial. Then, at each point $p \in M \setminus N$, every radial curvature of (M, N) is bounded below by $K(d(N, p))$, where $d(N, *)$ is the oriented or usual distance function to N in M.

REMARK 1.2. Let (M, N) be referred to (M^*, N). Let $l_\pm$ be defined in (1.1) for (M, N) and $l^*_\pm$ for (M^*, N). The condition (3) requires that

$$l_\pm \leq l^*_\pm\,. \tag{1.4}$$

The above inequality is automatically satisfied when the reference space is (1), (2), (4), or (5) of Theorem 1.1 and the open Möbius strip model with $\perp N$ being non-trivial. However, (1.4) is not necessarily satisfied for other warped product models. Therefore, we agree that (1.4) is assumed in (3) of the above definition.

In Section 3 we establish the Alexandrov-Toponogov comparison theorem for (M, N) referred to (M^*, N). Generalized geodesic triangles of the form $\triangle(N\,x\,y)$ are discussed. In Section 4 we prove the maximal diameter theorem for compact manifolds and the rigidity theorem for non-compact manifolds. Finally, in Section 5 we discuss collapsing phenomena of radially curved manifolds whose limit spaces do not have constant dimensions. Such a phenomenon is seen in the two-dimensional case [24]. It should be emphasized that the conditions on radial curvature make it possible to deal with such collapsing phenomena.

2. Characterization of warped product models. We first state relevant properties of $C(A)$ for a compact submanifold $A \subset M$. Let $\Gamma(A, x)$ for $x \in M$ be the set of all minimizing geodesics from A to x.

LEMMA 2.1. *Setting $\rho := d(A, C(A))$, we have the following.*

(1) (Berger [4], Omori [19]) *If $q \in C(A)$ and $\gamma_0, \gamma_1 \in \Gamma(A, q)$ satisfy $d(A, q) = \rho$ and $\dot\gamma_0(0) \neq \pm\dot\gamma_1(0)$, then the geodesic γ_λ, $\lambda \in [0, 1]$, defined by*

$$\gamma_\lambda(t) := \exp_q(\rho - t)v_\lambda\,, \quad t \in [0, \rho]\,, \quad v_\lambda := \frac{-(1-\lambda)\dot\gamma_0(\rho) - \lambda\dot\gamma_1(\rho)}{\|(1-\lambda)\dot\gamma_0(\rho) + \lambda\dot\gamma_1(\rho)\|}\,,$$

belongs to $\Gamma(A, q)$. In particular, q is a focal point to A along γ_λ.

(2) *If $q \in C(A)$ satisfies $d(A, q) = \rho$ and q is not a focal point to A along $\gamma \in \Gamma(A, q)$, then $\dot\gamma(2\rho) \in \perp A$. Moreover, $\Gamma(A, q)$ consists of exactly two elements.*

(3) (Berger [3]) *If $p \in M$ attains a local maximum of the usual distance function to A, then there exists, for every $\xi \in M_p$, a $\sigma \in \Gamma(A, p)$ such that $\langle \xi, -\dot\sigma(l)\rangle \geq 0$. In particular, we have $p \in C(A)$.*

The following basic lemma and Lemma 2.1 are useful in the proof of Theorem 1.1. Their proof is basically the same as that developed in [11] and is omitted here. Let M be a complete manifold with non-empty boundary $\partial M = N$. Let $B(N, a)$ for $a > 0$ denote the metric a-neighborhood around N. Then $\perp N$ is trivial and we define ρ and l by (1.2). With this notation we have

LEMMA 2.2 (Basic lemma). *If the radial curvature of (M, N) depends only on the distance to $N = \partial M$, then we have the following.*

(1) *$ds_M^2 = dt^2 + f^2(t)ds_N^2(x)$ on $B(N, \rho)$, where f satisfies* (1.3).

(2) *If $\rho < l$, then $K(t) = K(\rho)$ holds for all $t \in [\rho, l)$.*

PROOF OF THEOREM 1.1. It follows from Lemma 2.2 that $\rho_\pm = l_\pm$ if $\perp N$ is trivial, and $\rho = l$ if $\perp N$ is non-trivial. In the case where $l_\pm = \infty$ (or $l = \infty$), the conclusion is straightforward.

Suppose that $l_+ < \infty$ and $f(l_+) = 0$. If $q \in C(N)$, then Lemma 2.1 implies that q is the focal point to N along every geodesic in $\Gamma(N, q)$ and the multiplicity is $n - 1$. In particular, we have $C(N) = \{q\}$. Thus, M_+ is diffeomorphic to a closed n-disk, whose boundary N is isometric to the standard $(n-1)$-sphere of constant curvature $f(0)^{-2}K_N$.

