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Comparison of Different Confidence Intervals of Intensities for an Open Queueing Network with Feedback

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ABSTRACT

In this paper we propose a consistent and asymptotically normal estimator (CAN) of intensities ρ_1, ρ_2 for a queueing network with feedback (in which a job may return to previously visited nodes) with distribution-free inter-arrival and service times. Using this estimator and its estimated variance, some $100(1-\alpha)\%$ asymptotic confidence intervals of intensities are constructed. Also bootstrap approaches such as Standard bootstrap, Bayesian bootstrap, Percentile bootstrap and Bias-corrected and accelerated bootstrap are also applied to develop the confidence intervals of intensities. A comparative analysis is conducted to demonstrate performances of the confidence intervals of intensities for a queueing network with short run data.

Keywords: Coverage Percentage; Relative Coverage; Bayesian Bootstrap; Bias-Corrected and Accelerated Bootstrap; Percentile Bootstrap; Standard Bootstrap

1. Introduction

Consider a queueing network of a computer system with feedback (in which a job may return to previously visited nodes) as shown in **Figure 1**. This queueing network consists of a CPU node and an Input/Output (I/O) node. External jobs arrive at the CPU node according to the rate λ. After service completion at CPU node, the job proceeds to the I/O node with probability p_1, and with probability p_0 the job departs from the system, where $p_0 = 1 - p_1$. Jobs leaving the I/O node are always fed back to the CPU node (see **Figure 1**). The service times at each node are with rates μ_1 and μ_2 respectively. The successive service times at both nodes are assumed to be mutually independent and independent of the state of the system. The traffic intensity at the CPU node and I/O node is given by

$$\rho_1 = \frac{\lambda}{p_0\mu_1}, \rho_2 = \frac{p_1\lambda}{p_0\mu_2} \tag{1}$$

respectively. Intensity ρ_1 and ρ_2 can be interpreted as expected number of arrivals per mean service time. The condition for stability of the system is both ρ_1, ρ_2 are less unity.

Basic properties of queueing networks are introduced in Disney [1]. Burke [2], Beautler and Melamed [3] showed that the input process to a service center in a network with feedback is not Poisson in general. It is for this reason that Jacksons result is remarkable. Jacksons [4] theorem states that each node behaves like an independent queue.

The product form solution to open network of Markovian queues with feedback is also discussed in Jackson [4]. Simon and Foley [5], Melamed [6] pointed out that computation of response time distribution is difficult even for Jacksonian networks without feedback. Disney and Kiessel [7] discussed traffic process in queueing networks thorough Markov renewal approach. Thiruvaiyaru, Basawa and Bhat [8] established Maximum likelihood estimators of the parameters of an open Jackson network are derived, and their joint asymptotic normality. The problem of estimation for tandem queues is discussed as a special case of the Jackson system. These results are valid when the system is not necessarily in equilibrium. Thiruvaiyaru and Basawa [9] considered the problem of estimation for the parameters in a Jackson's type queueing network with the arrival at each node following renewal process and service time distribution being arbitrary. Open queueing networks are useful in studying the behavior of computer communication networks (Kleinrock [10]). More approach to queueing network analysis

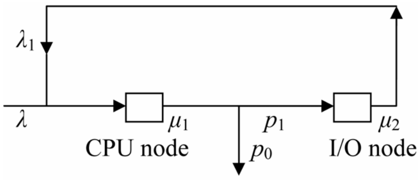

Figure 1. An open queueing network with feedback.

developed by Buzen and Denning [11]. Efron [12-14] the greatest statistician in the field of nonparametric resampling approach, originally developed and proposed the bootstrap, which is a resampling technique that can be effectively applied to estimate the sampling distribution of any statistic. Specifically, one can utilize the bootstrap method to approximate the sampling distribution of a statistic defined by a random sample from a population with unknown probability distribution. And due to the popularity of PC and statistical software, today the bootstrap becomes the most powerful nonparametric estimation procedure. Based upon the bootstrap resampling technique, most statisticians utilize the standard bootstrap (SB), percentile bootstrap (PB), and bias-corrected and accelerated bootstrap (BCaB) approaches to produce confidence intervals for practical problems.

Besides the standard bootstrap (SB) technique, Rubin [15] presented the Bayesian bootstrap (BB) technique of resampling. Miller [16] showed that the SB can be regarded as an extension of the jackknife. The BB is a natural Bayesian analogue of the SB. The BB simulates the posterior distribution of parameters under particular model specifications, whereas the SB simulates the estimated sampling distribution of a statistic estimating the parameters. Both SB and BB can be applied to construct confidence intervals of intensity for a queueing system with distribution-free inter-arrival and service times.

Chu and Ke [17] constructed new confidence intervals of mean response time for an M/G/1 FCFS queueing system. Also, they performed the accuracy of bootstrap confidence intervals through calculating the coverage probability and the average length of confidence intervals. Chu and Ke [18] proposed a consistent and asymptotically normal (CAN) estimator of the mean response time for a G/M/1 queueing system, which is based on the fixed point of empirical Laplace function. Ke and Chu [19] proposed a consistent and asymptotically normal estimator of intensity for a queueing system with distribution-free interarrival and service times. Also, they computed confidence intervals, testing statistical hypothesis of intensity and power function associated with it in this paper. Ke and Chu [20] constructed new confidence intervals of intensity for a queueing system, which are based on different bootstrap methods. They also performed the accuracy of these bootstrap confidence intervals through calculating the coverage probability and the expected length of confidence intervals. They first proposed bootstrapping technique and concept of relative coverage to queueing system. They studied five estimation approaches of intensity for a queueing system with distribution free inter-arrival and service times for short run. They have introduced a new measure called relative coverage to assess the efficient performances of confidence intervals.

In this paper we propose non parametric interval estimation approach to intensities ρ_1, ρ_2 for a open queueing network with feedback. In Section 2 we prove that the natural estimators $\hat{\rho}_1$, $\hat{\rho}_2$ of intensities ρ_1 and ρ_2 are strongly consistent and asymptotically normal (CAN). Based on the asymptotical normality of $\hat{\rho}_1$, $\hat{\rho}_2$, we can construct a CAN confidence interval of ρ_1 and ρ_2. Next in Section 3 we establish the SB confidence interval of ρ_1, ρ_2 via the standard bootstrap approach. In Section 4 we developed the derivation of the BB confidence interval of ρ_1, ρ_2 in terms of the Bayesian bootstrap approach. The percentile bootstrap (PB) confidence intervals of ρ_1, ρ_2 are developed in Section 5. In Section 6 we developed the bias-corrected and accelerated bootstrap (BCaB) confidence intervals. A numerical simulation study is conducted in Section 7 to demonstrate performances of the interval estimation approaches for an open queueing network with feedback using short run data. All simulation results are shown by appropriate tables for illustrating performances of the five interval estimation approaches. Finally, we make some conclusions in Section 8.

2. Nonparametric Statistical Inference of Intensities

Let X_1 and Y_1 be nonnegative random variables represents the inter-arrival and service time at CPU node. Similarly X_2 and Y_2 be nonnegative random variables represents the inter-arrival and service time at I/O node respectively. Given that a job just completed CPU node burst, it will next request I/O node service with probability p_1 and with probability p_0, where $p_0 = 1 - p_1$ departs from the system. The random variables at CPU node and I/O node are independent. The intensities are defined as follows:

$$\rho_1 = \frac{p_0 \mu_{Y_1}}{\mu_{X_1}} \text{ and } \rho_2 = \frac{p_0 \mu_{Y_2}}{p_1 \mu_{X_2}}, \qquad (2)$$

where μ_{X_1}, μ_{X_2} denote the mean inter-arrival times at CPU node and I/O node respectively. Similarly μ_{Y_1}, μ_{Y_2} denote the mean service times at CPU node and I/O node respectively. Equation (2) is equivalent to Equation (1).

2.1. Estimating Intensities

Assume that $X_{11}, X_{12}, \cdots, X_{1n}$ is a random sample drawn from X_1 and $p_0Y_{11}, p_0Y_{12}, \cdots, p_0Y_{1n}$ is a random sample drawn from Y_1. Let $\left(X_{1i}, p_0Y_{1i}\right)$ represents inter-arrival time and service time for the i^{th} customer of CPU node. Similarly assume $p_1X_{21}, p_1X_{22}, \cdots, p_1X_{2m}$ is a random sample drawn from X_2 and $p_0Y_{21}, p_0Y_{22}, \cdots, p_0Y_{2m}$ is a random sample drawn from Y_2. Let $\left(p_1X_{2i}, p_0Y_{2i}\right)$ represents inter-arrival time and service time for the i^{th} customer of I/O node.

Let $\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{Y}_2$ be the sample means of X_1, X_2, Y_1, Y_2 respectively.

$$\bar{X}_1 = \frac{1}{n}\sum_{i=1}^{n} X_{1i},\ \bar{X}_2 = \frac{1}{m}\sum_{i=1}^{m} p_1X_{2i}$$
$$\bar{Y}_1 = \frac{1}{n}\sum_{i=1}^{n} p_0Y_{1i},\ \bar{Y}_2 = \frac{1}{m}\sum_{i=1}^{m} p_0Y_{2i}$$

According to the Strong Law of Large Numbers, we know that $\bar{X}_1, \bar{X}_2, \bar{Y}_1, \bar{Y}_2$ are strongly consistent estimator of $\mu_{X_1}, \mu_{X_2}, \mu_{Y_1}, \mu_{Y_2}$ respectively. Thus a strongly consistent estimator of intensities are given by

$$\hat{\rho}_1 = \frac{\bar{Y}_1}{\bar{X}_1},\ \hat{\rho}_2 = \frac{\bar{Y}_2}{\bar{X}_2} \tag{3}$$

In practical queueing network, the true distributions of X_1, X_2, Y_1, Y_2 are rarely known, so the exact distributions of $\hat{\rho}_1, \hat{\rho}_2$ cannot be derived. But under the assumption that X_1 and Y_1, X_2 and Y_2 being independent, the asymptotic distributions of $\hat{\rho}_1, \hat{\rho}_2$ can be developed by the following procedures.

Firstly, according to the Central Limit Theorem (see [21] p. 234), we have

$$\begin{aligned}&\sqrt{n}\left(\bar{X}_1 - \mu_{X_1}\right) \xrightarrow{D} N\left(0, \sigma_{X_1}^2\right) \text{ and}\\ &\sqrt{n}\left(\bar{Y}_1 - p_0\mu_{Y_1}\right) \xrightarrow{D} N\left(0, \sigma_{Y_1}^2\right)\end{aligned} \tag{4}$$

where $\sigma_{X_1}^2$ and $\sigma_{Y_1}^2$ are variances of X_1 and Y_1, respectively.

Also,

$$\begin{aligned}&\sqrt{m}\left(\bar{X}_2 - p_1\mu_{X_2}\right) \xrightarrow{D} N\left(0, \sigma_{X_2}^2\right) \text{ and}\\ &\sqrt{m}\left(\bar{Y}_2 - p_0\mu_{Y_2}\right) \xrightarrow{D} N\left(0, \sigma_{Y_2}^2\right),\end{aligned} \tag{5}$$

where $\sigma_{X_2}^2$ and $\sigma_{Y_2}^2$ are variances of X_2 and Y_2, respectively, and $\xrightarrow{D}$ denotes convergence in distribution.

