Math. Z.
DOI 10.1007/s00209-009-0480-4

Mathematische Zeitschrift

Comparison of relative cohomology theories with respect to semidualizing modules

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Received: 26 July 2007 / Accepted: 2 October 2008

Abstract We compare and contrast various relative cohomology theories that arise from resolutions involving semidualizing modules. We prove a general balance result for relative cohomology over a Cohen–Macaulay ring with a dualizing module, and we demonstrate the failure of the naive version of balance one might expect for these functors. We prove that the natural comparison morphisms between relative cohomology modules are isomorphisms in several cases, and we provide a Yoneda-type description of the first relative Ext functor. Finally, we show by example that each distinct relative cohomology construction does in fact result in a different functor.

Keywords Auslander class · Balance · Bass class · Gorenstein homological dimensions · Relative cohomology · Relative homological algebra · Semi-dualizing · Semidualizing

Mathematics Subject Classification (2000) 13D02 · 13D05 · 13D07

Dedicated to the memory of Anders Juel Frankild.

T. Sharif was supported by a grant from IPM (No. 83130311).

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Published online: 12 February 2009

0 Introduction

The study of relative homological algebra was initiated by Butler and Horrocks [9] and Eilenberg and Moore [12] and has been revitalized recently by a number of authors, notably, Enochs and Jenda [14] and Avramov and Martsinkovsky [7]. The basic idea behind this construction is to consider resolutions of a module M over a ring R, where the modules in the resolutions are taken from a fixed class $\mathcal{X}$. One restricts focus to those resolutions X, called proper $\mathcal{X}$-resolutions, with good enough lifting properties to make them unique up to homotopy equivalence, and this yields well-defined functors

$$\operatorname{Ext}^n_{\mathcal{X}R}(M,-)=\operatorname{H}_{-n}(\operatorname{Hom}_R(X,-)).$$

Dually, one considers proper $\mathcal{X}$-coresolutions to define the functors $\operatorname{Ext}^n_{R\mathcal{X}}(-,M)$. Consult Sect. 1 for precise definitions.

In this article we investigate relative cohomology theories that arise from dualities with respect to semidualizing modules: when R is commutative and noetherian, a finitely generated R-module C is *semidualizing* when $\operatorname{Ext}^{\geqslant 1}_R(C,C)=0$ and $\operatorname{Hom}_R(C,C)\cong R$. Examples include projective R-modules of rank 1 and, when R is a Cohen–Macaulay ring of finite Krull dimension that is a homomorphic image of a Gorenstein ring, a dualizing module.

A semidualizing R-module C gives rise to several distinguished classes of modules. For instance, one has the class $\mathcal{P}_C$ of C-projective modules and the class $\mathcal{GP}_C$ of G_C-projective modules, which we use for resolutions. For coresolutions, we consider the class $\mathcal{I}_C$ of C-injective modules and the class $\mathcal{GI}_C$ of G_C-injective modules. As there is no risk of confusion in these cases, the corresponding relative cohomology functors are denoted $\operatorname{Ext}^n_{\mathcal{P}_C}(-,-)$, $\operatorname{Ext}^n_{\mathcal{GP}_C}(-,-)$, $\operatorname{Ext}^n_{\mathcal{I}_C}(-,-)$ and $\operatorname{Ext}^n_{\mathcal{GI}_C}(-,-)$. Detailed definitions can be found in Sect. 3.

Our investigation into these functors focuses on two questions: What conditions on a pair of modules (M,N) guarantee that the corresponding outputs of two of these functors are isomorphic? And when are these functors different?

As to the first question, Sect. 5 focuses on the issue of balance, motivated by the fact that one can compute the "absolute" cohomology $\operatorname{Ext}^n_R(M,N)$ in terms of a projective resolution of M or an injective resolution of N. This section begins with Example 5.3 which shows that the naive version of balance for relative cohomology fails in general: even if $\mathcal{P}_C\text{-}\operatorname{pd}_R(M)$ and $\mathcal{I}_C\text{-}\operatorname{id}_R(N)$ are both finite, one can have $\operatorname{Ext}^n_{\mathcal{P}_C}(M,N)\ncong\operatorname{Ext}^n_{\mathcal{I}_C}(M,N)$ and $\operatorname{Ext}^n_{\mathcal{GP}_C}(M,N)\ncong\operatorname{Ext}^n_{\mathcal{GI}_C}(M,N)$. It turns out that the correct balance result in this setting, stated next, uses coresolutions with respect to the semidualizing module $C^\dagger=\operatorname{Hom}_R(C,D)$ where D is a dualizing module. This result is contained in Proposition 5.4 and Theorem 5.7.

Theorem A *Let R be a Cohen–Macaulay ring with a dualizing module, and let C, M and N be R-modules with C semidualizing.*

(a) *If $\mathcal{P}_C\text{-}\operatorname{pd}_R(M)<\infty$ and $\mathcal{I}_{C^\dagger}\text{-}\operatorname{id}_R(N)<\infty$, then there is an isomorphism*

$$\operatorname{Ext}^n_{\mathcal{P}_C}(M,N)\cong\operatorname{Ext}^n_{\mathcal{I}_{C^\dagger}}(M,N)$$

for each integer n.

(b) *If $\mathcal{GP}_C\text{-}\operatorname{pd}_R(M)<\infty$ and $\mathcal{GI}_{C^\dagger}\text{-}\operatorname{id}_R(N)<\infty$, then there is an isomorphism*

$$\operatorname{Ext}^n_{\mathcal{GP}_C}(M,N)\cong\operatorname{Ext}^n_{\mathcal{GI}_{C^\dagger}}(M,N)$$

for each integer n.

In addition, Sect. 4 gives conditions that yield isomorphisms $\mathrm{Ext}^n_{\mathcal{P}_C}(M,N) \cong \mathrm{Ext}^n_{\mathcal{GP}_C}(M,N)$ and $\mathrm{Ext}^n_{\mathcal{I}_C}(M,N) \cong \mathrm{Ext}^n_{\mathcal{GI}_C}(M,N)$. See Propositions 4.2 and 4.4.

Section 6 deals with the even more interesting question of the differences between these functors. The next result summarizes our findings from this section and shows that each reasonably comparable pair of relative cohomology functors is distinct.