Suppose next that $l_+ < \infty$ and $f(l_+) > 0$. Then Lemma 2.1 again implies that every point $q \in C(N)$ has the property that $\Gamma(N, q)$ consists of exactly two elements. Let $\gamma_x : [0, 2l_+] \to M_+$ for each $x \in N$ be the geodesic with $\dot\gamma_x(0) \in \perp N$. Then we have $\dot\gamma_x(2l_+) \in \perp N$ and $\gamma_x(l_+) \in C(N)$. A fixed point free isometric involution $\iota : N \to N$ is well defined by $\iota(x) := \gamma_x(2l_+)$, $x \in N$. We then observe that $C(N)$ is isometric to $f(l_+)N/_{\{\iota, \iota^2 = \mathrm{id}\}}$. Setting

$\hat{M} := [0, 2l_+] \times_{\hat{f}} N$, where $\hat{f}$ and the radial curvature function $\hat{K} : [0, 2l_+] \to \boldsymbol{R}$ satisfy

$$\hat{f}(t) := \begin{cases} f(t)\,, & 0 \le t \le l_+\,, \\ f(2l_+ - t)\,, & l_+ \le t \le 2l_+\,, \end{cases} \tag{2.1}$$

$$\hat{K}(t) := \begin{cases} K(t)\,, & 0 \le t \le l_+\,, \\ K(2l_+ - t)\,, & l_+ \le t \le 2l_+\,, \end{cases} \tag{2.2}$$

we see that $\Phi : \hat{M} \to \hat{M}$ defined by

$$\Phi(t, x) := (2l_+ - t, \iota(x))\,, \quad (t, x) \in (0, 2l_+) \times N$$

is a fixed point free isometric involution and M_+ is isometric to $\hat{M}/_{\{\Phi, \Phi^2 = \mathrm{id}\}}$. The metric structure of M_- is now obtained in the above discussion. Taking account of all possibilities, we conclude the proof in this case.

The proof for the case where $l < \infty$ and $\perp N$ is non-trivial is essentially contained in the discussion above, and is omitted here.

Assume finally that M is compact, $M \setminus N$ is connected, and $\perp N$ is trivial. We then observe that $l_- = l_+ = \rho_+ = \rho_- < \infty$. It follows from Lemma 2.1 that there exist exactly two elements in $\Gamma(N, q)$ for every $q \in C(N)$. Two geodesics in $\Gamma(N, q)$ make an angle π at q and their initial vectors at points on N have opposite directions in $\perp N$. Thus, every level hypersurface $t^{-1}(\{a\})$, $a \in (-l, l)$, is isometric to $f(a)N$. This completes the proof of Theorem 1.1. □

REMARK 2.3. If $\perp N$ is non-trivial, then N is totally geodesic. In general, N is totally umbilic with principal curvature $\mu = f'(0)$.

3. Comparison theorems. We shall prove the Alexandrov-Toponogov comparison theorem for (M, N) referred to (M^*, N). From now on *we assume that N is totally geodesic in M and M^**. *A generalized geodesic triangle* $\triangle(N\,x\,y) \subset M$ is defined by a triple $\alpha, \beta, \gamma : [0, 1] \to M$ of minimizing geodesics (or N, x, y) as follows:

$$\dot{\alpha}(0), \dot{\beta}(0) \in \perp N\,, \quad \alpha(1) = \gamma(1) = y\,, \quad \beta(1) = \gamma(0) = x\,.$$

Here x, y are chosen in the same component, say M_+ of $M \setminus N$, and α, β are minimizing geodesics from N. We consider N as a corner of $\triangle(N\,x\,y)$. We say that $\triangle(N\,x\,y) \subset M$ *is a generalized narrow triangle* if and only if $\alpha(t) \in B(\beta(t), \delta)$ for all $t \in [0, 1]$, where $\delta > 0$ is the convexity radius on the compact set $\alpha[0, 1] \cup \beta[0, 1] \cup \gamma[0, 1]$.

THEOREM 3.1. *Assume that (M, N) is referred to (M^*, N). If a generalized narrow triangle $\triangle(N\,x\,y) \subset M$ admits the corresponding generalized narrow triangle $\triangle(N\,x^*y^*) \subset M^*$ such that*

$$d(N, x) = d(N, x^*)\,, \quad d(N, y) = d(N, y^*)\,, \quad d(x, y) = d(x^*, y^*)\,,$$

then we have

$$\angle Nxy \ge \angle Nx^*y^*\,, \quad \angle Nyx \ge \angle Ny^*x^*\,. \tag{3.1}$$

Note that the existence of the corresponding narrow triangle in M^*_+ is ensured if either $l^*_+ = \infty$ or $l^*_+ < \infty$ and $f(l^*_+) = 0$. In the case where $l^*_+ < \infty$ and $f(l^*_+) > 0$, it is not certain in general if $l_+ \le l^*_+$. It is shown later (see the last paragraph in Section 4) that we can take the corresponding triangle in a suitable finite cover of (M^*, N) so as to satisfy $l_\pm \le l^*_\pm$.

The Alexandrov convexity theorem is stated as follows.