Next note that

$$\begin{aligned}&\sqrt{n}\left(\hat{\rho}_1 - \rho_1\right)\\ &= \sqrt{n}\left(\frac{\bar{Y}_1}{\bar{X}_1} - \frac{p_0\mu_{Y_1}}{\mu_{X_1}}\right)\\ &= \frac{\sqrt{n}\left[\mu_{X_1}\left(\bar{Y}_1 - p_0\mu_{Y_1}\right) - p_0\mu_{Y_1}\left(\bar{X}_1 - \mu_{X_1}\right)\right]}{\mu_{X_1}\bar{X}_1}.\end{aligned}$$

Also,

$$\begin{aligned}&\sqrt{m}\left(\hat{\rho}_2 - \rho_2\right)\\ &= \sqrt{m}\left(\frac{\bar{Y}_2}{\bar{X}_2} - \frac{p_0\mu_{Y_2}}{p_1\mu_{X_2}}\right)\\ &= \frac{\sqrt{m}\left[p_1\mu_{X_2}\left(\bar{Y}_2 - p_0\mu_{Y_2}\right) - p_0\mu_{Y_2}\left(\bar{X}_2 - p_1\mu_{X_2}\right)\right]}{p_1\mu_{X_2}\bar{X}_2}.\end{aligned} \tag{6}$$

Therefore by the Slutsky's theorem [see [21] p. 227], we get

$$\sqrt{n}\left(\hat{\rho}_1 - \rho_1\right) \xrightarrow{D} N\left(0, \sigma_1^2\right),$$

where $\sigma_1^2 = \dfrac{\mu_{X_1}^2\sigma_{Y_1}^2 + p_0^2\mu_{Y_1}^2\sigma_{X_1}^2}{\mu_{X_1}^4}$.

And

$$\sqrt{m}\left(\hat{\rho}_2 - \rho_2\right) \xrightarrow{D} N\left(0, \sigma_2^2\right) \tag{7}$$

where $\sigma_2^2 = \dfrac{p_1^2\mu_{X_2}^2\sigma_{Y_2}^2 + p_0^2\mu_{Y_2}^2\sigma_{X_2}^2}{p_1^2\mu_{X_2}^4}$.

Now, set

$$\begin{aligned}&\hat{\sigma}_1^2 = \frac{\bar{X}_1^2 S_{Y_1}^2 + p_0^2\bar{Y}_1^2 S_{X_1}^2}{\bar{X}_1^4} \text{ and}\\ &\hat{\sigma}_2^2 = \frac{p_1^2\bar{X}_2^2 S_{Y_2}^2 + p_0^2\bar{Y}_2^2 S_{X_2}^2}{p_1^2\bar{X}_2^4},\end{aligned} \tag{8}$$

where

$$S_{X_1}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_{1i} - \bar{X}_1\right)^2,\ S_{Y_1}^2 = \frac{1}{n}\sum_{i=1}^{n}\left(p_0Y_{1i} - \bar{Y}_1\right)^2$$
$$S_{X_2}^2 = \frac{1}{m}\sum_{i=1}^{m}\left(p_1X_{2i} - \bar{X}_2\right)^2 \text{ and } S_{Y_2}^2 = \frac{1}{m}\sum_{i=1}^{m}\left(p_0Y_{2i} - \bar{Y}_2\right)^2.$$

Then $\hat{\sigma}_1^2, \hat{\sigma}_2^2$ are strongly consistent estimators of σ_1^2, σ_2^2 respectively. Applying the Slutsky's theorem once again, we deduce that

$$\begin{aligned}&\frac{\sqrt{n}\left(\hat{\rho}_1 - \rho_1\right)}{\hat{\sigma}_1} \xrightarrow{D} N(0,1) \text{ and}\\ &\frac{\sqrt{m}\left(\hat{\rho}_2 - \rho_2\right)}{\hat{\sigma}_2} \xrightarrow{D} N(0,1)\end{aligned} \tag{9}$$

Thus $\hat{\rho}_1, \hat{\rho}_2$ are strongly consistent and asymptotically normal (CAN) estimators with approximate variances $\dfrac{\hat{\sigma}_1^2}{n}, \dfrac{\hat{\sigma}_2^2}{m}$ respectively.

2.2. Confidence Intervals

Using the CAN estimators $\hat{\rho}_1, \hat{\rho}_2$ and its associated approximate variances $\dfrac{\hat{\sigma}_1^2}{n}, \dfrac{\hat{\sigma}_2^2}{m}$ we construct a confidence intervals of intensities ρ_1 & ρ_2 for a open queueing network with feedback. Let z_α be the upper α^{th} quantile of the standard normal distribution, by the asymptotic distribution of $\frac{\sqrt{n}\left(\hat{\rho}_1-\rho_1\right)}{\hat{\sigma}_1}$ & $\frac{\sqrt{m}\left(\hat{\rho}_2-\rho_2\right)}{\hat{\sigma}_2}$ in expression (9), an approximate $100(1-\alpha)\%$ confidence interval of ρ_1 & ρ_2 are obtained as

$$1-\alpha \approx P\left(-z_{\alpha/2} \le \frac{\sqrt{n}\left(\hat{\rho}_1-\rho_1\right)}{\hat{\sigma}_1} \le z_{\alpha/2}\right)$$

$$= P\left(\hat{\rho}_1 - \frac{z_{\alpha/2}\hat{\sigma}_1}{\sqrt{n}} \le \rho_1 \le \hat{\rho}_1 + \frac{z_{\alpha/2}\hat{\sigma}_1}{\sqrt{n}}\right)$$

Consequently, an approximate $100(1-\alpha)\%$ confidence interval of ρ_1 is

$$\left(\hat{\rho}_1 - \frac{z_{\alpha/2}\hat{\sigma}_1}{\sqrt{n}}, \hat{\rho}_1 + \frac{z_{\alpha/2}\hat{\sigma}_1}{\sqrt{n}}\right). \quad (10)$$

Similarly, an approximate $100(1-\alpha)\%$ confidence interval of ρ_2 is

$$\left(\hat{\rho}_2 - \frac{z_{\alpha/2}\hat{\sigma}_2}{\sqrt{m}}, \hat{\rho}_2 + \frac{z_{\alpha/2}\hat{\sigma}_2}{\sqrt{m}}\right). \quad (11)$$

3. Standard Bootstrap Confidence Intervals of Intensities

Now the bootstrap confidence intervals are developed as follows:

Let $x_{11}, x_{12}, \cdots, x_{1n}$ be a random sample of n observations taken from the population X_1 and $p_0y_{11}, p_0y_{12}, \cdots, p_0y_{1n}$ be a random sample of n observations taken from the population Y_1. According to the bootstrap procedure, a simple random sample $x_{11}^*, x_{12}^*, \cdots, x_{1n}^*$ can be taken from the empirical distribution function of $x_{11}, x_{12}, \cdots, x_{1n}$ called a bootstrap sample from $x_{11}, x_{12}, \cdots, x_{1n}$. Also, we can draw a bootstrap sample $p_0y_{11}^*, p_0y_{12}^*, \cdots, p_0y_{1n}^*$ from $p_0y_{11}, p_0y_{12}, \cdots, p_0y_{1n}$. It follows from Equation (2) that an estimate of intensity ρ_1 can be calculated from bootstrap samples as

$$\hat{\rho}_1^* = \frac{\overline{y}_1^*}{\overline{x}_1^*}, \quad (12)$$

where $\overline{x}_1^*$ and $\overline{y}_1^*$ are the sample means of $x_{11}^*, x_{12}^*, \cdots, x_{1n}^*$ and $p_0y_{11}^*, p_0y_{12}^*, \cdots, p_0y_{1n}^*$ respectively and $\hat{\rho}_1^*$ is called a bootstrap estimate of ρ_1. The above resampling process can be repeated N_1 times. The N_1 bootstrap estimates $\hat{\rho}_{11}^*, \hat{\rho}_{12}^*, \cdots, \hat{\rho}_{1N_1}^*$ can be computed from the bootstrap resamples. Averaging the N_1 bootstrap estimates we obtain that

$$\hat{\rho}_{N_1} = \frac{1}{N_1}\sum_{i=1}^{N_1}\hat{\rho}_{1i}^* \quad (13)$$

is the bootstrap estimate of ρ_1. And the standard deviation of $\hat{\rho}_1$ can be estimated by

$$sd\left(\hat{\rho}_{N_1}\right) = \left\{\frac{1}{N_1-1}\sum_{i=1}^{N_1}\left(\hat{\rho}_{1i}^* - \hat{\rho}_{N_1}\right)^2\right\}^{1/2}. \quad (14)$$

Because the central limit theorem implies that the distribution of $\hat{\rho}_1$ is approximately normal. A $100(1-\alpha)\%$ SB confidence interval for ρ_1 is

$$\left(\hat{\rho}_1 - z_{\alpha/2}sd\left(\hat{\rho}_{N_1}\right), \hat{\rho}_1 + z_{\alpha/2}sd\left(\hat{\rho}_{N_1}\right)\right), \quad (15)$$

Similarly $p_1x_{21}, p_1x_{22}, \cdots, p_1x_{2m}$ is a random sample of m observation drawn from population X_2 and p_0y_{21}, $p_0y_{22}, \cdots, p_0y_{2m}$ is a sample of m observations taken from the population Y_2. According to the bootstrap procedure, a simple random sample $p_1x_{21}^*, p_1x_{22}^*, \cdots, p_1x_{2m}^*$ can be taken from the empirical distribution function of $p_1x_{21}, p_1x_{22}, \cdots, p_1x_{2m}$ called a bootstrap sample from $p_1x_{21}, p_1x_{22}, \cdots, p_1x_{2m}$. Also, we can draw a bootstrap sample $p_0y_{21}^*, p_0y_{22}^*, \cdots, p_0y_{2m}^*$ from $p_0y_{21}, p_0y_{22}, \cdots$, p_0y_{2m}. An estimate of intensity ρ_2 can be calculated from bootstrap samples as

$$\hat{\rho}_2^* = \frac{\overline{y}_2^*}{\overline{x}_2^*}, \quad (16)$$

where $\overline{x}_2^*$ and $\overline{y}_2^*$ are the sample means of $p_1x_{21}^*$, $p_1x_{22}^*, \cdots, p_1x_{2m}^*$ and $p_0y_{21}^*, p_0y_{22}^*, \cdots, p_0y_{2m}^*$ respectively and $\hat{\rho}_2^*$ is called a bootstrap estimate of ρ_2. The above resampling process can be repeated M_1 times. The M_1 bootstrap estimates $\hat{\rho}_{21}^*, \hat{\rho}_{22}^*, \cdots, \hat{\rho}_{2M_1}^*$ can be computed from the bootstrap resamples. Averaging the M_1 bootstrap estimates we obtain that

$$\hat{\rho}_{M_1} = \frac{1}{M_1}\sum_{i=1}^{M_1}\hat{\rho}_{2i}^* \quad (17)$$

is the bootstrap estimate of ρ_2. And the standard deviation of $\hat{\rho}_2$ can be estimated by

$$sd\left(\hat{\rho}_{M_1}\right) = \left\{\frac{1}{M_1-1}\sum_{i=1}^{M}\left(\hat{\rho}_{2i}^* - \hat{\rho}_{M_1}\right)^2\right\}^{1/2}. \quad (18)$$

Because the central limit theorem implies that the distribution of $\hat{\rho}_2$ is approximately normal. A $100(1-\alpha)\%$ SB confidence interval for ρ_2 is

$$\left(\hat{\rho}_2 - z_{\alpha/2}sd\left(\hat{\rho}_{M_1}\right), \hat{\rho}_2 + z_{\alpha/2}sd\left(\hat{\rho}_{M_1}\right)\right), \quad (19)$$

4. Bayesian Bootstrap Confidence Intervals of Intensities

The Bayesian bootstrap is analogous to the standard bootstrap. Each BB replication generates a posterior probability for each x_{1i}. Specifically, one BB replication is generated by drawing $n-1$ uniform $(0,1)$ random numbers $r_1, r_2, \cdots, r_{n-1}$, ordering them, and calculating the gaps $w_{1i} = r_{(i)} - r_{(i-1)}$, $i = 1, 2, \cdots, n$, where $r_{(0)} = 0$ and

$r_{(n)}=1$. Then $w_{1i}=(w_{11},w_{12},\cdots,w_{1n})$ is the vector of probabilities attached to the inter-arrival data values $x_{11},x_{12},\cdots,x_{1n}$ in that BB replication. Considering all BB replications gives the BB distribution of the distribution of X_1 and thus of any parameter of this distribution. Hence for μ_{X_1} (the mean of X_1), in each BB replication we calculate μ_{X_1} as if w_{1i} were the probability that $X_1=x_i$ that is, we calculate $\bar{x}_1^{**}=\sum_{i=1}^{n}w_{1i}x_{1i}$. The distribution of the values of $\bar{x}_1^{**}$ overall BB replications is the BB distribution of μ_{X_1}.