Theorem B *Let $(R,\mathfrak{m})$ be a local ring, and let B, C be semidualizing R-modules.*

(a) *Assume $C \ncong R$. Then one has*

$$\mathrm{Ext}_{\mathcal{P}_C}(-,-) \ncong \mathrm{Ext}_R(-,-) \ncong \mathrm{Ext}_{\mathcal{GP}_C}(-,-)$$
$$\mathrm{Ext}_{\mathcal{I}_C}(-,-) \ncong \mathrm{Ext}_R(-,-) \ncong \mathrm{Ext}_{\mathcal{GI}_C}(-,-).$$

If there exist $y, z \in \mathfrak{m}$ such that $\mathrm{Ann}_R(y) = zR$ and $\mathrm{Ann}_R(z) = yR$, then

$$\mathrm{Ext}_{\mathcal{P}_C}(-,-) \ncong \mathrm{Ext}_{\mathcal{GP}_C}(-,-) \qquad \mathrm{Ext}_{\mathcal{I}_C}(-,-) \ncong \mathrm{Ext}_{\mathcal{GI}_C}(-,-).$$

(b) *Assume $\mathcal{GP}_C\text{-}\mathrm{pd}_R(B) < \infty$ and $C \ncong B$. Then one has*

$$\mathrm{Ext}_{\mathcal{P}_C}(-,-) \ncong \mathrm{Ext}_{\mathcal{P}_B}(-,-) \ncong \mathrm{Ext}_{\mathcal{GP}_C}(-,-)$$

$$\mathrm{Ext}_{\mathcal{GI}_B}(-,-) \ncong \mathrm{Ext}_{\mathcal{I}_C}(-,-) \ncong \mathrm{Ext}_{\mathcal{I}_B}(-,-) \ncong \mathrm{Ext}_{\mathcal{GI}_C}(-,-).$$

If $\mathrm{depth}(R) \geqslant 1$, *then*

$$\mathrm{Ext}_{\mathcal{GP}_C}(-,-) \ncong \mathrm{Ext}_{\mathcal{GP}_B}(-,-) \qquad \mathrm{Ext}_{\mathcal{GI}_C}(-,-) \ncong \mathrm{Ext}_{\mathcal{GI}_B}(-,-).$$

If C admits a proper $\mathcal{GP}_B$-resolution, then

$$\mathrm{Ext}_{\mathcal{P}_C}(-,-) \ncong \mathrm{Ext}_{\mathcal{GP}_B}(-,-).$$

As an aid for some of the computations in Theorem B we utilize a Yoneda-type characterization of relative cohomology modules. This is the subject of Sect. 2. In particular, the following result is contained in Theorem 2.3.

Theorem C *Let M and N be R-modules.*

(a) *If M admits a proper $\mathcal{X}$-resolution, then $\mathrm{Ext}_{\mathcal{X}R}(M,N)$ is in bijection with the set of equivalence classes of sequences $0 \to N \to T \to M \to 0$ that are exact and $\mathrm{Hom}_R(\mathcal{X},-)$-exact.*
(b) *If N admits a proper $\mathcal{Y}$-coresolution, then $\mathrm{Ext}_{R\mathcal{Y}}(M,N)$ is in bijection with the set of equivalence classes of sequences $0 \to N \to T \to M \to 0$ that are exact and $\mathrm{Hom}_R(-,\mathcal{Y})$-exact.*

1 Categories, resolutions, and relative cohomology

We begin with some notation and terminology for use throughout this paper.

Definition/Notation 1.1 Throughout this work R is a commutative ring. Write $\mathcal{M} = \mathcal{M}(R)$ for the category of R-modules, and write $\mathcal{P} = \mathcal{P}(R)$, $\mathcal{F} = \mathcal{F}(R)$ and $\mathcal{I} = \mathcal{I}(R)$ for the subcategories of projective, flat and injective R-modules, respectively. We use the term "subcategory" to mean a "full, additive, and essential (closed under isomorphisms) subcategory." If $\mathcal{X}$ is a subcategory of $\mathcal{M}$, then $\mathcal{X}^f$ is the subcategory of finitely generated modules in $\mathcal{X}$.

Definition 1.2 We fix subcategories $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{W}$, and $\mathcal{V}$ of $\mathcal{M}$ such that $\mathcal{W} \subseteq \mathcal{X}$ and $\mathcal{V} \subseteq \mathcal{Y}$. Write $\mathcal{X} \perp \mathcal{Y}$ if $\operatorname{Ext}_R^{\geqslant 1}(X, Y) = 0$ for each module X in $\mathcal{X}$ and each module Y in $\mathcal{Y}$. For a module M in $\mathcal{M}$, write $M \perp \mathcal{Y}$ (resp., $\mathcal{X} \perp M$) if $\operatorname{Ext}_R^{\geqslant 1}(M, Y) = 0$ for each module Y in $\mathcal{Y}$ (resp., if $\operatorname{Ext}_R^{\geqslant 1}(X, M) = 0$ for each module X in $\mathcal{X}$). We say that $\mathcal{W}$ is a *cogenerator* for $\mathcal{X}$ if, for each module X in $\mathcal{X}$, there exists an exact sequence $0 \to X \to W \to X' \to 0$ such that W is in $\mathcal{W}$ and X' is in $\mathcal{X}$. The subcategory $\mathcal{W}$ is an *injective cogenerator* for $\mathcal{X}$ if $\mathcal{W}$ is a cogenerator for $\mathcal{X}$ and $\mathcal{X} \perp \mathcal{W}$. The terms *generator* and *projective generator* are defined dually.

Definition 1.3 An *R-complex* is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial_{n+1}^X} X_n \xrightarrow{\partial_n^X} X_{n-1} \xrightarrow{\partial_{n-1}^X} \cdots$$

such that $\partial_{n-1}^X \partial_n^X = 0$ for each integer n; the nth *homology module* of X is $\mathrm{H}_n(X) = \operatorname{Ker}(\partial_n^X)/\operatorname{Im}(\partial_{n+1}^X)$. We frequently identify R-modules with complexes concentrated in degree 0. The *suspension* (or *shift*) of X, denoted ΣX, is the complex with $(\Sigma X)_n = X_{n-1}$ and $\partial_n^{\Sigma X} = -\partial_{n-1}^X$.

The complex X is $\operatorname{Hom}_R(\mathcal{X}, -)$-exact if the complex $\operatorname{Hom}_R(X', X)$ is exact for each module X' in $\mathcal{X}$. Dually, it is $\operatorname{Hom}_R(-, \mathcal{X})$-exact if the complex $\operatorname{Hom}_R(X, X')$ is exact for each module X' in $\mathcal{X}$. It is $- \otimes_R \mathcal{X}$-exact if the complex $X' \otimes_R X$ is exact for each module X' in $\mathcal{X}$.

Definition 1.4 Let X, Y be R-complexes. The Hom-complex $\operatorname{Hom}_R(X, Y)$ is the R-complex defined as $\operatorname{Hom}_R(X, Y)_n = \prod_p \operatorname{Hom}_R(X_p, Y_{p+n})$ with nth differential $\partial_n^{\operatorname{Hom}_R(X,Y)}$ given by $\{f_p\} \mapsto \{\partial_{p+n}^Y f_p - (-1)^n f_{n-1} \partial_p^X\}$. A *morphism* is an element of $\operatorname{Ker}(\partial_0^{\operatorname{Hom}_R(X,Y)})$. Two morphisms $\alpha, \alpha' \colon X \to Y$ are *homotopic*, written $\alpha \sim \alpha'$, if the difference $\alpha - \alpha'$ is in $\operatorname{Im}(\partial_1^{\operatorname{Hom}_R(X,Y)})$. The morphism α is a *homotopy equivalence* if there is a morphism $\beta \colon Y \to X$ such that $\beta\alpha \sim \mathrm{id}_X$ and $\alpha\beta \sim \mathrm{id}_Y$.