THEOREM 3.2. *Under the same assumption as in Theorem* 3.1, *if* $\gamma, \gamma^* : [0,1] \to M, M^*$ *are the edges of* $\triangle(N\,x\,y)$, $\triangle(N\,x^*\,y^*)$ *opposite to* N *such that* $\gamma(0) = x$, $\gamma(1) = y$ *and* $\gamma^*(0) = x^*$, $\gamma^*(1) = y^*$, *then*

$$d(N, \gamma(t)) \ge d(N, \gamma^*(t))\,, \quad t \in [0,1]\,. \tag{3.2}$$

The Clairaut relation for geodesics on (M^*, N) plays an important role in developing comparison geometry. Our model does not have the trigonometric rule for geodesic triangles. If $\gamma^* : \boldsymbol{R} \to M^*$ is a geodesic transversal to a meridian, then the Clairaut relation implies that $\dot\gamma^*(s)$ is linearly independent of $\nabla t(\gamma^*(s))$ for all $s \in \boldsymbol{R}$. Thus, we obtain a ruled surface $\mathcal{S}(\gamma^*) \subset M^*$ generated by the meridians passing through each point of $\gamma^*(\boldsymbol{R})$. If $\mathrm{Pr}_2 : M^* \to N$ is the second projection such that $\mathrm{Pr}_2(t, x) := x$, $(t, x) \in M^*$, then $x : \boldsymbol{R} \to N$ defined by $x(s) := \mathrm{Pr}_2(\gamma^*(s))$ is a regular smooth curve, which may be considered as the base curve of $\mathcal{S}(\gamma^*)$. We define a regular parameterization $\varphi : \boldsymbol{R}^2 \to \mathcal{S}(\gamma^*) \subset M^*$ of $\mathcal{S}(\gamma^*)$ by $\varphi(t, u) := (t, x(u))$, $(t, u) \in \boldsymbol{R}^2$. The metric on $\mathcal{S}(\gamma^*)$ induced through φ is given by $ds^2_{\mathcal{S}(\gamma^*)} = dt^2 + f^2(t)\,du^2$, $(t, u) \in \boldsymbol{R}^2$. We then have the Clairaut relation for a warped product model (M^*, N).

LEMMA 3.3 (The Clairaut relation). *Let* $\gamma^* : \boldsymbol{R} \to M^*$ *be a geodesic transversal to a meridian and* $\mathcal{S}(\gamma^*) \subset M^*$ *the ruled surface generated by* γ^*. *If we set* $\gamma^*(s) = (t(s), x(s)) \in M^*$ *and*

$$\alpha(s) := \angle(\dot\gamma^*(s), \nabla t(\gamma^*(s)))\,, \quad s \in \boldsymbol{R}\,,$$

then there exists a constant $C(\gamma^*)$ *depending only on* γ^* *such that*

$$f(t(s)) \sin\alpha(s) = C(\gamma^*)\,, \quad s \in \boldsymbol{R}\,. \tag{3.3}$$

To each generalized triangle $\triangle(N\,x^*\,y^*) \subset M^*$ we assign the ruled surface $\mathcal{S}(\gamma^*)$, where $\gamma^* : [0,1] \to M^*$ is the edge of $\triangle(N\,x^*\,y^*)$ opposite to N. Since each point of $\mathcal{S}(\gamma^*)$ admits a radial direction ∇t, the Gaussian curvature on $\mathcal{S}(\gamma^*)$ does not exceed the radial curvature of M^*. Our discussion of the angle comparison is applied to the ruled surface $\mathcal{S}(\gamma^*)$.

PROOF OF THEOREMS 3.1 AND 3.2. For a sufficiently small fixed $\varepsilon > 0$ we define a warped product model M^*_ε whose radial curvature function K_ε is defined by $K_\varepsilon := K - \varepsilon$. We may consider that the generalized narrow triangle $\triangle(Nx^*_\varepsilon y^*_\varepsilon) \subset M^*_\varepsilon$ corresponding to $\triangle(N\,x\,y)$ exists. Let $\mathcal{S}(\gamma^*_\varepsilon) \subset M^*_\varepsilon$ be the ruled surface generated by the edge $\gamma^*_\varepsilon : [0,1] \to M^*_\varepsilon$ of $\triangle(Nx^*_\varepsilon y^*_\varepsilon)$ opposite to N. Let $\triangle(Nx^*_\varepsilon \hat y_\varepsilon) \subset \mathcal{S}(\gamma^*_\varepsilon)$ be another generalized narrow triangle such that

$$\angle yxN = \angle \hat y_\varepsilon x^*_\varepsilon N\,, \quad d(y, x) = d(\hat y_\varepsilon, x^*_\varepsilon)\,, \quad d(N, x) = d(N, x^*_\varepsilon)\,. \tag{3.4}$$