Also, generating a vector of probabilities $v_1=(v_{11},v_{12},\cdots,v_{1n})$ attached to the service time data values $p_0y_{11},p_0y_{12},\cdots,p_0y_{1n}$ in a BB replication, and we calculate $\bar{y}_1^{**}=p_0\sum_{i=1}^{n}v_{1i}y_{1i}$ for μ_{Y_1} (the mean of Y_1). Then in terms of equation (2) an estimate of intensity ρ_1 can be calculated from BB replications as

$$\hat{\rho}_1^{**}=\frac{\bar{y}_1^{**}}{\bar{x}_1^{**}}, \qquad (20)$$

where $\hat{\rho}_1^{**}$ is called a Bayesian bootstrap estimate of ρ_1. The above BB process can be repeated N_1 times. The N_1 BB estimates, $\hat{\rho}_{11}^{**},\hat{\rho}_{12}^{**},\cdots,\hat{\rho}_{1N_1}^{**}$ can be computed from the BB replications. Averaging the N_1 BB estimates, we obtain that

$$\hat{\rho}'_{BB}=\frac{1}{N_1}\sum_{j=1}^{N_1}\hat{\rho}_{1j}^{**}, \qquad (21)$$

is the BB estimate of ρ_1. And the standard deviation of $\hat{\rho}_1$ can be estimated by

$$sd(\hat{\rho}'_{BB})=\left\{\frac{1}{N_1-1}\sum_{j=1}^{N_1}\left(\hat{\rho}_{1j}^{**}-\hat{\rho}'_{\mathrm{BB}}\right)^2\right\}^{\frac{1}{2}}. \qquad (22)$$

Applying the asymptotical normality of $\hat{\rho}_1$, a $100(1-\alpha)\%$ BB confidence interval for ρ_1 is

$$\left(\hat{\rho}_1-z_{\alpha/2}sd(\hat{\rho}'_{BB}),\hat{\rho}_1+z_{\alpha/2}sd(\hat{\rho}'_{BB})\right). \qquad (23)$$

Similarly each BB replication generates a posterior probability for each x_{2i}. Specifically, one BB replication is generated by drawing $m-1$ uniform $(0,1)$ random numbers $r_1,r_2,\cdots,r_{m-1}$, ordering them and calculating the gaps $w_{2i}=r_{(i)}-r_{(i-1)},i=1,2,\cdots,m$, where $r_{(0)}=0$ and $r_{(m)}=1$. Then $w_{2i}=(w_{21},w_{22},\cdots,w_{2m})$ is the vector of probabilities attached to the inter-arrival data values $p_1x_{21},p_1x_{22},\cdots,p_1x_{2m}$ in that BB replication. Considering all BB replications gives the BB distribution of the distribution of X_2 and thus of any parameter of this distribution. Hence for μ_{X_2} (the mean of X_2), in each BB replication we calculate μ_{X_2} as if w_{2i} were the probability that $X_2=x_{2i}$ that is, we calculate $\bar{x}_2^{**}=p_1\sum_{i=1}^{m}w_{2i}x_{2i}$. The distribution of the values of $\bar{x}_2^{**}$ over all BB replications is the BB distribution of μ_{X_2}.

Also, generating a vector of probabilities $v_2=(v_{21},v_{22},\cdots,v_{2m})$ attached to the service time data values $p_0y_{21},p_0y_{22},\cdots,p_0y_{2m}$ in a BB replication, and we calculate $\bar{y}_2^{**}=p_0\sum_{i=1}^{m}v_{2i}y_{2i}$ for μ_{Y_2} (the mean of Y_2). Then in terms of Equation (2) an estimate of intensity ρ_2 can be calculated from BB replications as

$$\hat{\rho}_2^{**}=\frac{\bar{y}_2^{**}}{\bar{x}_2^{**}}, \qquad (24)$$

where $\hat{\rho}_2^{**}$ is called a Bayesian bootstrap estimate of ρ_2. The above BB process can be repeated M_1 times. The M_1 BB estimates, $\hat{\rho}_{21}^{**},\hat{\rho}_{22}^{**},\cdots,\hat{\rho}_{2M_1}^{**}$ can be computed from the BB replications. Averaging the M_1 BB estimates, we obtain that

$$\hat{\rho}''_{BB}=\frac{1}{M_1}\sum_{j=1}^{M_1}\hat{\rho}_{2j}^{**}, \qquad (25)$$

is the *BB* estimate of ρ_2. And the standard deviation of $\hat{\rho}_2$ can be estimated by

$$sd(\hat{\rho}''_{BB})=\left\{\frac{1}{M_1-1}\sum_{j=1}^{M_1}\left(\hat{\rho}_{2j}^{**}-\hat{\rho}''_{BB}\right)^2\right\}. \qquad (26)$$

Applying the asymptotical normality of $\hat{\rho}_2$, a $100(1-\alpha)\%$ BB confidence interval for ρ_2 is

$$\left(\hat{\rho}_2-z_{\alpha/2}sd(\hat{\rho}''_{BB}),\hat{\rho}_2+z_{\alpha/2}sd(\hat{\rho}''_{BB})\right) \qquad (27)$$

5. Percentile Bootstrap Confidence Intervals of Intensities

Let $\hat{\rho}_{11}^{*},\hat{\rho}_{12}^{*},\cdots,\hat{\rho}_{1N_1}^{*}$ and $\hat{\rho}_{21}^{*},\hat{\rho}_{22}^{*},\cdots,\hat{\rho}_{2M_1}^{*}$ call the bootstrap distribution of $\hat{\rho}_1,\hat{\rho}_2$ respectively. Let $\hat{\rho}_1^{*}(1)$, $\hat{\rho}_1^{*}(2),\cdots,\hat{\rho}_1^{*}(N_1)$ and $\hat{\rho}_2^{*}(1),\hat{\rho}_2^{*}(2),\cdots,\hat{\rho}_2^{*}(M_1)$ be order statistics of $\hat{\rho}_{11}^{*},\hat{\rho}_{12}^{*},\cdots,\hat{\rho}_{1N_1}^{*}$ and $\hat{\rho}_{21}^{*},\hat{\rho}_{22}^{*},\cdots,\hat{\rho}_{2M_1}^{*}$ respectively. Then utilizing the $100(\alpha/2)$th and $100(1-\alpha/2)$th percentage points of the bootstrap distribution, a $100(1-\alpha)\%$ PB confidence interval for ρ_1, ρ_2 are obtained as

$$\left(\hat{\rho}_1^{*}\left(\left[N_1\left(\frac{\alpha}{2}\right)\right]\right),\hat{\rho}_1^{*}\left(\left[N_1\left(1-\frac{\alpha}{2}\right)\right]\right)\right), \qquad (28)$$

$$\left(\hat{\rho}_2^{*}\left(\left[M_1\left(\frac{\alpha}{2}\right)\right]\right),\hat{\rho}_2^{*}\left(\left[M_1\left(1-\frac{\alpha}{2}\right)\right]\right)\right), \qquad (29)$$

where [*x*] denotes the greatest integer less than or equal to *x*.

6. Bias-Corrected and Accelerated Bootstrap Confidence Intervals of Intensities

The bootstrap distribution $\hat{\rho}_{11}^{*},\hat{\rho}_{12}^{*},\cdots,\hat{\rho}_{1N_1}^{*}$ and $\hat{\rho}_{21}^{*}$,

$\hat{\rho}_{22}^{*},\cdots,\hat{\rho}_{2M_1}^{*}$ may be biased. This method is designed to correct this potential bias of the bootstrap designed. Set

$$p' = \sum_{j=1}^{N_1} \frac{I\left(\hat{\rho}_{1j}^{*} < \hat{\rho}_1\right)}{N_1} \quad \text{and} \quad p'' = \sum_{j=1}^{M_1} \frac{I\left(\hat{\rho}_{2j}^{*} < \hat{\rho}_2\right)}{M_1},$$

where $I(\cdot)$ is the indicator function. Define $\hat{z}_0 = \phi^{-1}(p')$ and $\hat{z}_1 = \phi^{-1}(p'')$, where ϕ^{-1} denotes the inverse function of the standard normal distribution ϕ. Except for correcting the potential bias of the bootstrap distribution, we can accelerate convergence of bootstrap distribution. Let $\tilde{X}_1(i)$ and $p_0\tilde{Y}_1(i)$ denote the original samples with the i^{th} observation x_{1i} and $p_0 y_{1i}$ deleted, also let $\hat{\rho}_{1i}$ be the estimator of ρ_1 calculated by using $\tilde{X}_1(i)$ and $p_0\tilde{Y}_1(i)$ Define $\tilde{\rho}_1 = \frac{1}{n}\sum_{i=1}^{n}\hat{\rho}_{1i}$, Similarly $p_1\tilde{X}_2(i)$, $p_0\tilde{Y}_2(i)$ denote the original samples with the i^{th} observation $p_1 x_{2i}$ and $p_0 y_{2i}$ deleted, also $\hat{\rho}_{2i}$ be the estimator of ρ_2 calculated by using $p_1\tilde{X}_2(i)$ and $p_0\tilde{Y}_2(i)$.

Define $\tilde{\rho}_2 = \frac{1}{m}\sum_{i=1}^{m}\hat{\rho}_{2i}$

And

$$\hat{a}_1 = \frac{\sum_{i=1}^{n}\left(\tilde{\rho}_1 - \hat{\rho}_{1i}\right)^3}{\left\{6\left(\sum_{i=1}^{n}\left(\tilde{\rho}_1 - \hat{\rho}_{1i}\right)^2\right)^{\left(\frac{3}{2}\right)}\right\}}, \qquad \hat{a}_2 = \frac{\sum_{i=1}^{m}\left(\tilde{\rho}_2 - \hat{\rho}_{2i}\right)^3}{\left\{6\left(\sum_{i=1}^{m}\left(\tilde{\rho}_2 - \hat{\rho}_{2i}\right)^2\right)^{\left(\frac{3}{2}\right)}\right\}} \tag{30}$$

where $\hat{z}_0, \hat{z}_1, \hat{a}_1$, and $\hat{a}_2$ are named bias-correction and acceleration respectively.