A morphism of complexes $\alpha \colon X \to Y$ induces homomorphisms on homology modules $\mathrm{H}_n(\alpha) \colon \mathrm{H}_n(X) \to \mathrm{H}_n(Y)$, and α is a *quasiisomorphism* when each $\mathrm{H}_n(\alpha)$ is bijective. The *mapping cone* of α is the complex $\operatorname{Cone}(\alpha)$ defined as $\operatorname{Cone}(\alpha)_n = Y_n \oplus X_{n-1}$ with nth differential $\partial_n^{\operatorname{Cone}(\alpha)} = \begin{pmatrix} \partial_n^Y & \alpha_{n-1} \\ 0 & -\partial_{n-1}^X \end{pmatrix}$. Recall that α is a quasiisomorphism if and only if $\operatorname{Cone}(\alpha)$ is exact.

Definition 1.5 An R-complex X is *bounded* if $X_n = 0$ for $|n| \gg 0$. When $X_{-n} = 0 = \mathrm{H}_n(X)$ for all $n > 0$, the natural map $X \to \mathrm{H}_0(X) \cong M$ is a quasiisomorphism. In this event, X is an *$\mathcal{X}$-resolution* of M if each X_n is in $\mathcal{X}$, and the exact sequence

$$X^+ = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \to M \to 0$$

is the *augmented $\mathcal{X}$-resolution* of M associated to X. We write "projective resolution" in lieu of "$\mathcal{P}$-resolution". The *$\mathcal{X}$-projective dimension* of M is the quantity

$$\mathcal{X}\text{-}\operatorname{pd}(M) = \inf\{\sup\{n \geqslant 0 \mid X_n \neq 0\} \mid X \text{is an} \mathcal{X}\text{-resolution of} M\}.$$

The modules of $\mathcal{X}$-projective dimension 0 are the nonzero modules in $\mathcal{X}$. We let $\operatorname{res} \widehat{\mathcal{X}}$ denote the subcategory of R-modules M with $\mathcal{X}\text{-}\operatorname{pd}(M) < \infty$. One checks easily that $\operatorname{res} \widehat{\mathcal{X}}$ is additive and contains $\mathcal{X}$.

The terms $\mathcal{Y}$*-coresolution* and $\mathcal{Y}$*-injective dimension* are defined dually. The *augmented* $\mathcal{Y}$*-coresolution* associated to a $\mathcal{Y}$-coresolution Y is denoted ${}^{+}Y$, and the $\mathcal{Y}$-injective dimension of M is denoted $\mathcal{Y}\text{-}\operatorname{id}(M)$. The subcategory of R-modules N with $\mathcal{Y}\text{-}\operatorname{id}(N) < \infty$ is denoted $\operatorname{cores}\widehat{\mathcal{Y}}$; it is additive and contains $\mathcal{Y}$.

Following much of the literature, we write "injective resolution" in lieu of "$\mathcal{I}$-coresolution" and set $\operatorname{pd} = \mathcal{P}\text{-}\operatorname{pd}$ and $\operatorname{id} = \mathcal{I}\text{-}\operatorname{id}$.

Definition 1.6 An $\mathcal{X}$-resolution X is *proper* when the augmented resolution X^{+} is $\operatorname{Hom}_R(\mathcal{X}, -)$-exact. We let $\operatorname{res}\widetilde{\mathcal{X}}$ denote the subcategory of R-modules admitting a proper $\mathcal{X}$-resolution. One checks readily that $\operatorname{res}\widetilde{\mathcal{X}}$ is additive and contains $\mathcal{X}$. *Proper coresolutions* are defined dually. The subcategory of R-modules admitting a proper $\mathcal{Y}$-coresolution is denoted $\operatorname{cores}\widetilde{\mathcal{Y}}$; it is additive and contains $\mathcal{Y}$.

The next lemmata are standard or have standard proofs: for 1.7 see [4, pf. of (2.3)]; for 1.8 see [4, pf. of (2.1)]; for 1.9 argue as in [7, (4.3)] and [20, (1.8)]; and for the "Horseshoe Lemma" 1.10 see [7, (4.5)] and [14, pf. of (8.2.1)].

Lemma 1.7 *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be an exact sequence of* R*-modules.*

(a) *If* $M_3 \perp \mathcal{X}$*, then* $M_1 \perp \mathcal{X}$ *if and only if* $M_2 \perp \mathcal{X}$*. If* $M_1 \perp \mathcal{X}$ *and* $M_2 \perp \mathcal{X}$*, then* $M_3 \perp \mathcal{X}$ *if and only if the given sequence is* $\operatorname{Hom}_R(-, \mathcal{X})$*-exact.*
(b) *If* $\mathcal{X} \perp M_1$*, then* $\mathcal{X} \perp M_2$ *if and only if* $\mathcal{X} \perp M_3$*. If* $\mathcal{X} \perp M_2$ *and* $\mathcal{X} \perp M_3$*, then* $\mathcal{X} \perp M_1$ *if and only if the given sequence is* $\operatorname{Hom}_R(\mathcal{X}, -)$*-exact.*

Lemma 1.8 *If* $\mathcal{X} \perp \mathcal{Y}$*, then* $\mathcal{X} \perp \operatorname{res}\widehat{\mathcal{Y}}$ *and* $\operatorname{cores}\widehat{\mathcal{X}} \perp \mathcal{Y}$*.*

Lemma 1.9 *Let* M, M', N, N' *be* R*-modules.*

(a) *Let* $P \xrightarrow{\rho} M$ *be a projective resolution. Assume that* M *admits a proper* $\mathcal{W}$*-resolution* $W \xrightarrow{\gamma} M$ *and* M' *admits a proper* $\mathcal{X}$*-resolution* $X' \xrightarrow{\gamma'} M'$*. For each homomorphism* $f\colon M \to M'$ *there exist morphisms* $\overline{f}\colon W \to X'$ *and* $\widetilde{f}\colon P \to X'$ *unique up to homotopy such that* $f\gamma = \gamma'\overline{f}$ *and* $f\rho = \gamma'\widetilde{f}$*. If* f *is an isomorphism, then* $\overline{f}$ *and* $\widetilde{f}$ *are quasiisomorphisms. If* f *is an isomorphism and* $\mathcal{X} = \mathcal{W}$*, then* $\overline{f}$ *is a homotopy equivalence.*
(b) *Let* $N' \xrightarrow{\phi'} I'$ *be an injective resolution. Assume that* N *admits a proper* $\mathcal{Y}$*-coresolution* $N \xrightarrow{\delta} Y$ *and* N' *admits a proper* $\mathcal{V}$*-coresolution* $N' \xrightarrow{\delta'} V'$*. For each homomorphism* $g\colon N \to N'$ *there exist morphisms* $\overline{g}\colon Y \to V'$ *and* $\widetilde{g}\colon Y \to I'$ *unique up to homotopy such that* $\overline{g}\delta = \delta' g$ *and* $\widetilde{g}\delta = \phi' g$*. If* g *is an isomorphism, then* $\overline{g}$ *and* $\widetilde{g}$ *are quasiisomorphisms. If* g *is an isomorphism and* $\mathcal{V} = \mathcal{Y}$*, then* $\overline{g}$ *is a homotopy equivalence.*

Lemma 1.10 *Let* $0 \to M' \to M \to M'' \to 0$ *be an exact sequence of* R*-modules.*

(a) *Assume that* M' *and* M'' *have proper* $\mathcal{X}$*-resolutions* $X' \xrightarrow{\simeq} M'$ *and* $X'' \xrightarrow{\simeq} M''$ *and that the given sequence is* $\operatorname{Hom}_R(\mathcal{X}, -)$*-exact. Then* M *admits a proper* $\mathcal{X}$*-resolution* $X \xrightarrow{\simeq} M$ *such that there exists a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & 0 \\
 & & \downarrow{\simeq} & & \downarrow{\simeq} & & \downarrow{\simeq} & & \\
0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0
\end{array}$$

whose top row is degreewise split exact.