For a minimizing geodesic $\hat{\alpha}_\varepsilon : [0,1] \to \mathcal{S}(\gamma_\varepsilon^*)$ from N to $\hat{y}_\varepsilon = \hat{\alpha}_\varepsilon(1)$, we find a vector field Y_ε^* along the edge β_ε^* opposite to $\hat{y}_\varepsilon$ such that

$$\hat{\alpha}_\varepsilon(t) = \exp_{\beta_\varepsilon^*(t)} Y_\varepsilon^*(t)\,, \quad \|Y_\varepsilon^*\| < \delta\,, \quad t \in [0,1]\,.$$

We also have the corresponding vector field Y_ε along β by the trivial identification of parallel fields along β and β_ε^*. Setting for each $0 \le t \le 1$,

$$\sigma_t(s) := \exp_{\beta(t)} sY_\varepsilon(t)\,, \quad \sigma_t^*(s) := \exp_{\beta_\varepsilon^*(t)} sY_\varepsilon^*(t)\,, \quad s \in [0,1]\,,$$

we observe that the maps $V,\ V^* : [0,1] \times [0,1] \to M,\ \mathcal{S}(\gamma_\varepsilon^*)$ defined by

$$V(t,s) := \sigma_t(s)\,, \quad V^*(t,s) := \sigma_t^*(s)\,, \quad (t,s) \in [0,1] \times [0,1]$$

are geodesic variations along each geodesic $\sigma_t,\ \sigma_t^*$. In particular, both $t \mapsto V(t,1)$ and $t \mapsto V^*(t,1)$ are curves, to which the Berger comparison theorem is applied. If K_ε^* is the Gaussian curvature of $\mathcal{S}(\gamma_\varepsilon^*)$, then, for every $0 \le t \le 1$,

$$K_M(\dot{\beta}(t), Y_\varepsilon(t)) \ge K(d(N,\beta(t))) > K_\varepsilon(d(N,\beta(t))) \ge K_\varepsilon^*(\beta_\varepsilon^*(t))\,.$$

Here the last inequality follows from the fact that meridians are asymptotic lines on $\mathcal{S}(\gamma^*)$. Since $\alpha[0,1]$ is sufficiently close to $\beta[0,1]$, we have

$$\begin{aligned} K_M(dV_{(t,s)}(\partial/\partial t), dV_{(t,s)}(\partial/\partial s) &\ge K_{M_\varepsilon^*}(dV_{(t,s)}^*(\partial/\partial t), dV_{(t,s)}^*(\partial/\partial s)) \\ &= K_\varepsilon^*(\sigma_t^*(s)), \quad s,t \in [0,1]\,. \end{aligned}$$

The Jacobi fields along σ_t and σ_t^* associated with the geodesic variations $V(t,s)$ and $V^*(t,s)$ have the same initial condition. The Berger comparison theorem implies that

$$\begin{aligned} d(N,\hat{y}_\varepsilon) = L(\hat{\alpha}_\varepsilon) &= \int_0^1 \|\, dV_{(t,1)}^*(\partial/\partial t)\|dt \\ &\ge \int_0^1 \|\, dV_{(t,1)}(\partial/\partial t)\|dt \ge d(N,y)\,, \end{aligned}$$

and, hence, $d(N,\hat{y}_\varepsilon) \ge d(N,y_\varepsilon^*) = d(N,y)$. This proves that

$$\angle Nxy \ge \angle Nx_\varepsilon^* y_\varepsilon^*\,, \quad \angle Nyx \ge \angle Ny_\varepsilon^* x_\varepsilon^*\,. \tag{3.5}$$

Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be chosen such that for each $i = 1,\ldots,k$, $\triangle(N\gamma(t_{i-1})\gamma(t_i))$ is a generalized narrow triangle and admits the corresponding generalized triangle $\triangle(N\gamma_\varepsilon^*(t_{i-1})\gamma_\varepsilon^*(t_i)) \subset \mathcal{S}(\gamma_\varepsilon^*)$ satisfying (3.5). The broken geodesic with corners at $x_\varepsilon^* = \gamma_\varepsilon^*(t_0),\ldots,\gamma_\varepsilon^*(t_k)$ is convex in $\mathcal{S}(\gamma_\varepsilon^*)$. Therefore, the stretching technique on $\mathcal{S}(\gamma_\varepsilon^*)$ shows that if $\triangle(Nx_\varepsilon^*\hat{\gamma}_\varepsilon^*(t_i)) \subset \mathcal{S}(\gamma_\varepsilon^*)$ is the generalized triangle corresponding to $\triangle(Nx\gamma(t_i))$, then the sequence of angles $\{\angle Nx_\varepsilon^*\hat{\gamma}_\varepsilon^*(t_i)\}_{i=1,2,\ldots,k}$ at x_ε^* is monotone non-increasing in i. If $z = \gamma(t)$ and if $z_i^* \in x_\varepsilon^*\hat{\gamma}_\varepsilon^*(t_i)$ is the point corresponding to z on $\triangle(Nx_\varepsilon^*\hat{\gamma}_\varepsilon^*(t_i))$, then $d(N,z) \ge d(N,z_i^*)$ and the sequence $\{d(N,z_i^*)\}_i$ is monotone non-increasing in i. We conclude the proof by letting $\varepsilon \downarrow 0$. □