Thus a $100(1-\alpha)\%$ Bias-corrected and accelerated bootstrap (BCaB) Confidence Interval of intensities ρ_1, ρ_2 are constructed by

$$\left(\hat{\rho}_1^{*}\left(\left[N_1\alpha_1\right]\right), \hat{\rho}_1^{*}\left(\left[N_1\alpha_2\right]\right)\right) \tag{31}$$

$$\left(\hat{\rho}_2^{*}\left(\left[M_1\alpha_3\right]\right), \hat{\rho}_2^{*}\left(\left[M_1\alpha_4\right]\right)\right) \tag{32}$$

where

$$\alpha_1 = \phi\left\{\hat{z}_0 + \frac{\left(\hat{z}_0 - z_{\alpha/2}\right)}{1-\hat{a}_1\left(\hat{z}_0 - z_{\alpha/2}\right)}\right\}$$

$$\alpha_2 = \phi\left\{\hat{z}_0 + \frac{\left(\hat{z}_0 + z_{\alpha/2}\right)}{1-\hat{a}_1\left(\hat{z}_0 + z_{\alpha/2}\right)}\right\}$$

$$\alpha_3 = \phi\left\{\hat{z}_1 + \frac{\left(\hat{z}_1 - z_{\alpha/2}\right)}{1-\hat{a}_2\left(\hat{z}_1 - z_{\alpha/2}\right)}\right\}$$

$$\alpha_4 = \phi\left\{\hat{z}_1 + \frac{\left(\hat{z}_1 + z_{\alpha/2}\right)}{1-\hat{a}_2\left(\hat{z}_1 + z_{\alpha/2}\right)}\right\}$$

7. Simulation Study

To evaluate performances of the different interval estimation approaches mentioned above for an open queueing network with feedback using short run data, a numerical simulation study was undertaken. Most of the statisticians assess performances of interval estimations in terms of coverage percentages or average lengths of confidence intervals. However, through simulation study in the research work, we find that larger coverage percentages of confidence interval may often be due to wider standard deviation of interval estimation methods. Moreover, narrower confidence intervals may often lead to smaller coverage percentages. Hence, both coverage percentage and average length are not efficient for appraising interval estimation methods. In order to overcome above two shortcomings, we propose a measure called relative coverage to evaluate performances of interval estimation methods where,

$$\text{Relative coverage} = \frac{\text{Coverage percentage}}{\text{Average length}}.$$

The larger of the relative coverage implies the better performance of the corresponding confidence interval. In order to reach this goal, we set a continuous distribution with mean $1/\lambda$ on inter-arrival time X_1 and X_2. Also set continuous distribution with mean $1/\mu_1$ on the service time Y_1 at CPU node and continuous distribution with mean $1/\mu_2$ on the service time Y_2 at I/O node. The levels of p_0 considered in the simulation study are 0.1 to 0.9 where as levels of p_1 are 0.9 to 0.1, where p_0 is the probability that the job departs from the system and p_1 is the probability that after service completion at CPU node, the job proceeds to the I/O node. This means with probability $p_0 = 0.1$ the job departs from the system and with probability $p_1 = 0.9$, after service completion at CPU node, the job proceeds to the I/O node and so on. Also we have considered the values of ρ_1 and ρ_2 such that $\rho_1 < 1$ and $\rho_2 < 1$ for simulation study. Note that in **Table 1** wherever $\rho_1 \geq 1$ and $\rho_2 \geq 1$ such values of ρ_1 and ρ_2 are not considered for simulation study. The intensity parameters ρ_1 and ρ_2 are calculated using Equation (1). The different values of λ, μ_1, μ_2, p_0 and p_1 are considered for simulation study as shown in **Table 1**.

For different levels of ρ_1, random samples of inter-arrival times $X_{11}, X_{12}, \cdots, X_{1n}$ and service times $p_0 Y_{11}$,

Table 1. Different levels of intensity parameters considered in the simulation study.

$\lambda = 0.1, \mu_1 = 1, \mu_2 = 1$				$\lambda = 0.1, \mu_1 = 1, \mu_2 = 2$				$\lambda = 0.1, \mu_1 = 2, \mu_2 = 1$			
p_0	p_1	ρ_1	ρ_2	p_0	p_1	ρ_1	ρ_2	p_0	p_1	ρ_1	ρ_2
0.1	0.9	1	0.9	0.1	0.9	1	0.45	0.1	0.9	0.5	0.9
0.2	0.8	0.5	0.4	0.2	0.8	0.5	0.2	0.2	0.8	0.25	0.4
0.3	0.7	0.33	0.23	0.3	0.7	0.33	0.12	0.3	0.7	0.17	0.23
0.4	0.6	0.25	0.15	0.4	0.6	0.25	0.08	0.4	0.6	0.13	0.15
0.5	0.5	0.2	0.1	0.5	0.5	0.2	0.05	0.5	0.5	0.1	0.1
0.6	0.4	0.17	0.07	0.6	0.4	0.17	0.03	0.6	0.4	0.08	0.07
0.7	0.3	0.14	0.04	0.7	0.3	0.14	0.02	0.7	0.3	0.07	0.04
0.8	0.2	0.13	0.03	0.8	0.2	0.13	0.01	0.8	0.2	0.06	0.03
0.9	0.1	0.11	0.01	0.9	0.1	0.11	0.01	0.9	0.1	0.06	0.01
$\lambda = 0.5, \mu_1 = 1, \mu_2 = 1$				$\lambda = 0.5, \mu_1 = 1, \mu_2 = 2$				$\lambda = 0.5, \mu_1 = 2, \mu_2 = 1$			
p_0	p_1	ρ_1	ρ_2	p_0	p_1	ρ_1	ρ_2	p_0	p_1	ρ_1	ρ_2
0.1	0.9	5	4.5	0.1	0.9	5	2.25	0.1	0.9	2.5	4.5
0.2	0.8	2.5	2	0.2	0.8	2.5	1	0.2	0.8	1.25	2
0.3	0.7	1.67	1.17	0.3	0.7	1.67	0.58	0.3	0.7	0.83	1.17
0.4	0.6	1.25	0.75	0.4	0.6	1.25	0.38	0.4	0.6	0.63	0.75
0.5	0.5	1	0.5	0.5	0.5	1	0.25	0.5	0.5	0.5	0.5
0.6	0.4	0.83	0.33	0.6	0.4	0.83	0.17	0.6	0.4	0.42	0.33
0.7	0.3	0.71	0.21	0.7	0.3	0.71	0.11	0.7	0.3	0.36	0.21
0.8	0.2	0.63	0.13	0.8	0.2	0.63	0.06	0.8	0.2	0.31	0.13
0.9	0.1	0.56	0.06	0.9	0.1	0.56	0.03	0.9	0.1	0.28	0.06
$\lambda = 0.9, \mu_1 = 1, \mu_2 = 1$				$\lambda = 0.9, \mu_1 = 1, \mu_2 = 2$				$\lambda = 0.9, \mu_1 = 2, \mu_2 = 1$			
p_0	p_1	ρ_1	ρ_2	p_0	p_1	ρ_1	ρ_2	p_0	p_1	ρ_1	ρ_2
0.1	0.9	9	8.1	0.1	0.9	9	4.05	0.1	0.9	4.5	8.1
0.2	0.8	4.5	3.6	0.2	0.8	4.5	1.8	0.2	0.8	2.25	3.6
0.3	0.7	3	2.1	0.3	0.7	3	1.05	0.3	0.7	1.5	2.1
0.4	0.6	2.25	1.35	0.4	0.6	2.25	0.68	0.4	0.6	1.13	1.35
0.5	0.5	1.8	0.9	0.5	0.5	1.8	0.45	0.5	0.5	0.9	0.9
0.6	0.4	1.5	0.6	0.6	0.4	1.5	0.3	0.6	0.4	0.75	0.6
0.7	0.3	1.29	0.39	0.7	0.3	1.29	0.19	0.7	0.3	0.64	0.39
0.8	0.2	1.13	0.23	0.8	0.2	1.13	0.11	0.8	0.2	0.56	0.23
0.9	0.1	1	0.1	0.9	0.1	1	0.05	0.9	0.1	0.5	0.1

$p_0Y_{12},\cdots,p_0Y_{1n}$ are drawn from X_1 and Y_1 respectively. Also for each level of ρ_2 random samples of inter-arrival times $p_1X_{21},p_1X_{22},\cdots,p_1X_{2m}$ and service times $p_0Y_{21},p_0Y_{22},\cdots,p_0Y_{2m}$ are drawn from X_2 and Y_2 respectively. Next N = 1000 bootstrap resamples each of size n and m = 10, 20, 29 are drawn from the original samples, as well as N = 1000 BB replications are simulated for the original samples. According to Equations (10), (11), (15), (19), (23), (27)-(29), (31) and (32) in respective, we obtain CAN1, CAN2, SB1, SB2, BB1, BB2, PB1, PB2, BCaB1 and BCaB2 confidence intervals of intensities ρ_1 and ρ_2 with confidence level 90%. The above simulation process is replicated N = 1000 times and we compute coverage percentages, average lengths and relative coverage of the above mentioned confidence intervals. We utilize a PC Dual Core and apply Matlab®7.0.1 to accomplish all simulations.

Here M represents exponential distribution, E_4 a 4-stage Erlang distribution, H_4^{Pe} a 4-stage hyper-exponential distribution and H_4^{Po} a 4-stage hypo-exponential distribution.

Based on the above mentioned interval estimation approaches, the coverage percentage, average lengths and relative coverage of intensities ρ_1 and ρ_2 are shown in **Tables 3** to **7** for queueing network models (presented in **Table 2**) with short run data, we find that average lengths

Table 2. Different queueing network models simulated for study.

Queueing Networks	Models Simulated
$M/G/1$ to $G/M/1$	$M/E_4/1$ to $E_4/M/1$
	$M/H_4^{Pe}/1$ to $H_4^{Pe}/M/1$
$G/G/1$ to $G/G/1$	$E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$
	$E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$
	$H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$

Table 3. Simulation results of coverage percentage, average lengths, and relative coverage for 90% confidence intervals under queueing network. $M/E_4/1$ to $E_4/M/1$.