(b) *Assume that M' and M'' have proper $\mathcal{Y}$-coresolutions $M' \xrightarrow{\simeq} Y'$ and $M'' \xrightarrow{\simeq} Y''$ and that the given sequence is* $\operatorname{Hom}_R(-,\mathcal{Y})$*-exact. Then M admits a proper $\mathcal{Y}$-coresolution $M \xrightarrow{\simeq} Y$ such that there exists a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \longrightarrow & M & \longrightarrow & M'' & \longrightarrow & 0 \\ & & \simeq\downarrow & & \simeq\downarrow & & \simeq\downarrow & & \\ 0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0 \end{array}$$

whose bottom row is degreewise split exact.

Definition 1.11 Let M, M', N, N' be R-modules with homomorphisms $f: M \to M'$ and $g: N \to N'$. Assume that M admits a proper $\mathcal{X}$-resolution $X \xrightarrow{\gamma} M$, and for each integer n define the nth *relative $\mathcal{X}R$-cohomology module* as

$$\operatorname{Ext}^n_{\mathcal{X}R}(M,N) = \mathrm{H}_{-n}(\operatorname{Hom}_R(X,N)).$$

If M' also admits a proper $\mathcal{X}$-resolution $X' \xrightarrow{\gamma'} M'$, then let $\overline{f}: X \to X'$ be a chain map such that $f\gamma = \gamma'\overline{f}$ as in Lemma 1.9 and define

$$\begin{aligned} \operatorname{Ext}^n_{\mathcal{X}R}(f,N) &= \mathrm{H}_{-n}(\operatorname{Hom}_R(\overline{f},N))\colon\ \operatorname{Ext}^n_{\mathcal{X}R}(M',N) \to \operatorname{Ext}^n_{\mathcal{X}R}(M,N) \\ \operatorname{Ext}^n_{\mathcal{X}R}(M,g) &= \mathrm{H}_{-n}(\operatorname{Hom}_R(X,g))\colon\ \operatorname{Ext}^n_{\mathcal{X}R}(M,N) \to \operatorname{Ext}^n_{\mathcal{X}R}(M,N'). \end{aligned}$$

The nth *relative $R\mathcal{Y}$-cohomology* $\operatorname{Ext}^n_{R\mathcal{Y}}(-,-)$ is defined dually.

Remark 1.12 Lemma 1.9 shows that Definition 1.11 yields well-defined bifunctors

$$\operatorname{Ext}^n_{\mathcal{X}R}(-,-)\colon\ \operatorname{res}\widetilde{\mathcal{X}} \times \mathcal{M} \to \mathcal{M} \quad \text{and} \quad \operatorname{Ext}^n_{R\mathcal{Y}}(-,-)\colon \mathcal{M} \times \operatorname{cores}\widetilde{\mathcal{Y}} \to \mathcal{M}$$

and one checks the following natural equivalences readily.

$$\begin{gathered} \operatorname{Ext}^{\geqslant 1}_{\mathcal{X}R}(\mathcal{X},-) = 0 = \operatorname{Ext}^{\geqslant 1}_{R\mathcal{Y}}(-,\mathcal{Y}) \\ \operatorname{Ext}^0_{\mathcal{X}R}(-,-) \cong \operatorname{Hom}_R(-,-)|_{\operatorname{res}\widetilde{\mathcal{X}}\times\mathcal{M}} \\ \operatorname{Ext}^0_{R\mathcal{Y}}(-,-) \cong \operatorname{Hom}_R(-,-)|_{\mathcal{M}\times\operatorname{cores}\widetilde{\mathcal{Y}}} \\ \operatorname{Ext}^n_{\mathcal{P}\mathcal{M}}(-,-) \cong \operatorname{Ext}^n_R(-,-) \cong \operatorname{Ext}^n_{\mathcal{M}\mathcal{I}}(-,-). \end{gathered}$$

Definition 1.13 Let M, N be R-modules. Let $P \xrightarrow{\rho} M$ be a projective resolution. Assume that M admits a proper $\mathcal{W}$-resolution $W \xrightarrow{\gamma} M$ and a proper $\mathcal{X}$-resolution $X \xrightarrow{\gamma'} M$. Let $\overline{\mathrm{id}_M}\colon W \to X$ and $\widetilde{\mathrm{id}_M}\colon P \to X$ be quasiisomorphisms such that $\gamma = \gamma'\overline{\mathrm{id}_M}$ and $\rho = \gamma'\widetilde{\mathrm{id}_M}$, as in Lemma 1.9 (a), and set

$$\begin{aligned} \vartheta^n_{\mathcal{X}\mathcal{W}R}(M,N) &= \mathrm{H}_{-n}(\operatorname{Hom}_R(\overline{\mathrm{id}_M},N))\colon\ \operatorname{Ext}^n_{\mathcal{X}R}(M,N) \to \operatorname{Ext}^n_{\mathcal{W}R}(M,N) \\ \varkappa^n_{\mathcal{X}R}(M,N) &= \mathrm{H}_{-n}(\operatorname{Hom}_R(\widetilde{\mathrm{id}_M},N))\colon\ \operatorname{Ext}^n_{\mathcal{X}R}(M,N) \to \operatorname{Ext}^n_R(M,N). \end{aligned}$$

On the other hand, if N admits a proper $\mathcal{Y}$-coresolution and a proper $\mathcal{V}$-coresolution, then the following maps are defined dually

$$\begin{aligned} \vartheta^n_{R\mathcal{Y}\mathcal{V}}(M,N) &:\ \operatorname{Ext}^n_{R\mathcal{Y}}(M,N) \to \operatorname{Ext}^n_{R\mathcal{V}}(M,N) \\ \varkappa^n_{R\mathcal{Y}}(M,N) &:\ \operatorname{Ext}^n_{R\mathcal{Y}}(M,N) \to \operatorname{Ext}^n_R(M,N). \end{aligned}$$

