Open access • Journal Article • DOI:10.1029/96WR03848

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Published on: 01 Apr 1997 - Water Resources Research (John Wiley & Sons, Ltd)

Topics: Estimator, Generalized extreme value distribution, Shape parameter, Efficient estimator and Method of moments (statistics)

WATER RESOURCES RESEARCH, VOL. 33, NO. 4, PAGES 747–757, APRIL 1997

P671
96-97

Comparison of annual maximum series and partial duration series methods for modeling extreme hydrologic events
1. At-site modeling

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Abstract. Two different models for analyzing extreme hydrologic events, based on, respectively, partial duration series (PDS) and annual maximum series (AMS), are compared. The PDS model assumes a generalized Pareto distribution for modeling threshold exceedances corresponding to a generalized extreme value distribution for annual maxima. The performance of the two models in terms of the uncertainty of the T-year event estimator is evaluated in the cases of estimation with, respectively, the maximum likelihood (ML) method, the method of moments (MOM), and the method of probability weighted moments (PWM). In the case of ML estimation, the PDS model provides the most efficient T-year event estimator. In the cases of MOM and PWM estimation, the PDS model is generally preferable for negative shape parameters, whereas the AMS model yields the most efficient estimator for positive shape parameters. A comparison of the considered methods reveals that in general, one should use the PDS model with MOM estimation for negative shape parameters, the PDS model with exponentially distributed exceedances if the shape parameter is close to zero, the AMS model with MOM estimation for moderately positive shape parameters, and the PDS model with ML estimation for large positive shape parameters. Since heavy-tailed distributions, corresponding to negative shape parameters, are far the most common in hydrology, the PDS model generally is to be preferred for at-site quantile estimation.

Introduction

As an alternative to the annual maximum series (AMS) approach in hydrologic frequency modeling, the partial duration series (PDS) method, also denoted the peaks over threshold (POT) method, has been advocated. The classical PDS model comprises the assumptions of a Poisson distributed number of threshold exceedances and independent exponentially distributed exceedance magnitudes [*Shane and Lynn*, 1964; *Todorovic and Zelenhasic*, 1970]. This implies that the annual maxima follow the Gumbel (EV1) distribution. Alternative exceedance distributions have been proposed, including the gamma distribution [*Zelenhasic*, 1970], the Weibull distribution [*Miquel*, 1984; *Ekanayake and Cruise*, 1993], and the lognormal distribution [*Rosbjerg et al.*, 1991]. In recent years, several papers have focused on the generalized Pareto (GP) distribution [e.g., *Van Montfort and Witter*, 1986; *Hosking and Wallis*, 1987; *Fitzgerald*, 1989; *Davison and Smith*, 1990; *Wang*, 1991; *Rosbjerg et al.*, 1992; *Madsen et al.*, 1994, 1995]. The GP distribution reduces to the exponential distribution as a special case, and it implies the annual maxima to be generalized extreme value (GEV) distributed. The GEV distribution was recommended for flood frequency analysis in the U.K. *Flood Studies Report* [*Natural Environment Research Council*, 1975], and, since the introduction of the index-flood procedure based on probability-weighted moments estimation by *Wallis* [1980] and *Greis and Wood* [1981], it has gained much interest in regional frequency studies [e.g., *Hosking et al.*, 1985a; *Wallis and Wood*, 1985; *Lettenmaier et al.*, 1987; *Hosking and Wallis*, 1988; *Chowdhury et al.*, 1991; *Stedinger and Lu*, 1995].

An objection against the AMS method is that it considers only the annual maximum value, notwithstanding that secondary events in one year may exceed the annual maxima of other years. In addition, annual maximum floods observed in dry years may in some regions be very small, and inclusion of these events can significantly bias the outcome of the extreme value analysis. The PDS method avoids these drawbacks by considering all events above a certain threshold level. However, in spite of this, the PDS method seems to be much less applied in hydrologic studies than the AMS method, mainly due to difficulties in defining the PDS. While the AMS in general is readily obtained from the hydrologic time series (as long as the water year is properly defined), the extraction of peaks to include in the PDS analysis is by no means a straightforward task. First, consecutive peaks should be independent, and

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Paper number 96WR03848.
0043-1397/97/96WR-03848$09.00

hence some criteria to identify independent events must be defined. For instance, in the case of multiple peaks corresponding to the same hydrologic event, only the largest peak should be included in the PDS. Second, the method involves the selection of an appropriate threshold level, that is, a level which ensures as much relevant information as possible to be included in the analysis without violating basic statistical assumptions. The problem of independent peaks was addressed by the *U.S. Water Resources Council* [1982] and *Cunnane* [1979]. Consecutive peak floods are defined as independent if (1) the interevent time exceeds a critical time and (2) the interevent discharge drops below a critical flow. The problem of threshold selection was discussed by *Rosbjerg and Madsen* [1992], who compared different methods for defining the threshold level q_0. The method based on a predefined frequency factor k, $q_0 = E\{Q\} + kS\{Q\}$ where $E\{Q\}$ and $S\{Q\}$ are, respectively, the mean and the standard deviation of the basic data series, was recommended. It has been applied in flood frequency studies [*Rasmussen and Rosbjerg*, 1991] and in extreme precipitation studies [*Madsen et al.*, 1994].

Although practical advantages and drawbacks should be considered when choosing between the two model candidates, it should not be the main criterion. A model comparison should primarily be based on appropriate performance criteria, for example, the accuracy with which quantiles are estimated. *Cunnane* [1973] compared the PDS model with exponentially distributed exceedances (PDS/EXP) with the AMS model based on the EV1 distribution (AMS/EV1) using the variance of the T-year event estimator as a performance index. By comparing the asymptotic variance expressions based on maximum likelihood estimation, he showed that for return periods larger than about 20 years, the PDS estimator has a smaller variance than the AMS estimator if the PDS contains more than 1.65 exceedances on average per year. This value is often referred to as a general point beyond which the PDS model becomes more efficient than the AMS model [e.g., *Stedinger et al.*, 1993]. However, as will be shown in this paper, the result is valid only for PDS/EXP versus AMS/EV1 estimation, and, in addition, it depends on the estimation method.

The objective of the present study is to generalize *Cunnane*'s [1973] results by comparing the PDS model with GP distributed exceedances with the AMS model based on the GEV distribution in terms of the accuracy of T-year event estimators. A comparison of the PDS/GP and AMS/GEV models has previously been made by *Wang* [1991]. However, he considered only estimation in a PDS with an average number of events equal to the number of years of the sample period. We, more generally, compare estimation in the AMS model with that of the PDS model using a wide range of the number of events included in the PDS, and the results from the present study therefore provide more useful recommendations for practitioners. While Cunnane's results were based on maximum likelihood (ML) estimation, Wang used the method of probability weighted moments (PWM). Here ML and PWM as well as the classical method of moments (MOM) estimation are compared.

PDS and AMS Model Formulations

PDS/GP Model

The exceedance magnitudes in the PDS are assumed to be GP distributed. The GP distribution, introduced by *Pickands* [1975], has the cumulative distribution function (CDF)

$$F(q) = 1 - \exp\left(-\frac{q - q_0}{\alpha}\right) \qquad \kappa = 0$$
$$F(q) = 1 - \left(1 - \kappa\frac{q - q_0}{\alpha}\right)^{1/\kappa} \qquad \kappa \neq 0 \tag{1}$$

where α is the scale parameter, κ is the shape parameter, and q_0 is the threshold level. For $\kappa = 0$ the EXP distribution is obtained as a special case. For $\kappa \le 0$ the range of q is $q_0 \le q < \infty$, whereas for $\kappa > 0$ an upper bound exists: $q_0 \le q \le q_0 + \alpha/\kappa$. An important feature of the GP distribution in a PDS context is that a truncated GP distribution remains a GP distribution, implying that theoretically, the choice of threshold level is not critical for the assumption of the type of exceedance distribution. Assume that the GP distribution applies for the threshold level q_0, then for a higher threshold, $q_1 > q_0$, the distribution of the exceedance given $Q > q_1$ is

$$F(q|q > q_1) = \frac{F(q) - F(q_1)}{1 - F(q_1)} = 1 - \exp\left(-\frac{q - q_1}{\alpha}\right) \qquad \kappa = 0$$
$$F(q|q > q_1) = \frac{F(q) - F(q_1)}{1 - F(q_1)} = 1 - \left(1 - \kappa\frac{q - q_1}{\alpha - \kappa(q_1 - q_0)}\right)^{1/\kappa} \qquad \kappa \neq 0 \tag{2}$$

which is a GP distribution with the original shape parameter κ and scale parameter $\alpha - \kappa(q_1 - q_0)$. For the EXP distribution, the scale parameter remains unchanged.