Intensity Parameters	Estimation Approach	Coverage Percentage			Average Length			Relative Coverage		
		n = 10	n = 20	n = 29	n = 10	n = 20	n = 29	n = 10	n = 20	n = 29
p_0 = 0.2, p_1 = 0.8, ρ_1 = 0.5 and ρ_2 = 0.2	CAN1	0.878	0.895	0.878	0.611	0.416	0.345	**1.437**	**2.151**	**2.548**
	CAN2	0.840	0.881	0.871	0.220	0.162	0.135	3.816	5.431	6.465
	SB1	0.916	0.910	0.900	0.748	0.455	0.365	1.225	2.002	2.465
	SB2	0.835	0.875	0.869	0.217	0.161	0.134	3.843	5.425	6.480
	BB1	0.879	0.898	0.877	0.628	0.418	0.344	1.399	2.148	2.547
	BB2	0.817	0.864	0.856	0.205	0.157	0.131	**3.978**	**5.517**	6.514
	PB1	0.832	0.874	0.867	0.687	0.438	0.356	1.211	1.994	2.437
	PB2	0.831	0.874	0.869	0.214	0.159	0.133	3.892	5.490	**6.537**
	BCaB1	0.831	0.877	0.872	0.668	0.433	0.353	1.243	2.026	2.473
	BCaB2	0.837	0.871	0.871	0.214	0.160	0.133	3.913	5.454	6.535
p_0 = 0.9, p_1 = 0.1, ρ_1 = 0.11 and ρ_2 = 0.01	CAN1	0.870	0.885	0.868	0.139	0.092	0.076	**6.276**	**9.648**	**11.410**
	CAN2	0.832	0.878	0.867	0.006	0.004	0.004	134.515	195.434	232.230
	SB1	0.910	0.912	0.887	0.171	0.100	0.081	5.310	9.116	11.013
	SB2	0.827	0.879	0.866	0.006	0.004	0.004	135.068	196.738	232.756
	BB1	0.879	0.882	0.867	0.143	0.092	0.076	6.138	9.586	11.400
	BB2	0.807	0.871	0.858	0.006	0.004	0.004	**139.854**	**200.876**	**235.214**
	PB1	0.824	0.871	0.859	0.156	0.096	0.078	5.270	9.031	10.946
	PB2	0.826	0.869	0.866	0.006	0.004	0.004	137.508	197.084	235.030
	BCaB1	0.833	0.872	0.868	0.152	0.095	0.078	5.478	9.169	11.159
	BCaB2	0.823	0.871	0.865	0.006	0.004	0.004	136.752	196.995	234.219

Continued

$p_0 = 0.6$ $p_1 = 0.4$ $\rho_1 = 0.83$ and $\rho_2 = 0.17$	CAN1	0.851	0.876	0.866	1.012	0.697	0.579	**0.841**	**1.258**	**1.496**
	CAN2	0.829	0.891	0.891	0.186	0.135	0.112	4.459	6.600	7.935
	SB1	0.893	0.894	0.882	1.249	0.760	0.613	0.715	1.176	1.438
	SB2	0.826	0.890	0.887	0.184	0.134	0.112	4.492	6.631	7.925
	BB1	0.856	0.879	0.865	1.045	0.700	0.578	0.819	1.256	1.496
	BB2	0.814	0.886	0.878	0.173	0.131	0.110	**4.693**	**6.783**	**8.010**
	PB1	0.827	0.866	0.848	1.135	0.732	0.598	0.729	1.184	1.418
	PB2	0.830	0.884	0.886	0.181	0.132	0.111	4.595	6.673	7.993
	BCaB1	0.827	0.866	0.848	1.108	0.722	0.593	0.747	1.199	1.431
	BCaB2	0.824	0.883	0.881	0.181	0.133	0.111	4.553	6.636	7.934
$p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.56$ and $\rho_2 = 0.03$	CAN1	0.882	0.891	0.879	0.682	0.470	0.384	**1.293**	**1.895**	**2.286**
	CAN2	0.859	0.861	0.875	0.031	0.022	0.019	27.501	38.453	46.779
	SB1	0.915	0.912	0.893	0.841	0.515	0.408	1.088	1.771	2.189
	SB2	0.850	0.858	0.873	0.031	0.022	0.019	27.505	38.524	46.903
	BB1	0.889	0.892	0.878	0.704	0.472	0.385	1.262	1.889	2.283
	BB2	0.831	0.850	0.867	0.029	0.022	0.018	**28.516**	**39.337**	47.492
	PB1	0.847	0.861	0.852	0.767	0.496	0.397	1.104	1.737	2.144
	PB2	0.850	0.848	0.877	0.030	0.022	0.018	28.019	38.473	**47.602**
	BCaB1	0.846	0.856	0.852	0.746	0.490	0.394	1.133	1.749	2.163
	BCaB2	0.857	0.850	0.877	0.030	0.022	0.018	28.158	38.438	47.496
$p_0 = 0.1$ $p_1 = 0.9$ $\rho_1 = 0.5$ and $\rho_2 = 0.9$	CAN1	0.842	0.893	0.880	0.588	0.418	0.342	**1.431**	**2.138**	**2.570**
	CAN2	0.842	0.856	0.879	1.016	0.730	0.618	0.829	1.173	1.423
	SB1	0.894	0.909	0.899	0.716	0.456	0.362	1.249	1.993	2.481
	SB2	0.839	0.856	0.877	1.005	0.725	0.615	0.835	1.181	1.425
	BB1	0.850	0.890	0.875	0.605	0.420	0.342	1.406	2.119	2.560
	BB2	0.830	0.847	0.874	0.948	0.705	0.603	**0.875**	**1.201**	**1.449**
	PB1	0.810	0.866	0.874	0.658	0.439	0.353	1.230	1.972	2.475
	PB2	0.834	0.846	0.870	0.986	0.717	0.610	0.846	1.180	1.427
	BCaB1	0.810	0.865	0.875	0.639	0.433	0.349	1.267	1.998	2.504
	BCaB2	0.835	0.842	0.871	0.990	0.719	0.611	0.843	1.172	1.425

Continued

$p_0 = 0.4$ $p_1 = 0.6$ $\rho_1 = 0.63$ and $\rho_2 = 0.75$	CAN1	0.841	0.868	0.878	0.764	0.523	0.431	**1.101**	**1.659**	**2.037**
	CAN2	0.851	0.863	0.875	0.837	0.600	0.505	1.016	1.439	1.732
	SB1	0.891	0.888	0.890	0.944	0.572	0.455	0.944	1.552	1.955
	SB2	0.850	0.859	0.875	0.828	0.596	0.503	1.026	1.440	1.739
	BB1	0.849	0.871	0.873	0.790	0.526	0.431	1.075	1.657	2.025
	BB2	0.832	0.849	0.864	0.781	0.580	0.494	**1.066**	**1.465**	1.748
	PB1	0.810	0.849	0.863	0.864	0.551	0.444	0.937	1.540	1.943
	PB2	0.843	0.854	0.881	0.813	0.589	0.498	1.037	1.449	**1.767**
	BCaB1	0.817	0.853	0.862	0.840	0.544	0.441	0.973	1.569	1.957
	BCaB2	0.841	0.854	0.869	0.814	0.592	0.499	1.033	1.443	1.740
$p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.28$ and $\rho_2 = 0.06$	CAN1	0.853	0.887	0.896	0.334	0.231	0.193	**2.554**	3.841	4.639
	CAN2	0.831	0.873	0.862	0.063	0.045	0.037	13.279	19.278	23.249
	SB1	0.893	0.906	0.911	0.411	0.251	0.205	2.171	3.611	4.452
	SB2	0.828	0.869	0.859	0.062	0.045	0.037	13.405	19.311	23.232
	BB1	0.856	0.890	0.897	0.345	0.232	0.193	2.484	**3.844**	**4.645**
	BB2	0.815	0.861	0.861	0.058	0.044	0.036	**13.972**	**19.708**	**23.773**
	PB1	0.824	0.858	0.877	0.377	0.242	0.199	2.187	3.546	4.398
	PB2	0.838	0.872	0.861	0.061	0.044	0.037	13.825	19.620	23.521
	BCaB1	0.830	0.857	0.880	0.364	0.238	0.198	2.278	3.595	4.449
	BCaB2	0.837	0.873	0.862	0.061	0.045	0.037	13.803	19.574	23.512

Note that: 1) boldface denotes the greatest relative coverage among the five estimation approach; 2) Confidence intervals of ρ_1 under different estimation approaches are denoted by CAN1, SB1, BB1, PB1 and BCaB1; 3) Confidence intervals of ρ_2 under different estimation approaches are denoted by CAN2, SB2, BB2, PB2 and BCaB2.

Table 4. Simulation results of coverage percentage, average lengths, and relative coverage for 90% confidence intervals under queueing network. $M/H_4^{Pe}/1$ to $H_4^{Pe}/M/1$.

Intensity Parameters	Estimation Approach	Coverage Percentage			Average Length			Relative Coverage		
		$n = 10$	$n = 20$	$n = 29$	$n = 10$	$n = 20$	$n = 29$	$n = 10$	$n = 20$	$n = 29$
$p_0 = 0.2$ $p_1 = 0.8$ $\rho_1 = 0.5$ and $\rho_2 = 0.2$	CAN1	0.875	0.876	0.903	0.620	0.432	0.359	**1.412**	**2.026**	**2.514**
	CAN2	0.825	0.870	0.881	0.234	0.169	0.140	3.530	5.155	6.275
	SB1	0.910	0.896	0.916	0.752	0.472	0.380	1.210	1.900	2.413
	SB2	0.826	0.869	0.882	0.235	0.169	0.140	3.512	5.147	6.279
	BB1	0.876	0.875	0.899	0.635	0.435	0.359	1.379	2.011	2.502
	BB2	0.802	0.859	0.874	0.220	0.163	0.137	**3.649**	**5.256**	**6.359**
	PB1	0.845	0.874	0.895	0.688	0.454	0.371	1.228	1.923	2.414
	PB2	0.836	0.864	0.880	0.229	0.167	0.139	3.645	5.185	6.331
	BCaB1	0.841	0.875	0.891	0.669	0.448	0.367	1.257	1.951	2.427
	BCaB2	0.830	0.860	0.875	0.229	0.167	0.139	3.618	5.150	6.295

Continued

$p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.11$ and $\rho_2 = 0.01$	CAN1	0.887	0.900	0.885	0.140	0.097	0.080	**6.315**	**9.299**	11.041
	CAN2	0.824	0.866	0.884	0.006	0.005	0.004	127.283	186.510	225.524
	SB1	0.911	0.917	0.904	0.171	0.105	0.085	5.313	8.711	10.681
	SB2	0.824	0.863	0.885	0.006	0.005	0.004	127.040	185.644	225.329
	BB1	0.894	0.901	0.885	0.145	0.097	0.080	6.166	9.256	**11.057**
	BB2	0.805	0.855	0.879	0.006	0.005	0.004	**132.218**	**189.933**	229.041
	PB1	0.852	0.872	0.881	0.157	0.102	0.083	5.429	8.588	10.676
	PB2	0.815	0.851	0.890	0.006	0.005	0.004	128.530	185.849	**229.244**
	BCaB1	0.856	0.877	0.882	0.152	0.100	0.082	5.622	8.761	10.783
	BCaB2	0.812	0.860	0.889	0.006	0.005	0.004	127.997	187.752	228.755
$p_0 = 0.6$ $p_1 = 0.4$ $\rho_1 = 0.83$ and $\rho_2 = 0.17$	CAN1	0.871	0.883	0.876	1.082	0.730	0.599	**0.805**	**1.209**	1.463
	CAN2	0.834	0.873	0.897	0.199	0.143	0.117	4.186	6.097	7.688
	SB1	0.904	0.904	0.896	1.339	0.795	0.633	0.675	1.137	1.415
	SB2	0.831	0.872	0.896	0.200	0.144	0.117	4.156	6.075	7.668
	BB1	0.879	0.884	0.878	1.118	0.734	0.599	0.786	1.205	**1.467**
	BB2	0.816	0.863	0.889	0.187	0.139	0.114	**4.362**	**6.218**	**7.780**
	PB1	0.837	0.858	0.870	1.223	0.766	0.618	0.685	1.120	1.409
	PB2	0.824	0.869	0.895	0.195	0.142	0.116	4.219	6.140	7.746
	BCaB1	0.843	0.855	0.872	1.183	0.756	0.611	0.712	1.131	1.426
	BCaB2	0.816	0.866	0.890	0.196	0.142	0.116	4.172	6.110	7.690
$p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.56$ and $\rho_2 = 0.03$	CAN1	0.866	0.880	0.874	0.700	0.489	0.399	**1.238**	**1.800**	2.192
	CAN2	0.837	0.868	0.894	0.032	0.023	0.020	26.164	37.301	44.908
	SB1	0.897	0.900	0.896	0.855	0.532	0.421	1.049	1.691	2.126
	SB2	0.828	0.871	0.898	0.032	0.023	0.020	25.819	37.366	44.974
	BB1	0.870	0.881	0.880	0.720	0.492	0.398	1.209	1.790	**2.211**
	BB2	0.807	0.860	0.889	0.030	0.023	0.019	**26.877**	**38.147**	**45.675**
	PB1	0.844	0.868	0.864	0.784	0.513	0.411	1.076	1.693	2.103
	PB2	0.827	0.872	0.897	0.031	0.023	0.020	26.381	37.956	45.425
	BCaB1	0.845	0.868	0.865	0.763	0.506	0.407	1.107	1.716	2.123
	BCaB2	0.824	0.870	0.892	0.031	0.023	0.020	26.212	37.782	45.098