We shall prove the angle comparison theorem for generalized geodesic triangles on (M, N) referred to (M^*, N). Given a generalized triangle $\triangle(Nxy) \subset M_+$ with edges $\alpha, \beta, \gamma : [0, 1] \to M_+$ and its corresponding generalized triangle $\triangle(Nx^*y^*) \subset M^*_+$ with edges $\alpha^*, \beta^*, \gamma^* : [0, 1] \to M^*$, we always denote the edge opposite to N by γ, γ^*. The partition of $\triangle(Nx^*y^*)$ into narrow triangles $\{\triangle(N\gamma^*(t_{i-1})\gamma^*(t_i))\}_{i=1,2,\dots,k}$ forms the boundary of a locally convex region on $\mathcal{S}(\gamma^*)$. With this notation we state the following.

THEOREM 3.4. *Assume that (M^*, N) satisfies $f(l_+) = 0$ or $l_+ = \infty$, and that (M, N) is referred to (M^*, N). For every generalized triangle $\triangle(N\,x\,y) \subset M_+$ there exists a generalized geodesic triangle $\triangle(N\,x^*\,y^*) \subset \mathcal{S}(\gamma^*)$ such that*

$$d(N, x) = d(N, x^*)\,, \quad d(N, y) = d(N, y^*)\,, \quad d(x, y) = |x^*y^*|\,,$$
$$\angle Nxy \geq \angle Nx^*y^*\,, \quad \angle Nyx \geq \angle Ny^*x^*\,.$$

Here $|x^*y^*|$ denotes the length of the edge x^*y^* which is not necessarily minimizing.

4. The maximal diameter theorem and rigidity theorem. In [9] we have proved the maximal diameter theorem for a compact (M, o) referred to an $\boldsymbol{S}^n$-model (M^*, o^*). Note that if (M, N) is referred to (M^*, N) and if $f(\pm l^*_\pm) = 0$, then $l_\pm \leq l^*_\pm$. However, we do not know in general whether $l_\pm \leq l^*_\pm$ holds when $f(\pm l^*_\pm) > 0$. The maximal diameter theorem follows from Theorem 3.1. Its proof is omitted here.

THEOREM 4.1 (The maximal diameter theorem). *We have the following.*

(i) *Let (M^*, N) be an $\boldsymbol{S}^n$-model with warping function $f : (-l^*_-, l^*_+) \to \boldsymbol{R}$ satisfying $f'(0) = 0$. If (M, N) is referred to (M^*, N), then $l_\pm \leq l^*_\pm$ and the equality holds if and only if M is isometric to M^*.*

(ii) *Let (M^*, N) be an $\boldsymbol{R}P^n$-model with non-trivial $\perp N$. If (M, N) is referred to (M^*, N), then $l \leq l^*$ and the equality holds if and only if M is isometric to M^*.*

The volume comparison theorem due to Heintze et al. [7, 16] implies the following.

THEOREM 4.2 (The maximal volume theorem). *Let (M^*, N) be a Klein-bottle model or a torus model and (M, N) be referred to (M^*, N) such that $l_\pm \leq l^*_\pm$. Then*

$$\mathrm{vol}(M_\pm) \leq \mathrm{vol}(M^*_\pm)\,,$$

and the equalities hold simultaneously if and only if (M, N) or its double cover is isometric to a torus model.

We shall discuss the rigidity theorems for non-compact manifolds. Sakai [21, 22, 23] has established the rigidity theorem for cylinder models under the condition of Ricci curvature. The following theorem has been proved in [25] when (M^*, o^*) is an $\boldsymbol{R}^n$-model.

THEOREM 4.3. *Let (M^*, N) be the reference space of (M, N) such that both M_+ and M_- are non-compact. Then M is diffeomorphic to M^* if one of the following is satisfied*:

$$\liminf_{t\downarrow-\infty} f(t) = \liminf_{t\uparrow\infty} f(t) = 0\,, \tag{4.1}$$

$$\lim \sup_{t\to\pm\infty} f(t) < \infty\,. \tag{4.2}$$

Moreover, M is isometric to M^* if

$$\int_{-\infty}^{0} f^{-2}(t)dt = \int_{0}^{\infty} f^{-2}(t)dt = \infty\,. \tag{4.3}$$

