COMPARISON OF EXCLUSION VALUES FOR LUMBER STRENGTH

W. G. Warren

Department of the Environment, Canadian Forestry Service,
Western Forest Products Laboratory, Vancouver, British Columbia, V6T 1X2

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ABSTRACT

There exists a temptation to utilize the distribution-free methods of ASTM D2915-70T for the comparison of the strengths of lumber populations, a use outside the intent of the standard. It is shown that the precision of such procedure is highly dependent on the form of the parent distribution of strength. Since this form is in general unknown, the confidence to be ascribed to inferences concerning near-minimum values of two lumber populations is also unknown and may well be inadequate.

Additional keywords: statistical analysis, models.

INTRODUCTION

It is well recognized that the mean strength of a population of lumber is of secondary importance to some "near-minimum" value, commonly the so-called 5% exclusion limit, from which allowable stresses are derived. Two populations may, of course, be equal in mean but differ in the fifth percentile; conversely, two populations may be equal in the fifth percentile but differ in mean. Thus the comparison of the means of two populations of lumber does not necessarily afford information concerning their relative merit as a structural material. It therefore becomes pertinent to consider the precision of comparisons between exclusion-limit estimates. A well-defined theory exists for the comparison of means; comparable theory for the case of exclusion limits appears to be lacking.

ASTM Standard D2915-70T, in taking a distribution-free approach, provides for the use of the interpolated fifth percentile from a sample of size 58, provided this value is no greater than 1.05 times the first-order statistic, $x_{1,58}$, (i.e. the lowest of the 58 values). If this is not the case, then 1.05 $x_{1,58}$ may be used. In effect, this is saying that the interpolated fifth percentile can be used, provided the associated confidence interval is sufficiently narrow; if this is not the case, a conservative value is to be taken. In practice, it appears that the latter course will almost always be called for. Accordingly, there is a temptation to use the first-order statistic of a sample of size 58 for comparison of exclusion limits of different populations. This, however, is outside the intent of ASTM D2915-70T and, as will be shown below, is a potentially misleading procedure.

It must be realized that $x_{1,58}$ is not, in the usual sense, an estimate of the fifth percentile. Its use arises as follows: let $x_{0.05}$ denote the true, but unknown, fifth percentile. Under a distribution-free approach, we wish to select an order statistic $x_{i,n}$ such that

$$P(x_{i,n} \leqslant x_{0.05}) \geqslant 0.95 \qquad (1)$$

i.e. such that the probability that $x_{i,n}$ is, at most, the fifth percentile is at least 95%. It turns out that the smallest sample size with which this can be achieved is $n = 58$, and with this the first-order statistic, $i = 1$, must be used. The real meaning of the above is that, if we follow this procedure, then about 95% of the time the value chosen will not exceed the true fifth percentile. How close it is to the true fifth percentile is not specified, nor is the scatter that will arise in repeated samplings of the same population. It is, in fact, what is called a lower 95% content tolerance limit with 95% confidence coefficient.

Unfortunately, the appearance of the term "95% confidence" seems to have given some people the impression that a reasonably high level of precision is associated with the value, as a measure of the fifth percentile. In reality the precision of the first-order statistic as the estimate of anything is not a factor in the definition. The first-order statistic may well be serviceable as a near-minimum value for use with a single population because, as noted, the value will exceed the true fifth percentile on the average, only one time in twenty. But this property has little relevance to the question of whether one population is "better" than another as regards their near-minimum values.

If the first-order statistic is subject to a relatively large but unrecognized sampling error, then there is a danger that, on the basis of this statistic, one population would be judged as "better" than another when there was, in fact, negligible difference or even a real difference in the other direction. The possibility of such an event cannot be denied; whether, however, it is of practical significance is open to question. In this paper, therefore, the relevant properties of the first-order statistic will be investigated quantitatively, under assumptions that appear to be reasonably realistic for lumber strength distributions. The arguments are highly technical and should not concern the general reader; it is necessary that they be outlined to provide the basis for the results obtained and the conclusions consequent thereon. In this way some feeling can be gained for the consequences of using the ASTM distribution-free procedure to compare populations.