Continued

$p_0 = 0.1$ $p_1 = 0.9$ $\rho_1 = 0.5$ and $\rho_2 = 0.9$	CAN1	0.859	0.889	0.901	0.629	0.434	0.363	**1.366**	**2.046**	2.483
	CAN2	0.863	0.888	0.875	1.045	0.748	0.628	0.826	1.188	1.394
	SB1	0.908	0.911	0.918	0.776	0.473	0.384	1.170	1.927	2.391
	SB2	0.860	0.887	0.880	1.049	0.749	0.628	0.820	1.184	1.400
	BB1	0.861	0.890	0.903	0.649	0.437	0.363	1.326	2.037	**2.491**
	BB2	0.845	0.875	0.865	0.981	0.724	0.614	**0.861**	**1.208**	**1.409**
	PB1	0.831	0.866	0.895	0.706	0.456	0.375	1.178	1.899	2.390
	PB2	0.855	0.885	0.866	1.027	0.738	0.622	0.833	1.199	1.393
	BCaB1	0.839	0.865	0.892	0.687	0.451	0.371	1.221	1.919	2.404
	BCaB2	0.851	0.886	0.868	1.026	0.739	0.622	0.829	1.198	1.395
$p_0 = 0.4$ $p_1 = 0.6$ $\rho_1 = 0.63$ and $\rho_2 = 0.75$	CAN1	0.881	0.899	0.900	0.778	0.551	0.451	**1.133**	**1.632**	1.995
	CAN2	0.848	0.879	0.862	0.884	0.635	0.529	0.959	1.385	1.629
	SB1	0.909	0.920	0.913	0.954	0.601	0.478	0.953	1.532	1.912
	SB2	0.843	0.883	0.862	0.887	0.635	0.530	0.950	1.390	1.627
	BB1	0.881	0.900	0.903	0.804	0.554	0.451	1.096	1.624	**2.001**
	BB2	0.831	0.871	0.856	0.831	0.616	0.518	**1.000**	**1.415**	1.652
	PB1	0.849	0.877	0.891	0.873	0.579	0.466	0.972	1.515	1.913
	PB2	0.833	0.884	0.867	0.868	0.626	0.523	0.960	1.412	1.657
	BCaB1	0.848	0.888	0.886	0.850	0.570	0.461	0.997	1.558	1.920
	BCaB2	0.831	0.874	0.883	0.868	0.628	0.524	0.957	1.392	**1.685**
$p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.28$ and $\rho_2 = 0.06$	CAN1	0.859	0.876	0.891	0.347	0.242	0.200	**2.477**	**3.615**	**4.452**
	CAN2	0.849	0.848	0.872	0.067	0.047	0.039	12.746	18.094	22.514
	SB1	0.894	0.898	0.902	0.422	0.264	0.212	2.117	3.402	4.263
	SB2	0.851	0.845	0.871	0.067	0.047	0.039	12.716	18.010	22.461
	BB1	0.861	0.881	0.887	0.357	0.244	0.200	2.415	3.613	4.437
	BB2	0.831	0.845	0.867	0.063	0.045	0.038	**13.279**	**18.605**	22.872
	PB1	0.822	0.860	0.874	0.386	0.254	0.206	2.130	3.382	4.239
	PB2	0.848	0.849	0.880	0.065	0.046	0.038	12.964	18.347	**22.926**
	BCaB1	0.828	0.866	0.875	0.375	0.251	0.204	2.207	3.448	4.282
	BCaB2	0.849	0.849	0.882	0.066	0.046	0.038	12.959	18.325	22.919

Note that: 1) boldface denotes the greatest relative coverage among the five estimation approach; 2) Confidence intervals of ρ_1 under different estimation approaches are denoted by CAN1, SB1, BB1, PB1 and BCaB1; 3) Confidence intervals of ρ_2 under different estimation approaches are denoted by CAN2, SB2, BB2, PB2 and BCaB2.

Table 5. Simulation results of coverage percentage, average lengths, and relative coverage for 90% confidence intervals under queueing network: $E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$.

Intensity Parameters	Estimation Approach	Coverage Percentage			Average Length			Relative Coverage		
		$n = 10$	$n = 20$	$n = 29$	$n = 10$	$n = 20$	$n = 29$	$n = 10$	$n = 20$	$n = 29$
$p_0 = 0.2$ $p_1 = 0.8$ $\rho_1 = 0.5$ and $\rho_2 = 0.2$	CAN1	0.881	0.873	0.885	0.402	0.286	0.236	2.193	3.051	3.754
	CAN2	0.875	0.888	0.897	0.161	0.113	0.094	5.441	7.837	9.518
	SB1	0.873	0.868	0.886	0.400	0.285	0.235	2.183	3.042	3.777
	SB2	0.875	0.892	0.896	0.163	0.114	0.095	5.367	7.816	9.463
	BB1	0.851	0.861	0.879	0.375	0.277	0.230	**2.268**	**3.111**	3.817
	BB2	0.857	0.875	0.891	0.151	0.110	0.092	**5.678**	**7.987**	**9.696**
	PB1	0.849	0.870	0.889	0.392	0.282	0.232	2.164	3.082	**3.824**
	PB2	0.853	0.879	0.881	0.159	0.113	0.094	5.352	7.796	9.393
	BCaB1	0.849	0.868	0.882	0.392	0.282	0.232	2.167	3.077	3.794
	BCaB2	0.860	0.876	0.872	0.159	0.112	0.094	5.422	7.798	9.308
$p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.11$ and $\rho_2 = 0.01$	CAN1	0.871	0.878	0.893	0.090	0.063	0.053	9.722	13.893	16.935
	CAN2	0.856	0.891	0.888	0.004	0.003	0.003	195.200	281.820	338.563
	SB1	0.869	0.876	0.894	0.089	0.063	0.053	9.753	13.908	17.007
	SB2	0.859	0.889	0.889	0.004	0.003	0.003	193.152	279.335	337.997
	BB1	0.849	0.862	0.886	0.084	0.061	0.051	**10.148**	**14.137**	**17.206**
	BB2	0.834	0.875	0.881	0.004	0.003	0.003	**201.763**	**286.371**	**344.687**
	PB1	0.858	0.865	0.884	0.088	0.062	0.052	9.794	13.872	16.953
	PB2	0.848	0.891	0.878	0.004	0.003	0.003	195.057	283.504	336.941
	BCaB1	0.859	0.865	0.880	0.087	0.062	0.052	9.833	13.876	16.891
	BCaB2	0.842	0.882	0.877	0.004	0.003	0.003	194.536	281.382	337.204
$p_0 = 0.6$ $p_1 = 0.4$ $\rho_1 = 0.83$ and $\rho_2 = 0.17$	CAN1	0.878	0.874	0.870	0.661	0.471	0.395	1.329	1.855	2.203
	CAN2	0.849	0.882	0.889	0.132	0.094	0.080	6.432	9.341	11.167
	SB1	0.872	0.869	0.867	0.658	0.470	0.393	1.325	1.851	2.204
	SB2	0.849	0.882	0.891	0.134	0.095	0.080	6.356	9.282	11.156
	BB1	0.847	0.854	0.859	0.617	0.455	0.385	**1.372**	**1.875**	**2.230**
	BB2	0.822	0.864	0.879	0.124	0.091	0.078	**6.635**	**9.473**	**11.335**
	PB1	0.865	0.866	0.869	0.646	0.464	0.390	1.338	1.866	2.229
	PB2	0.843	0.869	0.883	0.131	0.094	0.079	6.454	9.279	11.154
	BCaB1	0.868	0.864	0.865	0.645	0.464	0.390	1.345	1.862	2.217
	BCaB2	0.839	0.868	0.875	0.130	0.093	0.079	6.464	9.291	11.078

Continued

$p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.56$ and $\rho_2 = 0.03$	CAN1	0.858	0.875	0.891	0.452	0.320	0.263	1.897	2.730	3.390
	CAN2	0.870	0.884	0.891	0.022	0.016	0.013	39.305	55.479	67.767
	SB1	0.858	0.874	0.890	0.450	0.320	0.262	1.906	2.731	3.397
	SB2	0.870	0.882	0.896	0.022	0.016	0.013	38.755	54.933	67.862
	BB1	0.836	0.863	0.881	0.422	0.309	0.256	**1.983**	**2.790**	3.441
	BB2	0.836	0.879	0.881	0.021	0.015	0.013	**40.177**	**57.074**	**68.727**
	PB1	0.847	0.876	0.895	0.441	0.317	0.260	1.919	2.767	**3.445**
	PB2	0.847	0.878	0.887	0.022	0.016	0.013	38.630	55.413	67.828
	BCaB1	0.847	0.873	0.890	0.441	0.317	0.260	1.919	2.757	3.429
	BCaB2	0.843	0.878	0.883	0.022	0.016	0.013	38.676	55.614	67.700
$p_0 = 0.1$ $p_1 = 0.9$ $\rho_1 = 0.5$ and $\rho_2 = 0.9$	CAN1	0.853	0.882	0.886	0.396	0.282	0.237	2.156	3.127	3.738
	CAN2	0.879	0.895	0.877	0.723	0.504	0.425	1.216	1.777	2.064
	SB1	0.852	0.880	0.889	0.393	0.281	0.236	2.167	3.129	3.761
	SB2	0.882	0.897	0.875	0.736	0.506	0.426	1.199	1.771	2.055
	BB1	0.829	0.866	0.877	0.369	0.272	0.231	**2.247**	**3.184**	**3.795**
	BB2	0.863	0.885	0.868	0.681	0.486	0.414	**1.268**	**1.822**	**2.095**
	PB1	0.850	0.869	0.887	0.386	0.278	0.234	2.203	3.124	3.785
	PB2	0.869	0.901	0.857	0.718	0.501	0.421	1.211	1.799	2.033
	BCaB1	0.843	0.869	0.883	0.384	0.278	0.234	2.194	3.126	3.767
	BCaB2	0.869	0.898	0.859	0.713	0.499	0.421	1.219	1.799	2.043
$p_0 = 0.4$ $p_1 = 0.6$ $\rho_1 = 0.63$ and $\rho_2 = 0.75$	CAN1	0.858	0.890	0.891	0.503	0.358	0.295	1.707	2.485	3.018
	CAN2	0.871	0.857	0.891	0.595	0.427	0.354	1.463	2.005	2.517
	SB1	0.852	0.892	0.888	0.500	0.357	0.294	1.704	2.497	3.018
	SB2	0.875	0.858	0.897	0.605	0.430	0.355	1.447	1.995	2.525
	BB1	0.825	0.880	0.883	0.469	0.346	0.288	**1.759**	**2.546**	**3.066**
	BB2	0.849	0.845	0.883	0.559	0.412	0.345	**1.517**	**2.051**	**2.559**
	PB1	0.832	0.878	0.891	0.491	0.353	0.291	1.693	2.487	3.057
	PB2	0.848	0.854	0.884	0.591	0.425	0.352	1.436	2.010	2.514
	BCaB1	0.823	0.873	0.883	0.491	0.353	0.291	1.678	2.472	3.031
	BCaB2	0.853	0.848	0.881	0.586	0.424	0.351	1.455	2.001	2.510