For the GP distribution in (1), the threshold level q_0 is determined a priori, implying that only the parameters α and κ have to be estimated from the PDS sample (given in terms of the exceedances $x_i = q_i - q_0$). Estimation of the parameters of the GP distribution was considered by *Hosking and Wallis* [1987] who compared ML, MOM, and PWM. *Rosbjerg et al.* [1992] compared MOM and PWM with focus on T-year event estimation from PDS. The moment of order r of the GP distribution exists provided $\kappa > -1/r$. The MOM estimators are obtained from the sample mean $\hat{\mu}$ and the sample variance $\hat{\sigma}^2$ of the exceedances

$$\hat{\alpha} = \frac{1}{2}\hat{\mu}\left(\frac{\hat{\mu}^2}{\hat{\sigma}^2} + 1\right)$$
$$\hat{\kappa} = \frac{1}{2}\left(\frac{\hat{\mu}^2}{\hat{\sigma}^2} - 1\right) \tag{3}$$

The PWM estimators, or the equivalent L-moment estimators [*Hosking*, 1990], are given by

$$\hat{\alpha} = \hat{\lambda}_1\left(\frac{1}{\hat{\tau}_2} - 1\right)$$
$$\hat{\kappa} = \frac{1}{\hat{\tau}_2} - 2 \tag{4}$$

where $\hat{\lambda}_1$ is an estimate of the first L moment (equal to the sample mean), and $\hat{\tau}_2 = \hat{\lambda}_2/\hat{\lambda}_1$ is an estimate of the L coefficient of variation (L-C_v). To estimate the L moments, unbiased estimators of the PWMs $\beta_r = E\{X[F(X)]^r\}$ are employed [*Landwehr et al.*, 1979; *Hosking and Wallis*, 1995];

$\lambda_1 = \beta_0$, $\lambda_2 = 2\beta_1 - \beta_0$. For the ML estimation procedure, no explicit solution for the GP parameter estimators exists. Here a numerical procedure is applied based on the modified Newton-Raphson algorithm used by *Hosking and Wallis* [1987].

The occurrence of peaks is assumed to be described by a Poisson process with an annual periodic intensity. Hence, the number of exceedances N in t years is Poisson distributed with probability function

$$P\{N(t) = n\} = \frac{(\lambda t)^n}{n!} \exp(-\lambda t) \qquad n = 0, 1, 2, \cdots \tag{5}$$

where λ is the mean annual number of threshold exceedances. For the Poisson distribution, MOM, PWM, and ML estimation are identical, and the estimator is given by

$$\hat{\lambda} = \frac{N}{t} \tag{6}$$

The T_P-year event, that is, the event which on the average is exceeded once in T_P years, is in a PDS context usually defined as the $(1 - 1/\lambda T_P)$ quantile in the distribution of the exceedances [e.g., *Rosbjerg*, 1985]. By inverting (1), one obtains

$$\begin{aligned} q_{T_P} &= q_0 + \alpha \ln(\lambda T_P) && \kappa = 0 \\ q_{T_P} &= q_0 + \frac{\alpha}{\kappa}\left[1 - \left(\frac{1}{\lambda T_P}\right)^{\kappa}\right] && \kappa \neq 0 \end{aligned} \tag{7}$$

The T_P-year event estimate is obtained from (7) by inserting the estimated PDS parameters.

AMS/GEV Model

The annual maximum distribution corresponding to the parent PDS/GP model can be derived as follows. Assume that a threshold level q_0 is chosen, corresponding to a mean annual number of exceedances, λ; then, at any higher threshold, $q > q_0$, the number of exceedances $N_q(t)$ is Poisson distributed with parameter

$$\lambda_q = \lambda[1 - F(q)] \tag{8}$$

The probability of the annual maximum being lower or equal to q is equal to the probability of having no exceedances above q in a single year. Hence, by combining (1), (5), and (8), one obtains

$$\begin{aligned} G(q) &= P\{N_q(1) = 0\} = \exp(-\lambda[1 - F(q)]) \\ &= \exp\left(-\exp\left[-\frac{q - [q_0 + \alpha \ln(\lambda)]}{\alpha}\right]\right) \qquad \kappa = 0 \end{aligned} \tag{9}$$

$$\begin{aligned} G(q) &= P\{N_q(1) = 0\} = \exp(-\lambda[1 - F(q)]) \\ &= \exp\left(-\left[1 - \kappa\,\frac{q - \left[q_0 + \frac{\alpha}{\kappa}(1 - \lambda^{-\kappa})\right]}{\alpha\lambda^{-\kappa}}\right]^{1/\kappa}\right) \qquad \kappa \neq 0 \end{aligned}$$

which is a GEV distribution with the same shape parameter κ as in the GP distribution. In particular, for $\kappa = 0$ the EV1 distribution for annual maxima corresponds to a PDS model with EXP distributed exceedances. The location parameter ξ and the scale parameter α^* of the GEV distribution are

$$\begin{aligned} \xi &= q_0 + \alpha \ln(\lambda) && \kappa = 0 \\ \xi &= q_0 + \frac{\alpha}{\kappa}(1 - \lambda^{-\kappa}) && \kappa \neq 0 \end{aligned} \tag{10}$$

$$\alpha^* = \alpha\lambda^{-\kappa} \tag{11}$$

Note that the GEV distribution obtained from the PDS/GP model is defined only for $q > q_0$. At $q = q_0$ the CDF of the annual maximum distribution is equal to $\exp(-\lambda)$, which corresponds to the probability of no exceedances in a year. Thus the parent PDS model provides no information about the AMS below the threshold level, and this makes a direct comparison of the two models complicated. *Wang* [1991] argued that for "fairness of comparison," the PDS/GP model should be compared to the AMS/GEV model censored at the PDS threshold level. In this study, however, we compare the two methods as they would be applied in practice, that is, the comparison is based on estimation in the complete AMS.

The MOM estimators of the GEV parameters are given by [e.g., *Stedinger et al.*, 1993]

$$\begin{aligned} \hat{\xi} &= \hat{\mu} + \frac{\hat{\alpha}^*}{\hat{\kappa}}[\Gamma(1 + \hat{\kappa}) - 1] \\ \hat{\alpha}^* &= \text{sign}\{\hat{\kappa}\}([\hat{\sigma}\hat{\kappa}]/\{[\Gamma(1 + 2\hat{\kappa}) - [\Gamma(1 + \hat{\kappa})]^2]^{1/2}\}) \\ \hat{\gamma} &= \text{sign}\{\hat{\kappa}\}(\{-\Gamma(1 + 3\hat{\kappa}) + 3\Gamma(1 + \hat{\kappa})\Gamma(1 + 2\hat{\kappa}) \\ &\quad - 2[\Gamma(1 + \hat{\kappa})]^3\}/\{[\Gamma(1 + 2\hat{\kappa}) - [\Gamma(1 + \hat{\kappa})]^2]^{3/2}\}) \end{aligned} \tag{12}$$

where sign$\{\hat{\kappa}\}$ is plus or minus 1, depending on the sign of $\hat{\kappa}$. The shape parameter estimator is obtained from the sample skewness $\hat{\gamma}$ by iteration. The rth moment of the GEV distribution exists if $\kappa > -1/r$, implying that the skewness tends to infinity for $\kappa \to -1/3$. Thus MOM estimation of κ is restricted to $\hat{\kappa} > -1/3$. The PWM estimators read [*Hosking et al.*, 1985b]

$$\begin{aligned} \hat{\xi} &= \hat{\lambda}_1 + \frac{\hat{\alpha}^*}{\hat{\kappa}}[\Gamma(1 + \hat{\kappa}) - 1] \\ \hat{\alpha}^* &= \frac{\hat{\lambda}_2\hat{\kappa}}{\Gamma(1 + \hat{\kappa})(1 - 2^{-\hat{\kappa}})} \\ \hat{\kappa} &= 7.8590c + 2.9554c^2 \qquad c = \frac{2}{\hat{\tau}_3 + 3} - \frac{\ln(2)}{\ln(3)} \end{aligned} \tag{13}$$

where the L-moment estimators $\hat{\lambda}_1$, $\hat{\lambda}_2$, and $\hat{\tau}_3 = \hat{\lambda}_3/\hat{\lambda}_2$ (L skewness) are obtained from the unbiased estimators of the first three PWMs ($\lambda_3 = 6\beta_2 - 6\beta_1 + \beta_0$). The ML estimators are determined numerically using a modified Newton-Raphson algorithm [*Hosking*, 1985].