Remark 1.14 Lemma 1.9 shows that Definition 1.13 describes well-defined natural transformations

$$\begin{aligned}
\vartheta^n_{\mathcal{X}\mathcal{W}R}(-,-) &: \operatorname{Ext}^n_{\mathcal{X}R}(-,-)|_{(\operatorname{res}\widetilde{\mathcal{W}}\cap\operatorname{res}\widetilde{\mathcal{X}})\times\mathcal{M}} \to \operatorname{Ext}^n_{\mathcal{W}R}(-,-)|_{(\operatorname{res}\widetilde{\mathcal{W}}\cap\operatorname{res}\widetilde{\mathcal{X}})\times\mathcal{M}} \\
\varkappa^n_{\mathcal{X}R}(-,-) &: \operatorname{Ext}^n_{\mathcal{X}R}(-,-) \to \operatorname{Ext}^n_R(-,-)|_{\operatorname{res}\widetilde{\mathcal{X}}\times\mathcal{M}} \\
\vartheta^n_{R\mathcal{Y}\mathcal{V}}(-,-) &: \operatorname{Ext}^n_{R\mathcal{Y}}(-,-)|_{\mathcal{M}\times(\operatorname{cores}\widetilde{\mathcal{V}}\cap\operatorname{cores}\widetilde{\mathcal{Y}})} \to \operatorname{Ext}^n_{R\mathcal{V}}(-,-)|_{\mathcal{M}\times(\operatorname{cores}\widetilde{\mathcal{V}}\cap\operatorname{cores}\widetilde{\mathcal{Y}})} \\
\varkappa^n_{R\mathcal{Y}}(-,-) &: \operatorname{Ext}^n_{R\mathcal{Y}}(-,-) \to \operatorname{Ext}^n_R(-,-)|_{\mathcal{M}\times\operatorname{cores}\widetilde{\mathcal{Y}}}
\end{aligned}$$

independent of resolutions and liftings. Note that the left-exactness of $\operatorname{Hom}_R(-,-)$ implies that each of these transformations is a natural isomorphism when $n \leq 0$.

Lemma 1.10 yields the following long exact sequences as in [7, (4.4),(4.6)].

Lemma 1.15 *Let M and N be R-modules, and consider an exact sequence*

$$\mathbf{L} = \quad 0 \to L' \xrightarrow{f'} L \xrightarrow{f} L'' \to 0.$$

(a) *Assume that the sequence $\mathbf{L}$ is $\operatorname{Hom}_R(\mathcal{X},-)$-exact. If M is in $\operatorname{res}\widetilde{\mathcal{X}}$, then $\mathbf{L}$ induces a functorial long exact sequence*

$$\cdots \to \operatorname{Ext}^n_{\mathcal{X}R}(M,L') \xrightarrow{\operatorname{Ext}^n_{\mathcal{X}R}(M,f')} \operatorname{Ext}^n_{\mathcal{X}R}(M,L) \xrightarrow{\operatorname{Ext}^n_{\mathcal{X}R}(M,f)}$$
$$\operatorname{Ext}^n_{\mathcal{X}R}(M,L'') \xrightarrow{\eth^n_{\mathcal{X}R}(M,\mathbf{L})} \operatorname{Ext}^{n+1}_{\mathcal{X}R}(M,L') \xrightarrow{\operatorname{Ext}^{n+1}_{\mathcal{X}R}(M,f')} \cdots.$$

(b) *Assume that the sequence $\mathbf{L}$ is $\operatorname{Hom}_R(\mathcal{X},-)$-exact. If the modules L', L, L'' are in $\operatorname{res}\widetilde{\mathcal{X}}$, then $\mathbf{L}$ induces a functorial long exact sequence*

$$\cdots \to \operatorname{Ext}^n_{\mathcal{X}R}(L'',N) \xrightarrow{\operatorname{Ext}^n_{\mathcal{X}R}(f,N)} \operatorname{Ext}^n_{\mathcal{X}R}(L,N) \xrightarrow{\operatorname{Ext}^n_{\mathcal{X}R}(f',N)}$$
$$\operatorname{Ext}^n_{\mathcal{X}R}(L',N) \xrightarrow{\eth^n_{\mathcal{X}R}(\mathbf{L},N)} \operatorname{Ext}^{n+1}_{\mathcal{X}R}(L'',N) \xrightarrow{\operatorname{Ext}^{n+1}_{\mathcal{X}R}(f,N)} \cdots.$$

(c) *Assume that the sequence $\mathbf{L}$ is $\operatorname{Hom}_R(-,\mathcal{Y})$-exact. If N is in $\operatorname{cores}\widetilde{\mathcal{Y}}$, then $\mathbf{L}$ induces a functorial long exact sequence*

$$\cdots \to \operatorname{Ext}^n_{R\mathcal{Y}}(L'',N) \xrightarrow{\operatorname{Ext}^n_{R\mathcal{Y}}(f,N)} \operatorname{Ext}^n_{R\mathcal{Y}}(L,N) \xrightarrow{\operatorname{Ext}^n_{R\mathcal{Y}}(f',N)}$$
$$\operatorname{Ext}^n_{R\mathcal{Y}}(L',N) \xrightarrow{\eth^n_{R\mathcal{Y}}(\mathbf{L},N)} \operatorname{Ext}^{n+1}_{R\mathcal{Y}}(L'',N) \xrightarrow{\operatorname{Ext}^{n+1}_{R\mathcal{Y}}(f,N)} \cdots.$$

(d) *Assume that the sequence $\mathbf{L}$ is $\operatorname{Hom}_R(-,\mathcal{Y})$-exact. If the modules L', L, L'' are in $\operatorname{cores}\widetilde{\mathcal{Y}}$, then $\mathbf{L}$ induces a functorial long exact sequence*

$$\cdots \to \operatorname{Ext}^n_{R\mathcal{Y}}(M,L') \xrightarrow{\operatorname{Ext}^n_{R\mathcal{Y}}(M,f')} \operatorname{Ext}^n_{R\mathcal{Y}}(M,L) \xrightarrow{\operatorname{Ext}^n_{R\mathcal{Y}}(M,f)}$$
$$\operatorname{Ext}^n_{R\mathcal{Y}}(M,L'') \xrightarrow{\eth^n_{R\mathcal{Y}}(M,\mathbf{L})} \operatorname{Ext}^{n+1}_{R\mathcal{Y}}(M,A') \xrightarrow{\operatorname{Ext}^{n+1}_{R\mathcal{Y}}(M,f')} \cdots.$$

2 Relative cohomology and extensions

In this section, we compare relative cohomology modules with sets of equivalence classes of module extensions, as in the classical Yoneda setting.

Definition/Notation 2.1 Let N and M be R-modules. An *extension* of M by N is an exact sequence

$$0 \to N \to T \to M \to 0$$

and this is *equivalent* to a second extension $0 \to N \to T' \to M \to 0$ if there exists a homomorphism $\tau\colon T \to T'$ making the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & T & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_N} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \mathrm{id}_M} & & \\ 0 & \longrightarrow & N & \longrightarrow & T' & \longrightarrow & M & \longrightarrow & 0. \end{array}$$

We set

$$\begin{aligned} \mathrm{e}_R(M,N) &= \{\text{equivalence classes of extensions of} M \text{ by} N\} \\ \mathrm{e}_{\mathcal{X}R}(M,N) &= \{\text{equivalence classes of} \operatorname{Hom}_R(\mathcal{X},-) - \text{exact extensions of} M \text{ by} N\} \\ \mathrm{e}_{R\mathcal{Y}}(M,N) &= \{\text{equivalence classes of} \operatorname{Hom}_R(-,\mathcal{Y})\text{-exact extensions of} M \text{ by} N\}. \end{aligned}$$

Remark 2.2 Because of the containments $\mathcal{W} \subseteq \mathcal{X}$ and $\mathcal{V} \subseteq \mathcal{Y}$ there are inclusions $\mathrm{e}_{\mathcal{W}R}(M,N) \subseteq \mathrm{e}_{\mathcal{X}R}(M,N) \subseteq \mathrm{e}(M,N)$ and $\mathrm{e}_{R\mathcal{V}}(M,N) \subseteq \mathrm{e}_{R\mathcal{Y}}(M,N) \subseteq \mathrm{e}(M,N)$.