Continued

p_0 = 0.9 p_1 = 0.1 ρ_1 = 0.28 and ρ_2 = 0.06	CAN1	0.878	0.880	0.874	0.227	0.157	0.131	3.875	5.588	6.657
	CAN2	0.862	0.904	0.899	0.045	0.032	0.026	19.163	28.082	34.053
	SB1	0.874	0.885	0.873	0.226	0.157	0.131	3.871	5.635	6.667
	SB2	0.864	0.903	0.897	0.046	0.032	0.027	18.942	27.925	33.823
	BB1	0.858	0.868	0.868	0.212	0.152	0.128	**4.056**	**5.710**	**6.781**
	BB2	0.841	0.892	0.892	0.042	0.031	0.026	**19.872**	**28.700**	**34.691**
	PB1	0.864	0.869	0.870	0.222	0.155	0.130	3.898	5.595	6.704
	PB2	0.840	0.880	0.881	0.045	0.032	0.026	18.840	27.545	33.535
	BCaB1	0.862	0.874	0.866	0.221	0.155	0.130	3.909	5.637	6.681
	BCaB2	0.839	0.880	0.887	0.044	0.032	0.026	18.928	27.640	33.815

Note that: 1) boldface denotes the greatest relative coverage among the five estimation approach; 2) Confidence intervals of ρ_1 under different estimation approaches are denoted by CAN1, SB1, BB1, PB1 and BCaB1; 3) Confidence intervals of ρ_2 under different estimation approaches are denoted by CAN2, SB2, BB2, PB2 and BCaB2.

Table 6. Simulation results of coverage percentage, average lengths, and relative coverage for 90% confidence intervals under queueing network: $E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$.

Intensity Parameters	Estimation Approach	Coverage Percentage			Average Length			Relative Coverage		
		n = 10	n = 20	n = 29	n = 10	n = 20	n = 29	n = 10	n = 20	n = 29
p_0 = 0.2 p_1 = 0.8 ρ_1 = 0.5 and ρ_2 = 0.2	CAN1	0.868	0.889	0.892	0.400	0.283	0.236	2.173	3.140	3.775
	CAN2	0.863	0.883	0.877	0.159	0.115	0.094	5.421	7.646	9.316
	SB1	0.865	0.888	0.895	0.397	0.282	0.236	2.179	3.149	3.796
	SB2	0.866	0.885	0.878	0.162	0.116	0.094	5.355	7.611	9.300
	BB1	0.845	0.883	0.883	0.373	0.273	0.230	**2.268**	**3.240**	**3.835**
	BB2	0.843	0.871	0.871	0.150	0.112	0.092	**5.632**	**7.805**	**9.498**
	PB1	0.846	0.884	0.891	0.390	0.279	0.234	2.171	3.169	3.812
	PB2	0.850	0.875	0.866	0.158	0.115	0.094	5.379	7.626	9.256
	BCaB1	0.847	0.874	0.889	0.388	0.279	0.234	2.181	3.138	3.806
	BCaB2	0.848	0.872	0.867	0.157	0.114	0.093	5.388	7.632	9.285
p_0 = 0.9 p_1 = 0.1 ρ_1 = 0.11 and ρ_2 = 0.01	CAN1	0.873	0.900	0.894	0.089	0.063	0.052	9.794	14.320	17.061
	CAN2	0.892	0.866	0.885	0.004	0.003	0.003	199.102	275.616	336.933
	SB1	0.874	0.899	0.891	0.089	0.063	0.052	9.846	14.339	17.045
	SB2	0.893	0.868	0.888	0.005	0.003	0.003	196.801	274.561	336.762
	BB1	0.849	0.886	0.881	0.083	0.061	0.051	**10.189**	**14.627**	**17.223**
	BB2	0.873	0.859	0.874	0.004	0.003	0.003	**206.607**	**283.242**	341.123
	PB1	0.865	0.897	0.889	0.087	0.062	0.052	9.934	14.486	17.162
	PB2	0.874	0.858	0.879	0.004	0.003	0.003	197.039	274.871	337.131
	BCaB1	0.863	0.897	0.885	0.087	0.062	0.052	9.929	14.497	17.097
	BCaB2	0.872	0.859	0.888	0.004	0.003	0.003	197.877	275.862	**341.255**

Continued

$p_0 = 0.6$ $p_1 = 0.4$ $\rho_1 = 0.83$ and $\rho_2 = 0.17$	CAN1	0.870	0.888	0.870	0.661	0.473	0.391	1.316	1.878	2.226
	CAN2	0.852	0.890	0.894	0.134	0.095	0.080	6.378	9.414	11.213
	SB1	0.867	0.884	0.871	0.658	0.471	0.390	1.318	1.875	2.236
	SB2	0.859	0.891	0.893	0.135	0.095	0.080	6.347	9.388	11.170
	BB1	0.851	0.872	0.864	0.618	0.456	0.381	**1.378**	**1.912**	**2.267**
	BB2	0.835	0.875	0.887	0.126	0.091	0.078	**6.651**	**9.588**	**11.401**
	PB1	0.863	0.886	0.868	0.646	0.466	0.386	1.336	1.901	2.247
	PB2	0.843	0.872	0.888	0.132	0.094	0.079	6.369	9.293	11.214
	BCaB1	0.862	0.885	0.863	0.645	0.466	0.386	1.336	1.900	2.236
	BCaB2	0.837	0.870	0.885	0.132	0.094	0.079	6.362	9.302	11.196
$p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.56$ and $\rho_2 = 0.03$	CAN1	0.868	0.892	0.881	0.451	0.313	0.260	1.924	2.848	3.389
	CAN2	0.855	0.889	0.888	0.022	0.016	0.013	38.912	56.159	67.891
	SB1	0.868	0.890	0.882	0.449	0.312	0.259	1.932	2.850	3.400
	SB2	0.855	0.886	0.889	0.022	0.016	0.013	38.389	55.660	67.803
	BB1	0.846	0.873	0.872	0.421	0.303	0.254	**2.009**	**2.885**	**3.438**
	BB2	0.837	0.871	0.881	0.021	0.015	0.013	**40.525**	**56.991**	**69.056**
	PB1	0.852	0.885	0.871	0.441	0.309	0.257	1.931	2.868	3.388
	PB2	0.841	0.891	0.884	0.022	0.016	0.013	38.677	56.728	68.080
	BCaB1	0.853	0.878	0.872	0.440	0.308	0.257	1.940	2.846	3.390
	BCaB2	0.839	0.892	0.886	0.022	0.016	0.013	38.827	56.980	68.421
$p_0 = 0.1$ $p_1 = 0.9$ $\rho_1 = 0.5$ and $\rho_2 = 0.9$	CAN1	0.856	0.889	0.896	0.393	0.283	0.239	2.178	3.137	3.754
	CAN2	0.866	0.895	0.888	0.727	0.508	0.427	1.191	1.763	2.081
	SB1	0.851	0.891	0.896	0.391	0.283	0.238	2.174	3.152	3.759
	SB2	0.862	0.889	0.889	0.738	0.510	0.428	1.168	1.742	2.077
	BB1	0.829	0.878	0.886	0.367	0.274	0.232	**2.259**	**3.206**	**3.816**
	BB2	0.838	0.882	0.882	0.684	0.490	0.416	**1.225**	**1.798**	**2.121**
	PB1	0.851	0.888	0.882	0.385	0.279	0.236	2.213	3.179	3.731
	PB2	0.836	0.876	0.874	0.722	0.504	0.424	1.158	1.739	2.064
	BCaB1	0.848	0.881	0.885	0.384	0.279	0.236	2.210	3.155	3.746
	BCaB2	0.840	0.875	0.880	0.716	0.502	0.423	1.173	1.742	2.083

Continued

p_0 = 0.4 p_1 = 0.6 ρ_1 = 0.63 and ρ_2 = 0.75	CAN1	0.864	0.901	0.904	0.510	0.362	0.294	1.694	2.486	3.076
	CAN2	0.878	0.889	0.896	0.607	0.424	0.351	1.447	2.096	2.551
	SB1	0.860	0.897	0.898	0.508	0.361	0.293	1.692	2.482	3.064
	SB2	0.879	0.889	0.898	0.618	0.426	0.353	1.423	2.086	2.545
	BB1	0.844	0.884	0.894	0.477	0.350	0.286	**1.769**	**2.526**	**3.121**
	BB2	0.863	0.876	0.885	0.571	0.410	0.342	**1.511**	**2.139**	**2.586**
	PB1	0.841	0.883	0.891	0.498	0.357	0.291	1.688	2.472	3.066
	PB2	0.857	0.882	0.885	0.602	0.421	0.350	1.424	2.096	2.530
	BCaB1	0.850	0.880	0.895	0.497	0.357	0.291	1.711	2.465	3.078
	BCaB2	0.854	0.876	0.886	0.599	0.420	0.349	1.426	2.084	2.541
p_0 = 0.9 p_1 = 0.1 ρ_1 = 0.28 and ρ_2 = 0.06	CAN1	0.867	0.880	0.884	0.216	0.159	0.130	4.009	5.550	6.780
	CAN2	0.897	0.894	0.893	0.045	0.031	0.027	20.111	28.392	33.686
	SB1	0.860	0.879	0.885	0.215	0.158	0.130	4.005	5.556	6.806
	SB2	0.896	0.891	0.897	0.045	0.032	0.027	19.852	28.156	33.765
	BB1	0.846	0.865	0.878	0.202	0.153	0.127	**4.193**	**5.647**	**6.907**
	BB2	0.875	0.880	0.880	0.042	0.030	0.026	**20.922**	**28.966**	**34.057**
	PB1	0.850	0.876	0.888	0.211	0.156	0.129	4.029	5.600	6.893
	PB2	0.874	0.879	0.896	0.044	0.031	0.026	19.771	28.134	34.043
	BCaB1	0.849	0.871	0.883	0.211	0.156	0.129	4.033	5.575	6.860
	BCaB2	0.874	0.873	0.893	0.044	0.031	0.026	19.880	28.019	34.021

Note that: 1) boldface denotes the greatest relative coverage among the five estimation approach; 2) Confidence intervals of ρ_1 under different estimation approaches are denoted by CAN1, SB1, BB1, PB1 and BCaB1; 3) Confidence intervals of ρ_2 under different estimation approaches are denoted by CAN2, SB2, BB2, PB2 and BCaB2.