ANALYSIS

To obtain a more concrete picture of the situation, it is necessary to determine the properties, in particular the mean and variance (or standard deviation), of the first-order statistic for several assumed parent distributions. This, in fact, has been attempted by Habermann (1973) by Monte Carlo methods for a few specific cases. Our approach differs in that it is analytic, although we shall concentrate on the same distributional forms as Habermann, namely the normal, log normal, Weibull and gamma, all of which have been used to model lumber strength distributions.

For convenience, we shall suppose, in all cases, that the mean and variance of the parent distribution are $\mu = 100$ and $\sigma^2 = 400$, respectively, (i.e. coefficient of variation = 20%). Results for other values can be obtained by appropriate rescaling.

Normal

First, let us suppose the parent population to be normally distributed and, as a benchmark, consider the parametric version of the lower 95% content tolerance limit with 95% confidence coefficient. This is given by $x_l = \bar{x} - ks$, where $\bar{x}$ and s are the usual sample estimates of mean and standard deviation and for sample size $n = 58$, k takes the value 2.031 (see, e.g. the tables of Owen 1962). According to Jennett and Welch (1939), x_l is distributed, approximately, as a normal variate with mean μ-$ka\sigma$ and variance $[1/n + b^2k^2/2(n-1)]\sigma^2$ where, for $n = 58$, $a \simeq 0.996$, $b^2 \simeq 0.996$. The mean of x_l is, thus,

$$E(x_\ell) = E(\bar{x} - ks) = \mu - ka\sigma$$

$$= 100 - 2.031 \times 0.996 \times 20$$

$$= 59.54 \qquad (2)$$

(which is approximately the 2.15 percentile), and the variance of x_l is

$$\mathrm{Var}(x_\ell) = \mathrm{Var}(\bar{x} - ks) = \left[\frac{1}{n} + \frac{b^2k^2}{2(n-1)}\right]\sigma^2$$

$$= \left[\frac{1}{58} + \frac{0.996 \times 2.031^2}{2 \times 57}\right] \times 400$$

$$= 21.3122 \left(= 4.6165^2\right) . \qquad (3)$$

In other words, the mean and standard deviation of the parametric tolerance limit are 59.54 and 4.6165, respectively, and the coefficient of variation is $4.6165/59.54 = 7.8\%$.

Because of the central role of the normal distribution in statistical theory, the prop-

TABLE 1. *Properties of the first-order statistic for a Weibull parent distribution* ($n = 58$)

k	$E(x_{1,58})$	$Var(x_{1,58})$	C.V.	Percentile	Actual 5th Percentile
2.0	66.86	$7.4200(=2.7240^2)$	4.1	1.40	71.52
2.5	62.59	$17.8746(=4.2278^2)$	6.8	1.31	69.32
3.0	59.30	$31.5255(=5.6148^2)$	9.5	1.25	67.87
3.508772	56.68	$47.0784(=6.8614^2)$	12.1	1.21	66.85
4.0	54.65	$62.5731(=7.9103^2)$	14.5	1.18	66.14
4.4	53.12	$76.4287(=8.7424^2)$	16.5	1.15	65.65
5.0	51.54	$93.0908(=9.6484^2)$	18.7	1.13	65.19

erties of order statistics in this case have been widely studied. Thus, either directly from published tables (e.g. Harter 1960) or by extrapolation, we have that the mean and variance of the first-order statistic of a standard normal variable, sample size n = 58, are $E(Z_{1,58}) = -2.30635$ and $Var(Z_{1,58}) = 0.20819$, respectively. With x normally distributed with mean μ and variance σ^2, it follows that

$$E(x_{1,58}) = \mu - 2.30635\sigma = 53.87$$

(which is approximately the 1.06 percentile)

and $Var(x_{1,58}) = 0.20819\sigma^2 = 83.2760(= 9.1256^2)$. That is, the mean and standard deviation of the distribution-free tolerance limit are 53.87 and 9.1256, respectively, and the coefficient of variation is 9.1256/53.87 = 16.9%. The actual value of the fifth percentile is 66.70.