There exists a bijection $\xi_{RMN}\colon \operatorname{Ext}^1_R(M,N) \to \mathrm{e}_R(M,N)$ whose construction we recall from [24, Ch. 7]. Let $P \xrightarrow{\simeq} M$ be a projective resolution. Each element in $\operatorname{Ext}^1_R(M,N)$ is represented by a homomorphism $\alpha\colon P_1 \to N$ such that $\alpha\partial_2^P = 0$, and each such α induces a map $\overline{\alpha}\colon P_1/\operatorname{Im}(\partial_2^P) \to N$. Taking a pushout yields the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_1/\operatorname{Im}(\partial_2^P) & \xrightarrow{\overline{\partial_1^P}} & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{\alpha}} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \mathrm{id}_M} & & \\ 0 & \longrightarrow & N & \longrightarrow & T & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

and $\xi_{RMN}([\alpha])$ is the equivalence class of the bottom row of this diagram.

Dually, one constructs a bijection $\xi'_{RMN}\colon \operatorname{Ext}^1_R(M,N) \to \mathrm{e}_R(M,N)$ using an injective resolution of N.

The next result extends the construction of Remark 2.2 to the relative setting and contains Theorem C from the introduction. The connecting maps $\varkappa^1_{\mathcal{X}R}$, $\vartheta^1_{\mathcal{X}\mathcal{W}R}$, $\varkappa^1_{R\mathcal{Y}}$ and $\vartheta^1_{R\mathcal{Y}\mathcal{V}}$ are described in Definition 1.13.

Theorem 2.3 *Let M and N be R-modules.*

(a) *Assume that M admits a proper $\mathcal{X}$-resolution. There is then a bijective map $\xi_{\mathcal{X}MN}\colon \operatorname{Ext}^1_{\mathcal{X}R}(M,N) \to \mathrm{e}_{\mathcal{X}R}(M,N)$ making the following diagram commute*

$$\begin{array}{ccc} \operatorname{Ext}^1_{\mathcal{X}R}(M,N) & \xrightarrow{\xi_{\mathcal{X}MN}} & \mathrm{e}_{\mathcal{X}R}(M,N) \\ \downarrow{\scriptstyle \varkappa^1_{\mathcal{X}R}(M,N)} & & \downarrow \\ \operatorname{Ext}^1_R(M,N) & \xrightarrow{\xi_{RMN}} & \mathrm{e}(M,N), \end{array}$$

where the rightmost vertical arrow is the natural inclusion. In particular, the comparison map $\varkappa^1_{\mathcal{X}R}(M,N)\colon \operatorname{Ext}^1_{\mathcal{X}R}(M,N) \to \operatorname{Ext}^1_R(M,N)$ is injective.

(b) *Assume that M admits a proper $\mathcal{X}$-resolution and a proper $\mathcal{W}$-resolution. The following diagram commutes*

$$\begin{array}{ccc} \operatorname{Ext}^1_{\mathcal{X}R}(M,N) & \xrightarrow{\xi_{\mathcal{X}MN}} & \mathrm{e}_{\mathcal{X}R}(M,N) \\ {\scriptstyle \vartheta^1_{\mathcal{X}\mathcal{W}R}(M,N)}\downarrow & & \downarrow \\ \operatorname{Ext}^1_{\mathcal{W}R}(M,N) & \xrightarrow{\xi_{\mathcal{W}MN}} & \mathrm{e}_{\mathcal{W}R}(M,N), \end{array}$$

where the rightmost vertical arrow is the natural inclusion. In particular the comparison map $\vartheta^1_{\mathcal{X}\mathcal{W}R}(M,N)$ is injective.

(c) *Assume that N admits a proper $\mathcal{Y}$-coresolution. There is then a bijective map $\xi'_{\mathcal{Y}MN}\colon \operatorname{Ext}^1_{R\mathcal{Y}}(M,N) \to \mathrm{e}_{R\mathcal{Y}}(M,N)$ making the following diagram commute*

$$\begin{array}{ccc} \operatorname{Ext}^1_{R\mathcal{Y}}(M,N) & \xrightarrow{\xi'_{\mathcal{Y}MN}} & \mathrm{e}_{R\mathcal{Y}}(M,N) \\ {\scriptstyle \varkappa^1_{R\mathcal{Y}}(M,N)}\downarrow & & \downarrow \\ \operatorname{Ext}^1_{R}(M,N) & \xrightarrow{\xi'_{RMN}} & \mathrm{e}(M,N), \end{array}$$

where the rightmost vertical arrow is the natural inclusion. In particular, the comparison map $\varkappa^1_{R\mathcal{Y}}(M,N)\colon \operatorname{Ext}^1_{R\mathcal{Y}}(M,N) \to \operatorname{Ext}^1_R(M,N)$ is injective.

(d) *Assume that N admits a proper $\mathcal{Y}$-coresolution and a proper $\mathcal{V}$-coresolution. The following diagram commutes*

$$\begin{array}{ccc} \operatorname{Ext}^1_{R\mathcal{Y}}(M,N) & \xrightarrow{\xi'_{\mathcal{Y}MN}} & \mathrm{e}_{R\mathcal{Y}}(M,N) \\ {\scriptstyle \vartheta^1_{R\mathcal{Y}\mathcal{V}}(M,N)}\downarrow & & \downarrow \\ \operatorname{Ext}^1_{R\mathcal{V}}(M,N) & \xrightarrow{\xi'_{\mathcal{V}MN}} & \mathrm{e}_{R\mathcal{V}}(M,N), \end{array}$$

where the rightmost vertical arrow is the natural inclusion. In particular the comparison map $\vartheta^1_{R\mathcal{Y}\mathcal{V}}(M,N)$ is injective.

Proof Our proof is modeled on the arguments of [24, Ch. 7]; instead of rewriting much of the work there, we simply sketch the proof, indicating how $\operatorname{Hom}_R(\mathcal{X},-)$-exactness is detected and used.

(a) Let $X \xrightarrow{\simeq} M$ be a proper $\mathcal{X}$-resolution and set $\overline{X} = X_1/\operatorname{Im}(\partial_2^X)$. Each element $[\alpha] \in \operatorname{Ext}^1_{\mathcal{X}R}(M,N)$ is represented by a homomorphism $\alpha\colon X_1 \to N$ such that $\alpha\partial_2^X = 0$, and each such α induces a map $\overline{\alpha}\colon \overline{X} \to N$. Taking a pushout yields the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} & 0 & \longrightarrow & \overline{X} & \xrightarrow{\overline{\partial_1^X}} & X_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & {\scriptstyle\overline{\alpha}}\downarrow & & {\scriptstyle\tau}\downarrow & & {\scriptstyle\mathrm{id}_M}\downarrow & & \\ \zeta = & 0 & \longrightarrow & N & \longrightarrow & T & \xrightarrow{\pi} & M & \longrightarrow & 0. \end{array} \tag{2.3.1}$$

We claim that the bottom row of this diagram is $\operatorname{Hom}_R(\mathcal{X},-)$-exact. To see this, first note that the properness of the resolution $X \xrightarrow{\simeq} M$ implies that the top row of (2.3.1) is

$\mathrm{Hom}_R(\mathcal{X}, -)$-exact. Fix an R-module X' in $\mathcal{X}$ and apply $\mathrm{Hom}_R(X', -)$ to the diagram (2.3.1) to yield the next commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}_R(X', \overline{X}) & \to & \mathrm{Hom}_R(X', X_0) & \longrightarrow & \mathrm{Hom}_R(X', M) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_{\mathrm{Hom}_R(X',M)}} & & \\ 0 & \to & \mathrm{Hom}_R(X', N) & \to & \mathrm{Hom}_R(X', T) & \xrightarrow{\mathrm{Hom}_R(X',\pi)} & \mathrm{Hom}_R(X', M). & & \end{array}$$

An easy diagram chase shows that the map $\mathrm{Hom}_R(X', \pi)$ is surjective, as desired.