The T-year event based on the AMS is defined as the $(1 - 1/T)$ quantile in the annual maximum distribution. From (9) one obtains

$$\begin{aligned} q_T &= \xi + \alpha^* y_T && \kappa = 0 \\ q_T &= \xi + \frac{\alpha^*}{\kappa}[1 - \exp(-\kappa y_T)] && \kappa \neq 0 \end{aligned} \tag{14}$$

where $y_T = -\ln[-\ln(1 - 1/T)]$ is the Gumbel reduced variate. In terms of the PDS parameters, (14) reads

$$q_T = q_0 + \alpha[\ln(\lambda) + y_T] \qquad \kappa = 0$$
$$q_T = q_0 + \frac{\alpha}{\kappa}[1 - \exp(-\kappa[\ln(\lambda) + y_T])] \qquad \kappa \neq 0 \tag{15}$$

which is seen to be different from the T_P-year event definition in (7). The difference, however, is negligible for large return periods [*Rosbjerg*, 1977]. In the comparative study the AMS-based T-year event definition is applied, and T-year event estimators for both models are then obtained from (14) and (15) by inserting the estimated parameters.

Comparison of PDS/EXP and AMS/EV1 Models

In the study by *Cunnane* [1973] the PDS/EXP model was compared with the AMS/EV1 model using the asymptotic variance of the T-year event estimator as a performance index. For both models, ML estimation was applied. Based on a Taylor series expansion of (15), Cunnane derived the following asymptotic expression for the variance of the PDS/EXP T-year event estimator:

$$\text{var}_{\text{PDS}}\{\hat{q}_T\} = \frac{\alpha^2}{\lambda t}[1 + (\ln(\lambda) + y_T)^2] \tag{16}$$

An improved expression was deduced by *Rosbjerg* [1985], implying a small sample correction factor to be multiplied on the second term in (16). The difference between Rosbjerg's formula and (16) vanishes for larger sample sizes. Note that since ML, MOM, and PWM estimators are identical in the EXP and Poisson distributions, (16) is also valid for MOM and PWM estimation.

The asymptotic variance of the AMS/EV1 T-year event estimator in the case of ML estimation is given by [*Kimball*, 1949]

$$\text{var}_{\text{AMS}}\{\hat{q}_T\} = \frac{\alpha^2}{t}[1.109 + 0.514y_T + 0.608y_T^2] \tag{17}$$

By comparing (16) and (17) it is seen that the ratio $\text{var}_{\text{PDS}}\{\hat{q}_T\}/\text{var}_{\text{AMS}}\{\hat{q}_T\}$ depends only on λ and T. Since the variance of the PDS estimator decreases as λ increases, it is to be expected that for a certain value of λ, say λ_e, the variance of the PDS estimator equals that of the AMS estimator. It is readily seen that as the return period T tends to infinity, λ_e tends to an asymptotic value of 1/0.608 = 1.64 which, except from a minor deviation at the last decimal, corresponds to the value reported by *Cunnane* [1973]. For T larger than about 20 years, λ_e is virtually equal to the asymptotic value.

In the case of MOM estimation, the asymptotic variance of the AMS/EV1 T-year event estimator reads [e.g., *Phien*, 1987]

$$\text{var}_{\text{AMS}}\{\hat{q}_T\} = \frac{\alpha^2}{t}[1.168 + 0.192y_T + 1.100y_T^2] \tag{18}$$

Phien [1987] also derived an expression for the asymptotic variance of the PWM T-year event estimator which, however, contains an error of the sign of the term including y_T. The correct formula for large t reads

$$\text{var}_{\text{AMS}}\{\hat{q}_T\} = \frac{\alpha^2}{t}[1.113 + 0.457y_T + 0.805y_T^2] \tag{19}$$

By comparing (16) with (18) and (19), respectively, it is seen that the asymptotic value of λ_e is 1/1.100 = 0.91 for MOM estimation and 1/0.805 = 1.24 for PWM estimation. However, contrary to ML estimation, λ_e for MOM and PWM estimation only slowly tends to the asymptotic value as T increases. For both estimation methods, λ_e is smaller than the asymptotic value for all T (e.g., for T = 100 years, λ_e = 0.79 for MOM estimation and λ_e = 1.17 for PWM estimation).

Since the variance of the PDS/EXP T-year event estimator is identical for all three estimation methods, the results given above reflect the efficiency of the different estimation procedures for the AMS/EV1 model. Thus for this model ML estimation is more efficient than PWM estimation, which in turn is more efficient than MOM estimation. Monte Carlo simulation studies performed by *Landwehr et al.* [1979] also revealed this general behavior of the efficiency of the different estimation methods for the EV1 model. Hence, as a general guideline, the PDS/EXP model is preferable provided $\lambda > 1.64$, otherwise the AMS/EV1 model with ML estimation should be used.

Comparison of PDS/GP and AMS/GEV Models

Asymptotic expressions for the variance of the PDS/GP and AMS/GEV T-year event estimators are given in the appendix. For all three estimation methods, the asymptotic variance of the PDS/GP T-year event estimator can be written

$$\text{var}_{\text{PDS}}\{\hat{q}_T\} = \frac{\alpha^2}{\lambda t}g_1(\kappa, \lambda, T) \tag{20}$$

Similarly, for the AMS/GEV estimator one obtains

$$\text{var}_{\text{AMS}}\{\hat{q}_T\} = \frac{\alpha^{*2}}{t}g_2(\kappa, T) = \frac{\alpha^2}{t\lambda^{2\kappa}}g_2(\kappa, T) \tag{21}$$

where (11) has been used. Comparison of (20) and (21) reveals that the variance ratio $\text{var}_{\text{PDS}}\{\hat{q}_T\}/\text{var}_{\text{AMS}}\{\hat{q}_T\}$ depends only on κ, λ, and T; that is, α of the parent PDS model can be chosen arbitrarily. Thus for a given return period and a given shape parameter the value of λ that yields equal asymptotic variances of the two estimators can be found. First, however, a Monte Carlo simulation study is performed to evaluate the small sample properties of the model comparison. In this case a new parameter, the number of recording years t, has to be taken into account.

Simulation Algorithm

The simulation is complicated by the fact that the GEV distribution is defined only for values larger than the threshold level applied in the PDS. Simulation of a PDS sample from a parent GP distribution corresponding to a chosen λ (and hence a threshold level q_0) does not provide sufficient information about the AMS sample, since the annual maximum will be less than q_0 in a single year with probability $\exp(-\lambda)$. Similarly, simulation of an AMS sample from a parent GEV does not provide adequate information about the corresponding PDS. To overcome this problem, one may exploit the fact that a truncated GP distribution remains a GP distribution with the shape parameter unchanged. The simulation algorithm is as follows:

1. Choose an initial value of the mean annual number of exceedances, λ_0, in the PDS that implies the probability of no exceedances in a year, $\exp(-\lambda_0)$, to be small. Also choose, independently of λ_0, a threshold level $q_0 \geq 0$ and the GP scale parameter $\alpha_0 > 0$.
2. The number of exceedances n_i in a single year is simulated from a parent Poisson distribution with parameter λ_0.