PROOF OF THEOREM 4.3. We concentrate our discussion on M_+^* and M_+. We first prove that if $\sigma^* : [0,\infty) \to M_+^*$ is an arbitrary fixed meridian, then there exists a unique asymptotic class of all rays on M_+^*. Indeed, if $x^* \in M_+^* \setminus \sigma^*[0,\infty)$ and if $\tau^* : [0,\infty) \to M_+^*$ is a ray from x^* asymptotic to σ^*, we then prove that $\dot\tau^*(0) = \nabla t(x^*)$. If (4.1) is satisfied, then the conclusion follows from $C(\tau^*) = 0$. Here $C(\tau^*)$ is the Clairaut constant of τ^*. Suppose that (4.2) and $\dot\tau^*(0) \neq \nabla t(x^*)$ are satisfied. Suppose further that (4.1) is not satisfied. Then there exist $a > 0$ and $t_0 \gg 1$ such that

$$dt^2 + f^2(t)du^2 \geq (1+a^2)dt^2\,, \quad t \geq t_0$$

holds along $\tau^*(s) = (t(s), u(s))$ (cf. [25, 26]). If $1 \ll s_0 < s_1$, then

$$s_1 - s_0 \geq \sqrt{1+a^2}(t(s_1) - t(s_0))\,.$$

On the other hand, (4.2) implies that there exists a constant $b > 0$ such that

$$d(\tau^*(s), \sigma^*(t(s))) \leq b\,, \quad s \geq 0\,.$$

Then, the triangle inequality implies that

$$\begin{aligned} d(\tau^*(s_0), \tau^*(s_1)) &= s_1 - s_0 \\ &\leq d(\tau^*(s_0), \sigma^*(t(s_0))) + t(s_1) - t(s_0) + d(\tau^*(s_1), \sigma^*(t(s_1))) \\ &\leq t(s_1) - t(s_0) + 2b\,. \end{aligned}$$

Thus, a contradiction is derived for a sufficiently large s_1 and, hence, (4.2) implies the uniqueness of the asymptotic class in M_+^*.

We next prove that (4.1) or (4.2) implies that if (M, N) is referred to (M^*, N), then $C(N) = \emptyset$ in M. Let $\sigma : [0,\infty) \to M_+$ be an arbitrary fixed ray from N. Let $x \in M_+ \setminus \sigma[0,\infty)$ and $\sigma_x : [0,\infty) \to M_+$ be a ray asymptotic to σ with $\sigma_x(0) = x$. Theorem 3.4 then implies that

$$\angle N\, x\, \sigma_x(t) = \pi\,, \quad t > 0\,.$$

In particular, every geodesic $\tau : [0,\infty) \to M_+$ with $\dot\tau(0) \in \perp N$ is a ray from N which is asymptotic to σ. This proves that $C(N) \cap M_+ = \emptyset$ and, hence, that M_+ is diffeomorphic to the normal bundle over N in M_+. Thus, the first part of the proof is concluded.

We finally show that if (4.3) is satisfied, then $K_M(\dot\sigma(t), X) = K(t)$ for every $X \in T_{\sigma(t)}M$ with $X \perp \dot\sigma(t)$. Let $Y_1(t), Y_2(t), \ldots, Y_{n-1}(t)$ be N-Jacobi fields along σ such that

$\{Y_1(0), Y_2(0), \dots, Y_{n-1}(0)\}$ forms an orthonormal basis of $T_{\sigma(0)}N$ and $Y_1'(0) = Y_2'(0) = \cdots = Y_{n-1}'(0) = 0$. Then $Y_1(t) \wedge Y_2(t) \wedge \cdots \wedge Y_{n-1}(t)$ gives the volume element at $\sigma(t)$ of the t-level hypersurface $N_t = \{x \in M_+ ;\ d(N, x) = t\}$ and, moreover, $J(t) := \{ Y_1(t) \wedge Y_2(t) \wedge \cdots \wedge Y_{n-1}(t) \}^{1/(n-1)}$ satisfies a well-known equation

$$J'' + \frac{\mathrm{Ric}(\dot\sigma(t))}{n-1} J = -\frac{1}{(n-1)^2} \sum_{i<j} (\mu_i - \mu_j)^2 J \,, \tag{4.4}$$

and $J(0) = 1,\ J'(0) = 0$ (see [5]). Here $\mu_1(t), \mu_2(t), \dots, \mu_{n-1}(t)$ are the principal curvatures of N_t at $\sigma(t)$ with respect to the unit normal $\dot\sigma(t)$. The corresponding function $J^*(t)$ for $N_t^* = \{x^* \in M_+^* ;\ d(N^*, x^*) = t\}$ satisfies

$$(J^*)'' + \frac{\mathrm{Ric}(\dot\sigma^*(t))}{n-1} J^* = 0 \,, \quad J^*(0) = 1 \,, \quad (J^*)'(0) = 0 \,. \tag{4.5}$$

Actually, $\mathrm{Ric}(\dot\sigma^*(t))/(n-1) = K(t)$ holds, and we see that $f = J^*$ from initial conditions. Applying [25, Lemma 3.1], we obtain $J = J^*$,

$$\frac{\mathrm{Ric}(\dot\sigma(t))}{n-1} + \frac{1}{(n-1)^2} \sum_{i<j} (\mu_i - \mu_j)^2 = K(t) \,,$$

and $\mu(t) := \mu_1(t) = \mu_2(t) = \cdots = \mu_{n-1}(t)$. The assumption for the radial curvature of (M, N) implies that $K_M(\dot\sigma(t), X) = K(t)$. Since the principal curvature $\mu^*(t)$ of N_t^* is given as $\mu^*(t) = f'(t)/f(t) = \mu(t)$, we see that N_t is isometric to N_t^* and, hence, that M is isometric to M^*. □

In the case where (M^*, N) is a torus model, the following corollary is a direct consequence of Theorem 4.3.