Table 7. Simulation results of coverage percentage, average lengths, and relative coverage for 90% confidence intervals under queueing network: $H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$.

Intensity Parameters	Estimation Approach	Coverage Percentage			Average Length			Relative Coverage		
		n = 10	n = 20	n = 29	n = 10	n = 20	n = 29	n = 10	n = 20	n = 29
p_0 = 0.2 p_1 = 0.8 ρ_1 = 0.5 and ρ_2 = 0.2	CAN1	0.858	0.858	0.901	0.430	0.306	0.256	1.993	2.799	3.518
	CAN2	0.864	0.870	0.895	0.174	0.122	0.102	4.977	7.119	8.803
	SB1	0.863	0.864	0.905	0.436	0.308	0.256	1.980	2.808	3.529
	SB2	0.869	0.872	0.899	0.175	0.123	0.102	4.952	7.101	8.821
	BB1	0.839	0.845	0.897	0.405	0.296	0.250	**2.071**	**2.855**	**3.592**
	BB2	0.841	0.857	0.886	0.163	0.118	0.099	**5.147**	**7.256**	**8.922**
	PB1	0.860	0.854	0.892	0.426	0.304	0.254	2.017	2.813	3.513
	PB2	0.850	0.869	0.882	0.171	0.121	0.101	4.960	7.176	8.738
	BCaB1	0.859	0.850	0.886	0.424	0.303	0.253	2.025	2.809	3.495
	BCaB2	0.846	0.868	0.881	0.170	0.121	0.101	4.966	7.180	8.740

Continued

$p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.11$ and $\rho_2 = 0.01$	CAN1	0.859	0.869	0.903	0.096	0.069	0.057	8.959	12.609	15.900
	CAN2	0.863	0.875	0.899	0.005	0.003	0.003	178.166	256.808	313.751
	SB1	0.861	0.872	0.902	0.097	0.069	0.057	8.882	12.572	15.840
	SB2	0.871	0.874	0.898	0.005	0.003	0.003	177.672	255.224	312.522
	BB1	0.840	0.857	0.896	0.090	0.067	0.055	**9.318**	**12.869**	**16.171**
	BB2	0.844	0.865	0.895	0.005	0.003	0.003	**185.562**	**262.597**	**320.580**
	PB1	0.840	0.857	0.906	0.095	0.068	0.056	8.868	12.521	16.068
	PB2	0.852	0.868	0.899	0.005	0.003	0.003	177.949	256.892	316.192
	BCaB1	0.844	0.862	0.902	0.094	0.068	0.056	8.948	12.625	16.029
	BCaB2	0.850	0.861	0.899	0.005	0.003	0.003	178.207	255.533	316.061
$p_0 = 0.6$ $p_1 = 0.4$ $\rho_1 = 0.83$ and $\rho_2 = 0.17$	CAN1	0.857	0.877	0.892	0.727	0.512	0.430	1.178	1.712	2.076
	CAN2	0.879	0.892	0.904	0.143	0.102	0.084	6.127	8.703	10.757
	SB1	0.864	0.878	0.894	0.737	0.513	0.431	1.173	1.710	2.073
	SB2	0.879	0.892	0.907	0.145	0.103	0.084	6.058	8.655	10.759
	BB1	0.842	0.865	0.886	0.684	0.495	0.419	**1.230**	**1.747**	**2.115**
	BB2	0.852	0.883	0.897	0.135	0.099	0.082	**6.312**	**8.894**	10.924
	PB1	0.849	0.859	0.883	0.720	0.507	0.427	1.179	1.696	2.067
	PB2	0.855	0.885	0.911	0.142	0.102	0.083	6.031	8.703	10.910
	BCaB1	0.840	0.860	0.882	0.716	0.505	0.426	1.173	1.702	2.069
	BCaB2	0.848	0.879	0.911	0.141	0.101	0.083	6.012	8.661	**10.941**
$p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.56$ and $\rho_2 = 0.03$	CAN1	0.867	0.883	0.881	0.475	0.340	0.281	1.826	2.598	3.138
	CAN2	0.868	0.883	0.874	0.024	0.017	0.014	36.033	52.147	62.267
	SB1	0.865	0.886	0.885	0.482	0.342	0.281	1.795	2.593	3.147
	SB2	0.869	0.887	0.871	0.024	0.017	0.014	35.563	52.065	61.943
	BB1	0.847	0.871	0.872	0.447	0.329	0.274	**1.896**	**2.650**	**3.186**
	BB2	0.843	0.871	0.864	0.023	0.016	0.014	**37.101**	**53.150**	**63.192**
	PB1	0.863	0.870	0.884	0.471	0.337	0.278	1.833	2.579	3.176
	PB2	0.857	0.878	0.872	0.024	0.017	0.014	35.875	52.258	62.681
	BCaB1	0.856	0.873	0.876	0.468	0.336	0.278	1.831	2.598	3.153
	BCaB2	0.860	0.874	0.866	0.024	0.017	0.014	36.209	52.085	62.304

Continued

$p_0 = 0.1$ $p_1 = 0.9$ $\rho_1 = 0.5$ and $\rho_2 = 0.9$	CAN1	0.864	0.884	0.892	0.423	0.309	0.254	2.044	2.857	3.514
	CAN2	0.881	0.899	0.872	0.785	0.552	0.459	1.122	1.628	1.901
	SB1	0.869	0.885	0.892	0.427	0.311	0.255	2.036	2.848	3.503
	SB2	0.884	0.897	0.879	0.793	0.555	0.459	1.115	1.617	1.913
	BB1	0.844	0.869	0.882	0.397	0.299	0.248	**2.128**	**2.905**	**3.558**
	BB2	0.864	0.886	0.863	0.739	0.534	0.448	**1.170**	**1.660**	**1.924**
	PB1	0.878	0.864	0.891	0.417	0.307	0.252	2.104	2.817	3.536
	PB2	0.859	0.888	0.863	0.775	0.548	0.455	1.108	1.621	1.897
	BCaB1	0.870	0.863	0.887	0.414	0.306	0.251	2.100	2.822	3.527
	BCaB2	0.852	0.887	0.865	0.770	0.546	0.455	1.106	1.624	1.903
$p_0 = 0.4$ $p_1 = 0.6$ $\rho_1 = 0.63$ and $\rho_2 = 0.75$	CAN1	0.861	0.878	0.894	0.541	0.382	0.319	1.592	2.297	2.803
	CAN2	0.861	0.872	0.898	0.639	0.460	0.385	1.347	1.896	2.334
	SB1	0.864	0.880	0.896	0.548	0.384	0.320	1.575	2.293	2.804
	SB2	0.867	0.867	0.899	0.647	0.462	0.386	1.340	1.877	2.332
	BB1	0.842	0.870	0.888	0.509	0.369	0.311	**1.656**	**2.359**	**2.853**
	BB2	0.837	0.858	0.892	0.601	0.444	0.375	**1.393**	**1.931**	**2.378**
	PB1	0.851	0.878	0.882	0.536	0.379	0.316	1.588	2.318	2.790
	PB2	0.853	0.861	0.893	0.633	0.455	0.382	1.347	1.890	2.340
	BCaB1	0.848	0.878	0.882	0.532	0.378	0.315	1.593	2.323	2.797
	BCaB2	0.858	0.864	0.892	0.631	0.454	0.381	1.360	1.901	2.343
$p_0 = 0.9$ $p_1 = 0.1$ $\rho_1 = 0.28$ and $\rho_2 = 0.06$	CAN1	0.866	0.887	0.879	0.239	0.169	0.142	3.623	5.246	6.207
	CAN2	0.866	0.876	0.903	0.048	0.034	0.028	17.862	25.724	32.167
	SB1	0.870	0.885	0.877	0.242	0.170	0.142	3.598	5.210	6.180
	SB2	0.872	0.877	0.901	0.049	0.034	0.028	17.762	25.637	32.002
	BB1	0.849	0.879	0.870	0.225	0.163	0.138	**3.768**	**5.380**	**6.297**
	BB2	0.852	0.871	0.891	0.046	0.033	0.027	**18.679**	**26.465**	**32.544**
	PB1	0.849	0.880	0.873	0.236	0.168	0.140	3.590	5.248	6.215
	PB2	0.855	0.876	0.905	0.048	0.034	0.028	17.833	25.945	32.452
	BCaB1	0.848	0.875	0.874	0.235	0.167	0.140	3.601	5.230	6.225
	BCaB2	0.852	0.877	0.896	0.048	0.034	0.028	17.850	26.028	32.199

Note that: 1) boldface denotes the greatest relative coverage among the five estimation approach; 2) Confidence intervals of ρ_1 under different estimation approaches are denoted by CAN1, SB1, BB1, PB1 and BCaB1; 3) Confidence intervals of ρ_2 under different estimation approaches are denoted by CAN2, SB2, BB2, PB2 and BCaB2.

are decreasing as p_0 approaches 1 (p_1 approaches 0) but both coverage percentages and relative coverage are increasing as p_0 approaches 1 (p_1 approaches 0).

Also we find that average lengths are decreasing with sample size n but both coverage percentages and relative coverage are increasing with sample size n. From **Tables 3** to **7** one can observe that the coverage percentage can approach to 90% when n increases up to 29.

1) In queueing network models $M/E_4/1$ to $E_4/M/1$ and $M/H_4^{Pe}/1$ to $H_4^{Pe}/M/1$, the estimation approach CAN and BB has the greatest relative coverage among the five confidence intervals for ρ_1 and ρ_2 respectively for different values of p_0 and p_1.

2) In queueing network models $E_4/H_4^{Pe}/1$ to $H_4^{Pe}/E_4/1$, $E_4/H_4^{Po}/1$ to $H_4^{Po}/E_4/1$ and $H_4^{Pe}/H_4^{Po}/1$ to $H_4^{Po}/H_4^{Pe}/1$, the estimation approach BB has the greatest relative coverage among the five confidence intervals for ρ_1 and ρ_2 for different values of p_0 and p_1.

3) Average lengths are decreasing as sample size n increases for both ρ_1 and ρ_2. Also relative coverage increases with n for ρ_1 and ρ_2.

4) Some poor coverage percentage of above confidence intervals with respect to the nominal level 90% may be due to small sample size n.

8. Conclusion

This paper provides the interval estimations of intensities ρ_1 and ρ_2 for an open queueing network with feedback. Different estimation approaches CAN, SB, BB, PB and BCaB are applied to produce confidence intervals for intensities ρ_1 and ρ_2. The relative coverage is adopted to understand, compare and assess performance of the resulted confidence intervals. From simulation study it is clear that CAN and BB method has the best performance on interval estimation of intensities ρ_1 and ρ_2 for $M/G/1$ to $M/G/1$ queueing network and BB method has the best performance on interval estimation of intensities ρ_1 and ρ_2 for $G/G/1$ to $G/G/1$ queueing network with short run data. And approach is easily applied to practical queueing network system such as all types of open, closed, mixed queueing networks as well as cyclic, retrial queueing models.

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