For a given value of the κ parameter, the n_i exceedances, x_j, $j = 1, 2, \ldots, n_i$, are drawn from a parent GP distribution with parameters α_0 and κ, and the annual maximum is then determined as $q_i = \max\{x_j\} + q_0$. This step is repeated t times to obtain an AMS sample of length t years and an initial PDS sample with Σn_i exceedances.

3. From the initial PDS sample a number of series corresponding to higher threshold levels are extracted. These PDS samples also have a parent GP distribution with the same κ as the initial parent distribution, and hence T-year event estimators based on the different PDS samples are directly comparable to the AMS T-year event estimator. If M new PDS samples are generated with mean annual number of exceedances denoted λ_k, $k = 1, 2, \ldots, M$, the corresponding threshold levels $q_k \geq q_0$ are (see (8))

$$q_k = q_0 + \alpha_0 \ln\left(\frac{\lambda_0}{\lambda_k}\right) \qquad \kappa = 0$$

$$q_k = q_0 + \frac{\alpha_0}{\kappa}\left[1 - \left(\frac{\lambda_k}{\lambda_0}\right)^{\kappa}\right] \qquad \kappa \neq 0 \tag{22}$$

4. The AMS/GEV T-year event estimator $\hat{q}_{T,\mathrm{AMS}}$ and the PDS/GP T-year event estimators $(\hat{q}_{T,\mathrm{PDS}})_k$, $k = 1, 2, \ldots, M$ are calculated in the cases of, respectively, ML, MOM, and PWM estimation following the previously described procedures.

5. Steps (2)–(4) are repeated a large number of times, and on the basis of the resulting samples of $\hat{q}_{T,\mathrm{AMS}}$ and $(\hat{q}_{T,\mathrm{PDS}})_k$, the root mean square errors $\mathrm{RMSE}\{\hat{q}_T\} = [E\{(E\{\hat{q}_T\} - q_T)^2\}]^{1/2}$ are calculated for the three estimation methods considered.

Results

Monte Carlo simulations were performed for sample sizes $t = 10$, 30, and 50 years; mean annual number of exceedances in the range $0.4 \leq \lambda \leq 15$; and shape parameters in the range $-0.3 \leq \kappa \leq 0.3$. For each sample size, 50,000 samples were generated, and the RMSE of the T-year event estimates corresponding to $T = 10$, 100, and 1000 years were calculated. Note that the RMSE ratio $\mathrm{RMSE}_{\mathrm{PDS}}\{\hat{q}_T\}/\mathrm{RMSE}_{\mathrm{AMS}}\{\hat{q}_T\}$ is independent of the basic parameters λ_0, q_0, and α_0, implying that the relative performance of the AMS/GEV method is independent of the coefficient of variation of the parent GEV distribution.

In the Newton-Raphson algorithm for ML estimation, the PWM estimators were used as initial values. The Newton-Raphson algorithm occasionally fails to converge, and such failures are mostly caused by the nonexistence of a maximum of the log likelihood function rather than by a bad choice of initial estimates [*Hosking et al.*, 1985b; *Hosking and Wallis*, 1987]. Nonconvergence of the ML algorithm was a particular problem for small sample sizes and large (positive) κ. Another problem of the ML algorithm related to small samples was observed. In some cases the ML procedure resulted in very extreme T-year event estimates implying unstable estimates of the RMSE. To circumvent this problem, it was decided to exclude samples that gave a T-year event estimate larger than the event corresponding to a return period of 10^4 times T. The proportion of samples excluded because of nonconvergence of the Newton-Raphson algorithm or generation of very unreasonable T-year event estimates was found significant only for $t = 10$ years. Also, in the case of MOM estimation some generated samples provided unreasonable T-year event estimates and hence unstable estimates of the RMSE. The proportion of samples excluded in this case, however, was at most 2–3 of 50,000 for all sample sizes considered.

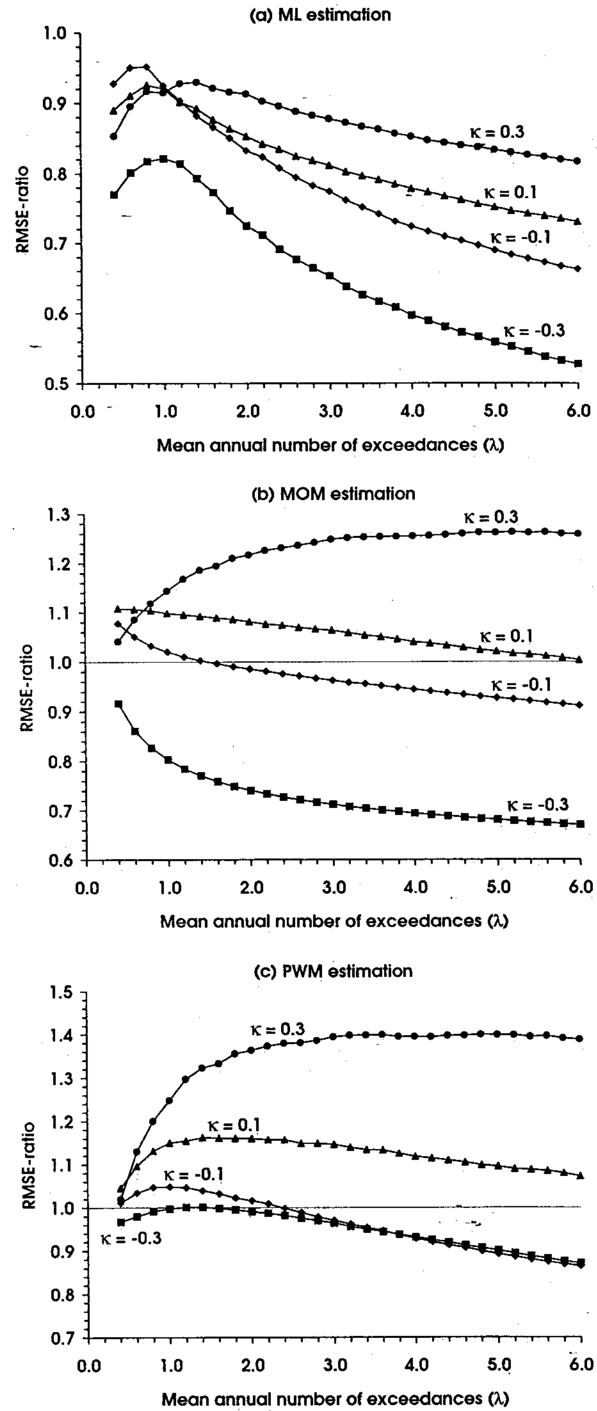

Figure 1. Ratio of simulated RMSE of T-year event estimators based on, respectively, the PDS/GP and the AMS/GEV model. Record length $t = 30$ years, and return period $T = 100$ years. (a) ML estimation, (b) MOM estimation, and (c) PWM estimation.