We define $\xi_{\mathcal{X}MN}([\alpha])$ to be the equivalence class $[\zeta]$ of the bottom row of the diagram (2.3.1). One now verifies readily (as in the proof of the classical result in [24, Ch. 7]) that this yields a well-defined function $\mathrm{Ext}^1_{\mathcal{X}R}(M, N) \to \mathrm{e}_{\mathcal{X}R}(M, N)$.

To show that $\xi_{\mathcal{X}MN}$ is bijective, we construct an inverse. Fix a $\mathrm{Hom}_R(\mathcal{X}, -)$-exact sequence $\zeta = (0 \to N \to T \to M \to 0)$. A standard lifting procedure as in Lemma 1.9 (a) yields the next commutative diagram with exact rows

$$\begin{array}{ccccccccc} X_2 & \xrightarrow{\partial_2^X} & X_1 & \xrightarrow{\partial_1^X} & X_0 & \to & M & \to & 0 \\ \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_M} & & \\ 0 & \to & N & \to & T & \to & M & \to & 0. \end{array}$$

The map α is thus a degree-1 cycle in $\mathrm{Hom}_R(X, N)$ and so gives rise to a cohomology class $[\alpha] \in \mathrm{Ext}^1_{\mathcal{X}R}(M, N)$. Again, one verifies that the assignment $[\zeta] \mapsto [\alpha]$ describes a well-defined function $\mathrm{e}_{\mathcal{X}R}(M, N) \to \mathrm{Ext}^1_{\mathcal{X}R}(M, N)$, and that this function is a two-sided inverse for $\xi_{\mathcal{X}MN}$; the reader may find [24, (7.18)] to be helpful.

The proof of part (a) will be compete once we verify $\xi_{RMN}\varkappa^1_{\mathcal{X}R} = \xi_{\mathcal{X}MN}$. Let $P \xrightarrow{\simeq} M$ be a projective resolution and set $\overline{P} = P_1 / \mathrm{Im}(\partial_2^P)$. Lemma 1.9 (a) yields the next commutative diagram with exact rows

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\partial_3^P} & P_2 & \xrightarrow{\partial_2^P} & P_1 & \xrightarrow{\partial_1^P} & P_0 & \to & M & \to & 0 \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \mathrm{id}_M} & & \\ \cdots & \xrightarrow{\partial_3^X} & X_2 & \xrightarrow{\partial_2^X} & X_1 & \xrightarrow{\partial_1^X} & X_0 & \to & M & \to & 0, \end{array}$$

which in turn induces another commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \overline{P} & \xrightarrow{\overline{\partial_1^P}} & P_0 & \to & M & \to & 0 \\ & & \downarrow{\scriptstyle \overline{f_1}} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \mathrm{id}_M} & & \\ 0 & \to & \overline{X} & \xrightarrow{\overline{\partial_1^X}} & X_0 & \to & M & \to & 0. \end{array} \tag{2.3.2}$$

Given $[\alpha] \in \mathrm{Ext}^1_{\mathcal{X}R}(M, N)$, construct the extension ζ as above. The diagram (2.3.2) combines with (2.3.1) to yield the next diagram

$$\begin{array}{rccccccccc} & 0 & \longrightarrow & \overline{P} & \xrightarrow{\partial_1^{\overline{P}}} & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & \Big\downarrow{\scriptstyle \overline{\alpha f_1}} & & \Big\downarrow{\scriptstyle \tau f_0} & & \Big\downarrow{\scriptstyle \mathrm{id}_M} & & \\ \zeta = & 0 & \longrightarrow & N & \longrightarrow & T & \xrightarrow{\pi} & M & \longrightarrow & 0. \end{array} \tag{2.3.3}$$

It follows from Definition 1.13 that $\varkappa^1_{\mathcal{X}R}([\alpha]) = [\alpha f_1]$. From [24, (7.18)] one concludes $\xi_{RMN}([\alpha f_1]) = [\zeta]$, and this yields the first equality in the following sequence

$$\xi_{RMN}(\varkappa^1_{\mathcal{X}R}([\alpha])) = [\zeta] = \xi_{\mathcal{X}MN}([\alpha])$$

while the second equality is by definition. This completes the proof of part (a).

Part (b) is proved as in the previous paragraph, using a proper $\mathcal{W}$-resolution in place of the projective resolution $P \to M$. The proofs of (c) and (d) are dual. □

3 Categories of interest

In this section we discuss the categories whose relative cohomology theories are of primary interest in this paper. Each category is defined in terms of a semidualizing module, the study of which was initiated independently (with different names) by Foxby [16], Golod [19], and Vasconcelos [28].

Definition/Notation 3.1 An R-module C is *semidualizing* if it satisfies the following conditions:

(1) C admits a (possibly unbounded) resolution by finite rank free R-modules,
(2) the natural homothety map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism, and
(3) $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$.

A finitely generated projective R-module of rank 1 is semidualizing. If R is Cohen–Macaulay, then D is *dualizing* if it is semidualizing and $\mathrm{id}_R(D)$ is finite. If C is semidualizing and D is dualizing, then [11, (2.12)] says that the R-module $C^{\dagger} = \mathrm{Hom}_R(C, D)$ is semidualizing, $\mathrm{Ext}_R^{\geqslant 1}(C, D) = 0$ and $C^{\dagger\dagger} \cong C$; see also [28, (4.11)].

Based on the work of Enochs and Jenda [13], the following notions were introduced and studied in this generality by Holm and Jørgensen [21] and White [29].

Definition 3.2 Let C be a semidualizing R-module. An R-module is *C-projective* (resp., *C-flat* or *C-injective*) if it is isomorphic to a module of the form $P \otimes_R C$ for some projective R-module P (resp., $F \otimes_R C$ for some flat R-module F or $\mathrm{Hom}_R(C, I)$ for some injective R-module I). We let $\mathcal{P}_C$, $\mathcal{F}_C$ and $\mathcal{I}_C$ denote the categories of C-projective, C-flat and C-injective R-modules, respectively.

A *complete $\mathcal{P}\mathcal{P}_C$-resolution* is a complex X of R-modules satisfying the following:

(1) X is exact and $\mathrm{Hom}_R(-, \mathcal{P}_C)$-exact, and
(2) X_i is projective when $i \geqslant 0$ and X_i is C-projective when $i < 0$.