Log normal

Suppose now that the strength distribution is log normal, but with mean and variance still 100 and 400, respectively. Then, by the Taylor series expansion for the expectation of functions of random variables (details from author on request) it can be shown that $E(x_{1,58}) = 71.14$ (which is approximately the 1.09 percentile) and $Var(x_{1,58}) = 20.5770(= 4.5362^2)$. That is, for the log normal, the distribution-free tolerance limit has mean 71.14, standard deviation 4.5362, and coefficient of variation 4.5362/71.14 = 6.4%. The actual value of the fifth percentile is 77.89.

Weibull

In contrast to the cases considered above, the Weibull distribution, in its general form, has three parameters—namely the location, scale, and shape. We shall write the density function as

$$f(x) = \frac{k(x-c)^{k-1}}{d^k} \exp\left[-\left(\frac{x-c}{d}\right)^k\right] \qquad (4)$$

where the location, scale, and shape parameters are c, d, and k, respectively. This distribution has mean

$$\mu = c + d\Gamma(1 + 1/k)$$

and variance

$$\sigma^2 = d^2[\Gamma(1 + 2/k) - \Gamma^2(1 + 1/k)].$$

We must, therefore, be given more than the mean and variance in order to determine the parameters. The most convenient approach would seem to be to select values for the shape parameter, k, and solve for the location and scale parameters. For example with k = 2 we have

$$100 = c + d\Gamma(1.5)$$
$$400 = d^2[\Gamma(2) - \Gamma^2(1.5)],$$

whence $c = 61.7388$

$$d = 43.1731.$$

Explicit formulae for the moments of the reduced log-Weibull distribution, and fairly extensive tables of the mean and variance, have been given by White (1967) and, from there, approximate values for the case in question can be derived by essentially the

TABLE 2. *Properties of the first-order statistic for a gamma parent distribution* ($n = 58$)

k	$E(x_{1,58})$	$Var(x_{1,58})$	C.V.	Percentile	Actual 5th Percentile
2	74.22	$2.1313(=1.4599^2)$	2.0	1.39	76.74
3	70.90	$5.3516(=2.3134^2)$	3.3	1.29	74.80
4	68.80	$8.5089(=2.9170^2)$	4.2	1.25	73.66
5	67.31	$11.3184(=3.3643^2)$	5.0	1.22	72.90

same techniques as used with the lognormal. For $k = 2$, we obtain

$$E(x_{1,58}) = 66.86$$

(which is approximately the 1.40 percentile)

and $\quad Var(x_{1,58}) = 7.4200(= 2.7240^2)$.

That is, the distribution-free tolerance limit has mean 66.86, standard deviation 2.7240, and coefficient of variation $2.7240/66.86 = 4.1\%$.

The process can be repeated for any value of k and in this way Table 1 has been obtained.

Gamma

The general form of the gamma distribution also has three parameters, which again may be identified as location, scale, and shape. We shall write the density function as

$$f(x) = \frac{\lambda^k (x - c)^{k-1}}{\Gamma(k)} \exp[-\lambda(x - c)]. \quad (5)$$

The mean is $\quad \mu = c + k/\lambda$

and the variance is

$$\sigma^2 = k/\lambda^2.$$

As for the Weibull, we may select values for k and solve for c and λ. For example, with $k = 2$

$$100 = c + 2/\lambda$$
$$400 = 2/\lambda^2,$$

whence $c = 71.7157, \quad \lambda = 0.070710.$

Explicit formulae for the moments of the order statistics for the standard gamma ($c = 0$, $\lambda = 1$, k a positive integer) and tables of the mean and variance of the first-order statistic for $n = 1(1)15$ have been given by Gupta (1960). Evaluation for larger values of n is demanding, and extrapolation to as far as $n = 58$, clearly precarious. Accordingly, the approximation formulae of Blom (1958) have been specialized for use here. The results are presented in Table 2.