COROLLARY 4.4. *If the infinite cover $(\hat M^*, N)$ of a torus model (M^*, N) is the reference space of (M, N), and if M_+ and M_- are non-compact, then M is isometric to $\hat M^*$.*

REMARK 4.5. Let (M^*, N) be a torus model with $M^* \backslash C(N) = (-l^*, l^*) \times_f N$, $f(-l^*) = f(l^*) > 0$, and its infinite cover $\boldsymbol{R} \times_{\tilde f} N$ have the property that $2l^*$ is the fundamental period of $\tilde f$. If (M, N) is referred to $\boldsymbol{R} \times_{\tilde f} N$ and if M is compact, we then choose the finite cover $(\hat M^*, N)$ of (M, N) and two positive integers k_+ and k_- such that

$$(k_- - 1)\, l^* < l_- \le k_- l^* \,, \quad (k_+ - 1)\, l^* < l_+ \le k_+ l^* \,,$$

where $\hat M^*$ is isometric to $\boldsymbol{S}^1((k_+ + k_-)\, l^*) \times_{\tilde f} N$, and $\boldsymbol{S}^1(r)$ is the circle of circumference r.

In principle, we can develop comparison geometry for (M, N) referred to a cylinder or Klein-bottle model $(\hat M^*, N)$.

5. Collapsing radially curved manifolds. To describe our collapsing phenomena, we shall introduce new models. When an $\boldsymbol{R}^n$-model defined on $\boldsymbol{R}^n$ is realized in $\boldsymbol{R}^{n+1}$ as a hypersurface of revolution around the x_{n+1}-axis, f is the profile curve parameterized by arc length measured from the base point, which is the point of intersection with the rotation axis. Then $f(t)$ is the radius of the standard $(n-1)$-sphere in the hyperplane orthogonal to the

x_{n+1}-axis. Roughly speaking, a singular model is obtained when the profile curve touches the rotation axis. More precisely, a *singular n-model $(\hat{M}, \hat{o})$ with base point at $\hat{o}$* is, by definition, a metric space whose pseudo metric $d\hat{s}^2$ around $\hat{o}$ is expressed as

$$d\hat{s}^2 = dt^2 + \hat{f}^2(t) ds^2_{\boldsymbol{S}^{n-1}}(\Theta), \quad (t, \Theta) \in (0, \infty) \times \boldsymbol{S}^{n-1}. \tag{5.1}$$

Here the warping function $\hat{f} : (0, \infty) \to \boldsymbol{R}$ of $\hat{M}$ is *non-negative, continuous on* $(0, \infty)$, *and smooth on the interior* $\mathrm{Int}(\mathrm{supp}(\hat{f}))$ *of* $\mathrm{supp}(\hat{f})$. Further, the radial curvature function $\hat{K}$ is defined and smooth on $\mathrm{Int}(\mathrm{supp}(\hat{f}))$ on which the following Jacobi equation is satisfied:

$$\hat{f}'' + \hat{K}\hat{f} = 0, \quad \hat{f}(0) = 0, \quad \hat{f}'(0) = 1. \tag{5.2}$$

Clearly, a singular n-model $(\hat{M}, \hat{o})$ has dimension 1 in a neighborhood of $(t, \Theta) \in (0, \infty) \setminus (\mathrm{supp}(\hat{f}))$ and dimension n in $\mathrm{Int}(\mathrm{supp}(\hat{f}))$.

We next consider a sequence of n-models $\{(M_j^*, o_j^*)\}_{j=1,2,\dots}$ converging to a singular model $(\hat{M}, \hat{o})$. Each model (M_j^*, o_j^*) has the metric $ds^2_{M_j^*}$ of the form (5.1) with the radial curvature function $K_j^* : [0, \infty) \to \boldsymbol{R}$ satisfying (5.2) for its warping function $f_j^* : (0, \infty) \to \boldsymbol{R}$. We assume that the radial curvature function $\hat{K}$ and the warping function $\hat{f}$ of a singular model $(\hat{M}, \hat{o})$ are obtained as the limits of the sequences $\{K_j^*\}_{l=1,2,\dots}$ and $\{f_j\}_{l=1,2,\dots}$ in the following way:

$$\lim_{j\to\infty} \frac{d^k f_j^*}{dt^k} = \frac{d^k \hat{f}}{dt^k} \quad \text{for } k = 0, 1, 2 \quad \text{and} \quad \lim_{j\to\infty} K_j^* = \hat{K}. \tag{5.3}$$

Here the convergence is uniform on every compact set of $\mathrm{Int}(\mathrm{supp}(\hat{f}))$. The pseudo metric of $(\hat{M}, \hat{o})$ is expressed as (5.1). *We say that a sequence $\{(M_j^*, o_j^*)\}_{j=1,2,\dots}$ of n-models converges to a singular model $(\hat{M}, \hat{o})$ if and only if* (5.3) *holds*.