For different values of κ, the RMSE of the PDS/GP and AMS/GEV models are compared in Figure 1 for $t = 30$ years and $T = 100$ years. If the RMSE ratio $\mathrm{RMSE}_{\mathrm{PDS}}\{\hat{q}_T\}/$

Table 1. Mean Annual Number of Exceedances in PDS to Obtain Equal Performance of PDS/GP and AMS/GEV T-Year Event Estimators for Different Shape Parameters κ, Record Lengths t, and Return Periods T

	$T = 10$		$T = 100$				$T = 1000$	
κ	$t = 30$	$t = \infty$	$t = 10$	$t = 30$	$t = 50$	$t = \infty$	$t = 30$	$t = \infty$
				ML Estimation				
−0.3	<0.4	0.70	<0.4	<0.4	1.1	2.31	<0.4	2.32
−0.2	<0.4	0.77	<0.4	<0.4	1.1	2.26	0.9	2.27
−0.1	<0.4	0.87	VU	<0.4	1.1	2.21	0.7	2.22
0.0	0.7	0.99	VU	<0.4	1.1	2.14	0.7	2.16
0.1	0.7	1.15	VU	<0.4	1.1	2.08	0.9	2.11
0.2	0.8	1.30	VU	<0.4	1.3	1.92	0.9	2.05
0.3	0.9	1.41	VU	<0.4	1.4	0.64	0.9	2.01
				MOM Estimation				
−0.3	<0.4	NA	0.6	<0.4	<0.4	NA	0.8	NA
−0.2	<0.4	NA	1.0	0.5	<0.4	NA	2.7	NA
−0.1	<0.4	0.12	2.1	1.4	1.3	0.01	5.2	0.01
0.0	0.7	0.91	5.8	3.5	3.0	0.14	4.5	0.91
0.1	1.1	1.07	10.3	6.3	5.5	3.66	5.0	2.60
0.2	>15	160	>15	>15	>15	19.4	8.5	6.73
0.3	>15	3840	>15	>15	>15	156	>15	19.8
				PWM Estimation				
−0.3	0.9	0.55	6.5	1.5	<0.4	0.98	1.3	1.18
−0.2	1.0	0.76	3.3	1.7	1.5	1.35	1.5	1.50
−0.1	1.0	0.87	2.9	2.4	2.2	2.17	1.9	2.06
0.0	>15	21.0	4.5	4.5	4.4	4.42	2.8	3.11
0.1	>15	110	7.9	9.9	10.9	12.2	4.3	5.44
0.2	>15	798	>15	>15	>15	49.9	7.0	12.2
0.3	>15	16,200	>15	>15	>15	455	>15	49.9

Results are obtained from simulations ($t = 10, 30, 50$) and asymptotic theory ($t = \infty$).
T and t are in years. VU, values are unreliable since more than 10% of the generated samples were excluded because of nonconvergence of the Newton-Raphson algorithm or generation of unreasonable T-year event estimates; NA, no approximate solution exists.

$\text{RMSE}_{\text{AMS}}\{\hat{q}_T\}$ in this figure is less than 1, the PDS/GP estimator is more efficient. The value of λ to obtain equal RMSE of the two estimators, λ_e, is shown in Table 1 for different combinations of t and T. For λ larger than λ_e in this table, the PDS/GP estimator is more efficient. In general, the PDS model becomes more efficient as λ increases, although the relative performance of the PDS model depends strongly on the estimation method and the shape parameter.

In the case of ML estimation, for $t = 30$ years and $T = 100$ years (see Figure 1a) the RMSE ratio is smaller than 1 irrespective of λ ($\lambda > 0.4$), implying that the PDS model should always be preferred. The value of λ_e increases slightly as κ and/or t increases (see Table 1). However, since typical λ values fall in the range of 2–5, the results indicate that for most practical purposes, the PDS/GP model provides the most efficient T-year event estimator for ML estimation. In the case of MOM estimation (see Figure 1b and Table 1), λ_e varies widely for κ values in the range $-0.3 \le \kappa \le 0.3$. The value of λ_e increases as κ increases, and for large (positive) κ a very large, and in practice unrealistic, number of exceedances is required to obtain equal model performance. For typical λ values obtained in practice, the PDS model is generally preferable for negative κ for $T = 100$ years. For $T = 10$ years the PDS model is more efficient also for slightly positive κ, and for $T = 1000$ years, only if $\kappa < -0.1$. The results from PWM estimation (see Figure 1c and Table 1) are virtually similar to those obtained for MOM estimation. Also in this case, the PDS model seems to be a realistic alternative to the AMS model only for negative κ. For very small sample sizes, however, λ_e increases for decreasing κ when $\kappa < -0.1$, and in this case the PDS model seems preferable only for $-0.2 \le \kappa \le 0$. For $T = 1000$ years the PDS model may be preferable also for slightly positive κ.

For all estimation methods, the RMSE ratio (and hence the RMSE of the PDS/GP T-year event estimator) for small values of λ is seen to decrease for decreasing λ. This somewhat surprising result suggests that the PDS model in some cases performs better than the AMS model also for small λ. This result was also found by *Wang* [1991] in the case of PWM estimation, and he argued that in certain circumstances it may be advantageous to increase the threshold level in order to reduce the sampling variance. However, since this behavior is observed only for very small λ values, it is of little practical relevance, and hence no such secondary λ_e values are reported in Table 1.

The asymptotic variance expressions given in the appendix have been evaluated for different return periods and shape parameters. The values of λ to obtain equal asymptotic variances of the two estimators are compared with the Monte Carlo simulated results in Table 1. In the case of ML estimation the asymptotic formulae provide poor estimates of λ_e for finite samples. The significant difference between asymptotic and simulated results in this case is caused by the generally bad evaluation of the variances and covariances of the parameter estimates when based on the expected information matrix. The inverse of the observed information matrix (i.e., the matrix evaluated at the estimated parameters) is a better estimator of the covariance matrix [*Prescott and Walden*, 1983]. For MOM estimation an approximate solution for the variance of the AMS/GEV T-year event estimator exists only for $\kappa > -1/6$. For κ close to $-1/6$, the unbounded variance of the AMS/GEV T-year event estimator causes the asymptotic variance ratio

Table 2. Standardized RMSE of T-year Event Estimators RMSE$\{\hat{q}_T\}/q_T$ for Estimation With the AMS/GEV Model for Different Shape Parameters κ, Record Lengths t, and Return Periods T

κ		$T = 10$, $t = 30$	$T = 100$, $t = 10$	$T = 100$, $t = 30$	$T = 100$, $t = 50$	$T = 1000$, $t = 30$
−0.3	ML	0.20	2.49	0.71	0.41	2.07
	MOM	0.32	0.56	0.44	0.37	0.56
	PWM	0.17	0.63	0.42	0.34	1.00
−0.1	ML	0.12	VU	0.34	0.23	0.64
	MOM	0.13	0.33	0.22	0.18	0.33
	PWM	0.12	0.45	0.26	0.20	0.54
0.1	ML	0.072	VU	0.16	0.11	0.39
	MOM	0.071	0.19	0.12	0.092	0.18
	PWM	0.071	0.27	0.14	0.11	0.26
0.3	ML	0.040	VU	0.071	0.048	0.14
	MOM	0.040	0.11	0.061	0.047	0.089
	PWM	0.041	0.15	0.072	0.054	0.12

T and t are in years. VU, values are unreliable since more than 10% of the generated samples were excluded because of nonconvergence of the Newton-Raphson algorithm or generation of unreasonable T-year event estimates.

$\text{var}_{\text{PDS}}\{\hat{q}_T\}/\text{var}_{\text{AMS}}\{\hat{q}_T\}$ to deteriorate and tend to zero. In practice, only for $\kappa > 0.1$, the asymptotic formulae provide reasonable results for moderate sample sizes. In the case of PWM estimation the results obtained by using the asymptotic formulae agree fairly well with the Monte Carlo–simulated values when sample sizes are not too small.

In summary, the results obtained in this section reveal that the PDS/GP model is generally the most efficient model for ML estimation. For MOM and PWM estimation the PDS/GP model is preferable for negative κ, whereas the AMS/GEV model is more efficient for positive κ. Important to note is that the above findings differ significantly from the results of *Cunnane*'s [1973] comparison between the PDS/EXP and AMS/EV1 models, and hence an interpretation of Cunnane's results beyond its assumptions may lead to erroneous conclusions.

Discussion

The above analysis has focused on the choice of model when a certain estimation method is applied. However, when choosing the most efficient T-year event estimator, one should consider not only the choice of PDS or AMS but also the choice of estimation method. Thus one has to compare the performance of six different estimators: PDS/GP-ML, PDS/GP-MOM, and PDS/GP-PWM for estimation in PDS and AMS/GEV-ML, AMS/GEV-MOM, and AMS/GEV-PWM for estimation in AMS.