IMPLICATIONS

The σ/μ ratio used in the above examples differs somewhat from that employed by Habermann (1973); nevertheless, the results are reasonably consistent with those of his Monte Carlo study. Although not stated in his text, in the case of the Weibull and of the gamma, he took the location parameter as zero. For example, with this constraint applied to the Weibull with $\mu = 100$ and $\sigma^2 = 400$, the value of k would be somewhat greater than 5; thus the relatively high variance of the first-order statistic obtained by Habermann is not surprising. A parallel argument can be applied to the gamma.

As has been demonstrated for the normal distribution, and is, of course, true in general, the distribution-free method yields, on the average, a more conservative value than the appropriate parametric method. More importantly, it is clear from the above that the properties of the first-order statistic not only are highly dependent on the form of the parent distribution, but also are very sensitive to changes in the shape parameter within a single family of distributions. It can also be shown that, although within a family the variance of the first-order statistic is a decreasing function of the degree of skewness of the distribution, equal values of skewness do not correspond to equal precisions between families.

In other words, unless the shape of the parent distribution is known, or can be safely assumed—and if this were the case one would try to avoid distribution-free methods—the precision of the distribution-free tolerance limit is unknown. For the cases studied, the coefficients of variation range from 2.0% (for the gamma with $k = 2$) to 18.7% (for the Weibull with $k = 5$). It follows that we are in no position to make inferences concerning the near-minimum values of two lumber populations on the basis of the distribution-free procedure based on ASTM D2915-70T, an application for which, it is again emphasized, the standard was not intended.

It must be pointed out, in all fairness, that tolerance-limit theory has not been well developed under parametric assumptions other than the normal. Thus, even if the form of the underlying distribution were known, one still might be tempted to use a distribution-free procedure. Indeed, ASTM 2915-70T does provide for increasing the sample size to 93 and using the second-order statistic. Although the precision will thereby be increased, it will still differ with the form of the underlying distribution. The same can be said of the Hanson and Koopmans (1964) tolerance limits, also studied briefly by Habermann.

For the comparison of two sets, a technique that would utilize more information would be the application of the Mann-Whitney test (see e.g. Siegel 1956) to, say, the first six-order statistics of each set. This has a certain intuitive appeal, but the properties of this test in such an application should be examined before it could be recommended for general use.

REFERENCES

BLOM, G. 1958. Statistical estimates and transformed beta-variables. J. Wiley and Sons, New York.

GUPTA, S. S. 1960. Order statistics from the gamma distribution. Technometrics 2:243–262.

HABERMANN, H. 1973. Tolerance intervals. I. Simulation studies of non-parametric tolerance intervals. Presented at meeting of Division 5, IUFRO, South Africa.

HANSON, D. L., AND L. H. KOOPMANS. 1964. Tolerance limits for hazard rates. Ann. Math. Stat. 35:1561–1570.

HARTER, H. L. 1960. Tables of range and studentized range. Ann. Math. Stat. 31:1122–1147.

JENNETT, W. J., AND B. L. WELCH. 1939. The control of proportion defective as judged by a single quality characteristic varying on a continuous scale. J. Roy. Stat. Suppl. 6:80–88.

OWEN, D. B. 1962. Handbook of statistical tables. Addison-Wesley, Reading, Mass.

SIEGEL, S. 1956. Non-parametric statistics for the behavioral sciences. McGraw Hill, New York.

WHITE, J. S. 1967. The moments of log-Weibull order statistics. General Motors Corporation, Res. Publ. GMR-717, Warren, Mich.