THEOREM 5.1. *Let $\{(M_j^*, o_j^*)\}_{j=1,2,\dots}$ be a sequence of models converging to a singular model $(\hat{M}, \hat{o})$. Let $\{(M_j, o_j)\}_{j=1,2,\dots}$ be a sequence of complete non-compactn-manifolds such that each (M_j, o_j) is referred to (M_j^*, o_j^*). Then the pointed Hausdorff limit of $\{(M_j, o_j)\}_{j=1,2,\dots}$ exists and is isometric to the singular n-model $(\hat{M}, \hat{o})$ if the warping functions satisfy*

$$\int_1^\infty (f_j^*)^{-2}(t)dt = \infty, \quad j = 1, 2, \dots. \tag{5.4}$$

A similar observation is made on a sequence of $\{(M_j, N)\}_{j=1,2,\dots}$ referred to warped product models. Let $\{(M_j^*, N)\}_{j=1,2\dots}$ be a sequence of cylinder models such that each (M_j^*, N) has its warping function $f_j^* : \boldsymbol{R} \to \boldsymbol{R}_+$ and radial curvature function $K_j : \boldsymbol{R} \to \boldsymbol{R}$, and $N \subset M_j^*$ is totally geodesic. Assume that there exists a non-negative piecewise smooth function $\hat{f} : \boldsymbol{R} \to \boldsymbol{R}_+$ and the radial curvature function $\hat{K} : \mathrm{Int}(\mathrm{supp}(\hat{f})) \to \boldsymbol{R}$ satisfying (5.3). The singular space $(\hat{M}, N)$ is obtained by the pointed Hausdorff limit of the sequence

$\{(M_j^*, N)\}_{j=1,2\dots}$, where the pseudo metric is expressed by

$$ds^2_{\hat{M}} = dt^2 + \hat{f}^2(t)\, ds^2_N(x)\,, \quad (t, x) \in (-\infty, \infty) \times N\,. \tag{5.5}$$

THEOREM 5.2. *Let $\{(M_j^*, N)\}_{j=1,2,\dots}$ be a sequence of cylinder models converging to a singular model $(\hat{M}, N)$. Let $\{(M_j, N)\}_{j=1,2,\dots}$ be a sequence of complete non-compact n-manifolds such that each (M_j, N) is referred to (M_j^*, N). We then have*

$$\lim_{j\to\infty} d_{pGH}(M_j, N) = (\hat{M}, N) \quad (\textit{isometric})$$

if the warping functions satisfy

$$\int_0^\infty (f_j^*)^{-2}(t)dt = \infty\,, \quad j = 1, 2, \dots\,.$$

The key point of the proof of Theorems 5.1 and 5.2 is the Sturm comparison theorem for the Jacobi equation (see [25, Lemma 3.1]) defined on $[0, \infty)$:

$$f_i''(t) + K_i(t) f_i(t) = 0\,, \quad f_i(0) = 0\,, \quad f_i'(0) = 1\,, \quad t \geq 0,\ i = 1, 2\,.$$

Assume that $K_1 \geq K_2$ and $f_1 > 0$ on $(0, \infty)$. Lemma 3.1 in [25] implies that if $\int_1^\infty f_2^{-2}(t)dt = \infty$, then $f_1 = f_2$ and $K_1 = K_2$ on $[0, \infty)$. By a slight modification of the proof of [25, Lemma 3.1], we observe that the same conclusion is valid for the initial conditions $f_i(0) = 1,\ f_i'(0) = 0,\ i = 1, 2$.

PROOF OF THEOREMS 5.1 AND 5.2. Let $\gamma_j : [0, \infty) \to M_j$ for $j = 1, 2, \dots$ be a ray with $\gamma_j(0) = o_j$ and Y_j a Jacobi field along γ_j such that

$$Y_j(0) = 0\,, \quad \|Y'(0)\| = 1\,, \quad j = 1, 2, \dots\,.$$

The radial curvature $K_{M_j}(\dot{\gamma}_j(t), Y_j(t))$ is bounded below by $K_j(t)$. We then observe that

$$K_{M_j}(\dot{\gamma}_j(t), Y_j(t)) = K_j(t)\,, \quad Y_j(t) = f_j(t)E_j(t)\,, \quad t \geq 0\,,$$

where E_j is the parallel field along γ_j such that $E_j(0) = Y_j'(0)$. We see from [25] that M_j is isometric to M_j^*. This proves Theorem 5.1.

The proof of Theorem 5.2 is now immediate from the above discussion and Theorem 4.3. □

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