Rosbjerg et al. [1992] compared the PDS/GP-MOM and PDS/GP-PWM methods and showed that except for very large sample sizes, which are rarely available in practice, MOM estimation is more efficient for all κ in the range $-0.5 < \kappa < 0.5$. *Hosking and Wallis* [1987] also applied the ML procedure and demonstrated that the PDS/GP-ML method is preferable only for larger samples and for $\kappa > 0.2$. In addition, they showed that for small return periods (not considered by *Rosbjerg et al.* [1992]), the PDS/GP-PWM procedure is the most efficient method. *Hosking et al.* [1985b] compared the AMS/GEV-ML and AMS/GEV-PWM methods and concluded that the PWM method is generally to be preferred for small to moderate sample sizes. To the authors' knowledge, no comparison has been made with the AMS/GEV-MOM estimation procedure.

The RMSE of the T-year event estimator of the three estimation methods for the AMS/GEV model based on the simulations described in the previous section are shown in Table 2. For $T = 100$ years and moderate sample sizes, MOM estimation has the smallest RMSE for $-0.25 < \kappa < 0.3$, PWM estimation is preferable for $\kappa < -0.25$, and ML estimation (when sample sizes are not too small) is preferable for $\kappa > 0.3$. As t decreases and/or T increases, the interval of κ values for which MOM estimation is more efficient tends to increase. For instance, for $T = 1000$ years, MOM estimation is more efficient for all shape parameters considered. For $T = 10$ years, PWM estimation is preferable for negative κ, whereas for positive κ the three methods have almost equal efficiency. In practice, when focus is on high quantile estimation, the simulations reveal that MOM estimation is preferable in most AMS/GEV cases, in accordance with the results of estimation in the PDS/GP model.

For $T = 100$ years and $t = 30$ years, the performance of the six different models are compared in Figure 2. In this case, for typical values of λ in the range 2–5, the following conclusions are obtained. For negative κ, the PDS/GP-MOM method is the most efficient method; for $0 < \kappa < 0.2$, the AMS/GEV-MOM method is preferable; and for $\kappa > 0.2$, the AMS/GEV-MOM method is preferable when λ is smaller than about 4, and the PDS/GP-ML method is preferable otherwise. For larger sample sizes the PDS/GP-ML method is preferable for $\kappa > 0.2$, irrespective of the value of λ. For larger return periods the AMS/GEV-MOM method becomes more efficient for slightly negative κ. Only for small return periods and small κ is PWM estimation preferable. For instance, for $T = 10$ years the PDS/GP-PWM method is the most efficient method for $\kappa < -0.2$ and λ smaller than about 3. In all other cases, the results obtained for $T = 100$ years apply.

Another aspect to consider is the particular case where κ is close to zero. *Rosbjerg et al.* [1992] evaluated the performance of the PDS/EXP model when the parent population is GP distributed. Despite the introduced model error, the EXP distribution, in terms of RMSE of the T-year event estimator, is

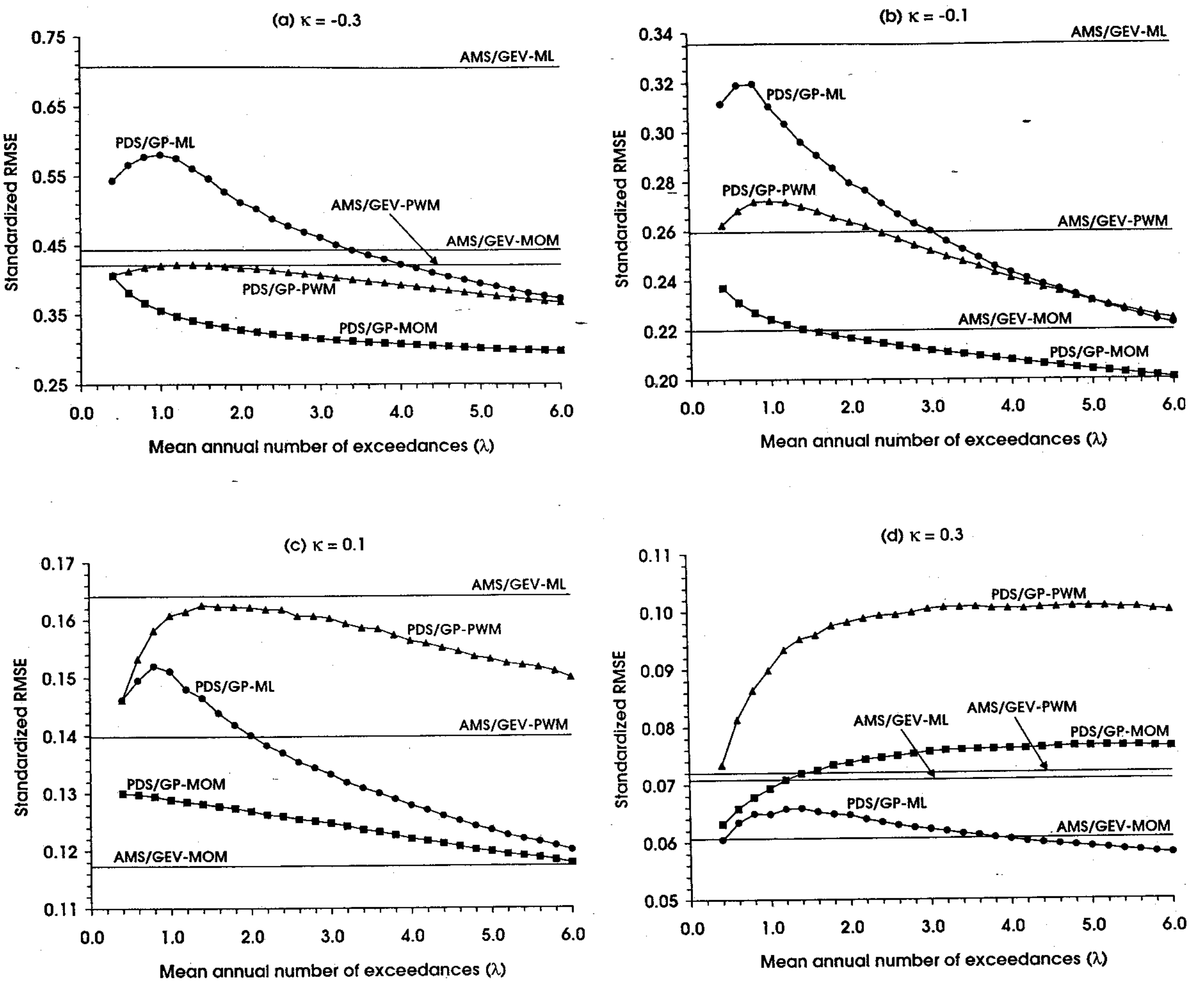

Figure 2. Standardized RMSE of T-year event estimator RMSE$\{\hat{q}_T\}/q_T$ for the six estimation methods considered. Record length $t = 30$ years, and return period $T = 100$ years; (a) $\kappa = -0.3$, (b) $\kappa = -0.1$, (c) $\kappa = 0.1$, and (d) $\kappa = 0.3$.

more efficient for moderate sample sizes when $-0.2 < \kappa < 0.1$ for MOM estimation and $-0.3 < \kappa < 0.1$ for PWM estimation. For increasing sample size the interval of κ values for preference of the PDS/EXP model decreases. In the case of estimation with AMS, *Lu and Stedinger* [1992] obtained virtually similar results when comparing the EV1 and GEV T-year event estimators for samples drawn from a parent GEV distribution. These results suggest that if no physical evidence implies a κ value significantly different from zero, the PDS/EXP or AMS/EV1 model should be applied when κ is close to zero. In this case, as shown above, the PDS/EXP model generally is preferable.

The RMSE as applied in the present study is commonly used as an overall measure of estimator performance. *Kroll and Stedinger* [1996] proposed another performance index based on the RMSE in log space (L-RMSE). With this index, underestimation errors will receive more weight than overestimation errors. Application of L-RMSE in our comparative study resulted in essentially similar conclusions as application of the RMSE criterion. The only difference was the relative performance of MOM and PWM estimation for very small κ ($\kappa < -0.3$), where the MOM estimator has a large negative bias. In this case the L-RMSE of the PDS/GP T-year event estimator for MOM and PWM estimation suggested equal performance of the estimators.

Conclusions

In this paper, estimation of T-year events based on PDS and AMS data, respectively, has been compared. It has been shown that a PDS with GP-distributed exceedances and a Poisson-distributed number of threshold exceedances implies the AMS to be GEV distributed. The special case of $\kappa = 0$ corresponds to PDS with EXP-distributed exceedances and to AMS with an EV1 distribution.

In the special case $\kappa = 0$ the comparison of the PDS/EXP and AMS/EV1 models by *Cunnane* [1973] has been reviewed by including also MOM and PWM estimation for the AMS/EV1 model. The PDS/EXP model is the most efficient model provided $\lambda > 1.64$; otherwise, the AMS/EV1 model with ML estimation is preferable.

The performance of the PDS/GP and AMS/GEV models for ML, MOM, and PWM estimation has been compared using Monte Carlo simulations as well as asymptotic theory. For all practical purposes the PDS model provides the most efficient T-year event estimator in the case of ML estimation. For

MOM and PWM estimation the PDS model is generally preferable for negative κ, whereas for positive κ the AMS model is more efficient. A comparison of the six different models reveals that in general, one should apply the PDS/GP-MOM model for negative κ, the AMS/GEV-MOM model for $0 < \kappa < 0.2$, and the PDS/GP-ML model for $\kappa > 0.2$. For small sample sizes, however, the ML procedure is not advisable, and in this case the AMS/GEV-MOM model should be applied also for $\kappa > 0.2$. When κ is close to zero, and no physical evidence suggests a κ value different from zero, the PDS/EXP model is preferable.

In conclusion, since heavy-tailed distributions, corresponding to negative κ, are far the most common in hydrologic applications (see, e.g., comprehensive flood frequency studies by *Farquharson et al.* [1987] and *Gustard et al.* 1989) the results obtained in this study suggest that the PDS model generally is to be preferred for at-site quantile estimation.

Appendix: Asymptotic Variance of *T*-Year Event Estimators

For the PDS/GP model the asymptotic variance of the T-year event estimator is obtained from a first-order Taylor series expansion of (15):

$$\mathrm{var}_{\mathrm{PDS}}\,\{\hat{q}_T\} = \left(\frac{\partial q_T}{\partial \alpha}\right)^2 \mathrm{var}\,\{\hat{\alpha}\} + \left(\frac{\partial q_T}{\partial \kappa}\right)^2 \mathrm{var}\,\{\hat{\kappa}\} + \left(\frac{\partial q_T}{\partial \lambda}\right)^2 \mathrm{var}\,\{\hat{\lambda}\} + 2\left(\frac{\partial q_T}{\partial \alpha}\right)\left(\frac{\partial q_T}{\partial \kappa}\right) \mathrm{cov}\,\{\hat{\alpha}, \hat{\kappa}\} \quad \text{(A1)}$$

where it has been assumed that the sample properties of the exceedance magnitudes are independent of the sample estimate of λ [e.g., *Rosbjerg*, 1985], that is, $\mathrm{cov}\,\{\hat{\lambda}, \hat{\alpha}\} = \mathrm{cov}\,\{\hat{\lambda}, \hat{\kappa}\} = 0$. The bias of the parameter estimates is usually of second order importance, and the partial derivatives in (A1) are therefore for simplicity evaluated at the population values. The variance of $\hat{\lambda}$ reads

$$\mathrm{var}\,\{\hat{\lambda}\} = \lambda/t \quad \text{(A2)}$$

where the small sample corrections given by *Rosbjerg et al.* [1991] have been neglected since they are relevant only for $N = \lambda t < 5$. *Smith* [1984] gave the asymptotic covariance matrix of the ML estimators of the GP parameters, which for $\kappa < 1/2$ reads

$$D\begin{bmatrix} \hat{\alpha} \\ \hat{\kappa} \end{bmatrix} = \frac{(1-\kappa)}{N} \begin{bmatrix} 2\alpha^2 & \alpha \\ \alpha & 1-\kappa \end{bmatrix} \quad \text{(A3)}$$

The asymptotic covariance matrix of the MOM estimators was obtained by *Hosking and Wallis* [1987]:

$$D\begin{bmatrix} \hat{\alpha} \\ \hat{\kappa} \end{bmatrix} = \frac{1}{N} \frac{(1+\kappa)^2}{(1+2\kappa)(1+3\kappa)(1+4\kappa)} \cdot \begin{bmatrix} 2\alpha^2(1+6\kappa+12\kappa^2) & \alpha(1+2\kappa)(1+4\kappa+12\kappa^2) \\ \alpha(1+2\kappa)(1+4\kappa+12\kappa^2) & (1+2\kappa)^2(1+\kappa+6\kappa^2) \end{bmatrix} \quad \text{(A4)}$$

Since the asymptotic variance of $\hat{\sigma}^2$ depends on the fourth moment, (A4) is valid only for $\kappa > -1/4$. *Hosking and Wallis* [1987] also obtained the asymptotic covariance matrix of the PWM estimators which for $\kappa > -1/2$ is given by

$$D\begin{bmatrix} \hat{\alpha} \\ \hat{\kappa} \end{bmatrix} = \frac{1}{N} \frac{1}{(1+2\kappa)(3+2\kappa)} \cdot \begin{bmatrix} \alpha^2(7+18\kappa+11\kappa^2+2\kappa^3) & \alpha(2+\kappa)(2+6\kappa+7\kappa^2+2\kappa^3) \\ \alpha(2+\kappa)(2+6\kappa+7\kappa^2+2\kappa^3) & (1+\kappa)(2+\kappa)^2(1+\kappa+2\kappa^2) \end{bmatrix} \quad \text{(A5)}$$

Rosbjerg et al. [1992] derived small sample correction factors for the variances and the covariances of the MOM and PWM estimators conditioned on the number of exceedances being greater than or equal to 2. The corrections, however, are relevant only for smaller sample sizes, $N < 20$, and can therefore usually be neglected.

For the AMS/GEV model the asymptotic variance of the T-year event estimator can be written [*Rosbjerg and Madsen*, 1995]

$$\mathrm{var}_{\mathrm{AMS}}\,\{\hat{q}_T\} = \frac{\alpha^{*2}}{t}\,[w_{11} + A(Aw_{22} + 2w_{12}) + B(Bw_{33} - 2w_{13} - 2Aw_{23})] \quad \text{(A6)}$$

where A and B are expressed in terms of κ and T. The w_{ij} in (A6) are elements of the asymptotic covariance matrix of the GEV parameter estimators

$$D\begin{bmatrix} \hat{\xi} \\ \hat{\alpha}^* \\ \hat{\kappa} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} \alpha^{*2}w_{11} & \alpha^{*2}w_{12} & \alpha^* w_{13} \\ \alpha^{*2}w_{12} & \alpha^{*2}w_{22} & \alpha^* w_{23} \\ \alpha^* w_{13} & \alpha^* w_{23} & w_{33} \end{bmatrix} \quad \text{(A7)}$$

and depend only on the shape parameter κ. In the case of ML estimation the asymptotic properties are obtained by inverting the Fisher information matrix given by *Prescott and Walden* [1980]. The algebraic form of the w_{ij} is very complicated. They are evaluated numerically for several values of κ in Table A1.

In the case of MOM estimation the asymptotic covariance matrix can be calculated as follows. The moments of the GEV distribution are given by

$$\mu = \xi + \frac{\alpha^*}{\kappa}[1 - \Gamma(1+\kappa)]$$

$$\mu_r = \left(\frac{\alpha^*}{\kappa}\right)^r \sum_{i=0}^{r} \binom{r}{i} (-1)^{r+i}\Gamma([r-i]\kappa + 1)[\Gamma(\kappa+1)]^i \qquad r = 2, 3, \cdots \quad \text{(A8)}$$

where μ_r is the central moment of order $r(\mu_2 = \sigma^2)$, which exists provided $\kappa > -1/r$. The sample estimators $(\hat{\mu}, \hat{\sigma}^2, \hat{\mu}_3)^T$ are asymptotically normally distributed with mean $(\mu, \sigma^2, \mu_3)^T$ and covariance matrix $t^{-1}\mathbf{V}$ where t is the number of observations. The elements of the covariance matrix, for convenience written in vector notation $\mathbf{V_M}$, are given by [*Kendall and Stuart*, 1963]

$$\begin{aligned} v_{M,1} &= t\ \mathrm{var}\,\{\hat{\mu}\} = \sigma^2 \\ v_{M,2} &= t\ \mathrm{var}\,\{\hat{\sigma}^2\} = \mu_4 - \sigma^4 \\ v_{M,3} &= t\ \mathrm{var}\,\{\hat{\mu}_3\} = \mu_6 - \mu_3^2 - 6\mu_4\sigma^2 + 9\sigma^6 \\ v_{M,4} &= t\ \mathrm{cov}\,\{\hat{\mu}, \hat{\sigma}^2\} = \mu_3 \\ v_{M,5} &= t\ \mathrm{cov}\,\{\hat{\mu}, \hat{\mu}_3\} = \mu_4 - 3\sigma^4 \\ v_{M,6} &= t\ \mathrm{cov}\,\{\hat{\sigma}^2, \hat{\mu}_3\} = \mu_5 - 4\mu_3\sigma^2 \end{aligned} \quad \text{(A9)}$$

Table A1. Elements of the Asymptotic Covariance Matrix of the GEV Parameter Estimators

κ	w_{11}	w_{12}	w_{13}	w_{22}	w_{23}	w_{33}
			ML Estimation			
−0.4	1.2909	0.7980	0.2637	1.0431	−0.0881	0.8344
−0.3	1.2866	0.6855	0.2712	0.9136	−0.0178	0.7366
−0.2	1.2784	0.5713	0.2730	0.8056	0.0451	0.6444
−0.1	1.2658	0.4551	0.2687	0.7188	0.1000	0.5578
0.0	1.2487	0.3368	0.2583	0.6533	0.1468	0.4770
0.1	1.2273	0.2156	0.2421	0.6083	0.1849	0.4008
0.2	1.2017	0.0919	0.2202	0.5839	0.2140	0.3304
0.3	1.1727	−0.0347	0.1933	0.5795	0.2333	0.2653
0.4	1.1413	−0.1644	0.1623	0.5945	0.2422	0.2058
			MOM Estimation			
−0.1	1.4641	1.4879	1.0299	8.9121	5.6607	4.3852
0.0	1.3180	0.5247	0.4252	1.2022	0.6217	0.8909
0.1	1.2404	0.2448	0.2677	0.6896	0.2459	0.4520
0.2	1.2144	0.0969	0.2437	0.5959	0.2222	0.3742
0.3	1.2160	−0.0165	0.2777	0.5888	0.2680	0.4305
0.4	1.2401	−0.1177	0.3580	0.6178	0.3337	0.5942
			PWM Estimation			
−0.4	1.6637	1.3355	1.1405	1.8461	1.1628	2.9092
−0.3	1.4153	0.8912	0.5640	1.2574	0.4442	1.4090
−0.2	1.3322	0.6727	0.3926	1.0013	0.2697	0.9139
−0.1	1.2915	0.5104	0.3245	0.8440	0.2240	0.6815
0.0	1.2686	0.3704	0.2993	0.7391	0.2248	0.5634
0.1	1.2551	0.2411	0.2966	0.6708	0.2447	0.5103
0.2	1.2474	0.1177	0.3081	0.6330	0.2728	0.5021
0.3	1.2438	−0.0023	0.3297	0.6223	0.3033	0.5294
0.4	1.2433	−0.1205	0.3592	0.6368	0.3329	0.5880

On the basis of the relations between the sample moments and the GEV parameter estimators $(\hat{\xi}, \hat{\alpha}^*, \hat{\kappa})^T$ (see (A8)), a first-order Taylor series expansion yields

$$t^{-1}V_M = \begin{bmatrix} \text{var}\,\{\hat{\mu}\} \\ \text{var}\,\{\hat{\sigma}^2\} \\ \text{var}\,\{\hat{\mu}_3\} \\ \text{cov}\,\{\hat{\mu}, \hat{\sigma}^2\} \\ \text{cov}\,\{\hat{\mu}, \hat{\mu}_3\} \\ \text{cov}\,\{\hat{\sigma}^2, \hat{\mu}_3\} \end{bmatrix} = D \begin{bmatrix} \text{var}\,\{\hat{\xi}\} \\ \text{var}\,\{\hat{\alpha}^*\} \\ \text{var}\,\{\hat{\kappa}\} \\ \text{cov}\,\{\hat{\xi}, \hat{\alpha}^*\} \\ \text{cov}\,\{\hat{\xi}, \hat{\kappa}\} \\ \text{cov}\,\{\hat{\alpha}^*, \hat{\kappa}\} \end{bmatrix} = DV_p \tag{A10}$$

where D is a matrix of partial derivatives given by

$$D = \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & 2a_{11}a_{12} & 2a_{11}a_{13} & 2a_{12}a_{13} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & 2a_{21}a_{22} & 2a_{21}a_{23} & 2a_{22}a_{23} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & 2a_{31}a_{32} & 2a_{31}a_{33} & 2a_{32}a_{33} \\ a_{11}a_{21} & a_{12}a_{22} & a_{13}a_{23} & a_{11}a_{22} + a_{12}a_{21} & a_{11}a_{23} + a_{31}a_{21} & a_{12}a_{23} + a_{13}a_{22} \\ a_{11}a_{31} & a_{12}a_{32} & a_{13}a_{33} & a_{11}a_{32} + a_{12}a_{31} & a_{11}a_{33} + a_{13}a_{31} & a_{12}a_{33} + a_{13}a_{32} \\ a_{21}a_{31} & a_{22}a_{32} & a_{23}a_{33} & a_{21}a_{32} + a_{22}a_{31} & a_{21}a_{33} + a_{23}a_{31} & a_{22}a_{33} + a_{23}a_{32} \end{bmatrix} \tag{A11}$$

The a_{ij} are elements of the matrix

$$A = \begin{bmatrix} \dfrac{\partial \mu}{\partial \xi} & \dfrac{\partial \mu}{\partial \alpha^*} & \dfrac{\partial \mu}{\partial \kappa} \\ \dfrac{\partial \sigma^2}{\partial \xi} & \dfrac{\partial \sigma^2}{\partial \alpha^*} & \dfrac{\partial \sigma^2}{\partial \kappa} \\ \dfrac{\partial \mu_3}{\partial \xi} & \dfrac{\partial \mu_3}{\partial \alpha^*} & \dfrac{\partial \mu_3}{\partial \kappa} \end{bmatrix} \tag{A12}$$

and can be calculated from (A8). Note that the derivative of the gamma function, $\Gamma(x)$, is $\Gamma(x)\psi(x)$ where $\psi(x)$ is the digamma function. The vector $\mathbf{V_P}$ containing the covariances of the GEV parameter estimators is finally obtained

$$V_P = t^{-1}D^{-1}V_M = t^{-1} \begin{bmatrix} \alpha^{*2}w_{11} \\ \alpha^{*2}w_{22} \\ w_{33} \\ \alpha^{*2}w_{12} \\ \alpha^* w_{13} \\ \alpha^* w_{23} \end{bmatrix} \tag{A13}$$

where the w_{ij} are functions of κ. The asymptotic variances and covariances of the MOM estimators involve the use of the central moment of order 6 implying that the asymptotic expressions are valid only for $\kappa > -1/6$. Values of w_{ij} for different shape parameters are given in Table A1.

The covariance matrix of the PWM estimators are obtained by using a procedure similar to the one described above for MOM estimation. The variances and the covariances of the estimated PWMs are given by *Hosking et al.* [1985b]. Numerically evaluated values of w_{ij} using these expressions are given in Table A1.

Acknowledgments. We are grateful to Richard Vogel and another but anonymous referee for very careful reviews of this paper and to Jery Stedinger, who pointed out an error in our presentation.

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(Received April 3, 1996; revised November 6, 1996;
accepted December 11, 1996.)