An R-module G is *G_C-projective* if there exists a complete $\mathcal{P}\mathcal{P}_C$-resolution X such that $G \cong \mathrm{Coker}(\partial_1^X)$, in which case X is a *complete $\mathcal{P}\mathcal{P}_C$-resolution of G*. We let $\mathcal{G}\mathcal{P}_C$ denote the subcategory of G_C-projective R-modules and set $\mathcal{G}\mathcal{P} = \mathcal{G}\mathcal{P}_R$. Projective R-modules and C-projective R-modules are G_C-projective.

The terms *complete $\mathcal{I}_C\mathcal{I}$-coresolution* and G_C*-injective* are defined dually, and $\mathcal{GI}_C$ is the subcategory of G_C-injective R-modules. An R-module that is injective or C-injective is G_C-injective.

Assume that R is noetherian. A *complete $\mathcal{FF}_C$-resolution* is a complex X of R-modules satisfying the following conditions:

(1) X is exact and $-\otimes_R \mathcal{I}_C$-exact, and
(2) X_i is flat when $i \geqslant 0$ and X_i is C-flat when $i < 0$.

An R-module G is G_C*-flat* if there exists a complete $\mathcal{FF}_C$-resolution X such that $G \cong \operatorname{Coker}(\partial_1^X)$, in which case X is a *complete $\mathcal{FF}_C$-resolution of G*. We let $\mathcal{GF}_C$ denote the subcategory of G_C-flat R-modules and set $\mathcal{GF} = \mathcal{GF}_R$. Flat R-modules (hence, projective R-modules) and C-flat R-modules are G_C-flat.

The G_C-flats are only used in this paper as a tool for verifying certain relations between G_C-projectives and G_C-injectives. These relations are contained in the next result which is essentially an assemblage of facts from [21].

Lemma 3.3 *Assume that R is noetherian. Let C, E and M be R-modules with C semidualizing and E faithfully injective.*

(a) *There is an inequality $\mathcal{GF}_C\text{-}\operatorname{pd}_R(M) \leq \mathcal{GP}_C\text{-}\operatorname{pd}_R(M)$, and so $\mathcal{GP}_C \subseteq \mathcal{GF}_C$.*
(b) *If M is G_C-flat, then $\operatorname{Hom}_R(M, E)$ is G_C-injective.*
(c) *If R has finite Krull dimension, then the quantities $\mathcal{GI}_C\text{-}\operatorname{id}_R(\operatorname{Hom}_R(M, E))$, $\mathcal{GP}_C\text{-}\operatorname{pd}_R(M)$ and $\mathcal{GF}_C\text{-}\operatorname{pd}_R(M)$ are simultaneously finite.*

Proof (a) Let $R \ltimes C$ denote the trivial extension of R by C and view M as an $R \ltimes C$-module via the natural surjection $R \ltimes C \to R$. In the next sequence

$$\mathcal{GF}_C\text{-}\operatorname{pd}_R(M) = \mathcal{GF}\text{-}\operatorname{pd}_{R\ltimes C}(M) \leq \mathcal{GP}\text{-}\operatorname{pd}_{R\ltimes C}(M) = \mathcal{GP}_C\text{-}\operatorname{pd}_R(M)$$

the equalities are from [21, (2.16)] and the inequality is from [10, (5.1.4)].

(b) Assume $M \in \mathcal{GF}_C$. From [21, (2.16)] we know that G is Gorenstein flat over $R \ltimes C$, and so [21, (2.15)] implies that $\operatorname{Hom}_R(M, E)$ is Gorenstein injective over $R \ltimes C$. An application of [21, (2.13.1)] implies $\operatorname{Hom}_R(M, E) \in \mathcal{GI}_C$.

(c) Set $(-)^\vee = \operatorname{Hom}_R(-, E)$. Using [21, (2.1),(2.16)] we have equalities

$$\mathcal{GI}_C\text{-}\operatorname{id}_R(M^\vee) = \mathcal{GI}\text{-}\operatorname{id}_{R\ltimes C}(M^\vee) = \mathcal{GF}\text{-}\operatorname{pd}_{R\ltimes C}(M) = \mathcal{GF}_C\text{-}\operatorname{pd}_R(M)$$

$$\mathcal{GP}_B\text{-}\operatorname{pd}_R(C) = \mathcal{GP}\text{-}\operatorname{pd}_{R\ltimes B}(C).$$

From [15, (3.4)] we conclude that $\mathcal{GF}\text{-}\operatorname{pd}_{R\ltimes C}(M)$ and $\mathcal{GP}\text{-}\operatorname{pd}_{R\ltimes C}(M)$ are simultaneously finite, and hence so are the six displayed quantities. □

The following equalities are taken from [27, (2.11)].

Fact 3.4 Let C and M be R-modules with C semidualizing.

(a) $\operatorname{pd}_R(M) = \mathcal{P}_C\text{-}\operatorname{pd}_R(C \otimes_R M)$ and $\mathcal{P}_C\text{-}\operatorname{pd}_R(M) = \operatorname{pd}_R(\operatorname{Hom}_R(C, M))$.
(b) $\mathcal{I}_C\text{-}\operatorname{id}_R(M) = \operatorname{id}_R(C \otimes_R M)$ and $\operatorname{id}_R(M) = \mathcal{I}_C\text{-}\operatorname{id}_R(\operatorname{Hom}_R(C, M))$.

Definition 3.5 Let M and N be R-modules such that $\mathcal{GP}_C\text{-}\operatorname{pd}_R(M) < \infty$ and $\mathcal{GI}_C\text{-}\operatorname{id}_R(N) < \infty$. From [29, (3.6) and its dual] there are exact sequences

$$0 \to K \to G_0 \to M \to 0 \qquad\qquad 0 \to N \to H_0 \to L \to 0$$

such that $\mathcal{P}_C\text{-}\operatorname{pd}_R(K)$ and $\mathcal{I}_C\text{-}\operatorname{id}_R(L)$ are finite, G_0 is G_C-projective, and H_0 is G_C-injective. The first exact sequence is called a *$\mathcal{GP}_C$-approximation* of M, and the second one is called a *$\mathcal{GI}_C$-coapproximation* of N.

Augmenting the $\mathcal{GP}_C$-approximation with a bounded $\mathcal{P}_C$-resolution of K yields a bounded $\mathcal{GP}_C$-resolution $G \xrightarrow{\simeq} M$ such that $G_n \in \mathcal{P}_C$ for each $n \geqslant 1$. Such a resolution is called a *bounded strict $\mathcal{GP}_C$-resolution*. Dually, N admits a *bounded strict $\mathcal{GI}_C$-coresolution*.

The next definition was first introduced by Auslander and Bridger [2,3] in the case $C = R$, and in this generality by Golod [19] and Vasconcelos [28].

Definition 3.6 Assume that R is noetherian, and let C be a semidualizing R-module. A finitely generated R-module H is *totally C-reflexive* if

(1) $\operatorname{Ext}_R^{\geqslant 1}(H, C) = 0 = \operatorname{Ext}_R^{\geqslant 1}(\operatorname{Hom}_R(H, C), C)$, and
(2) the natural biduality map $H \to \operatorname{Hom}_R(\operatorname{Hom}_R(H, C), C)$ is an isomorphism.

A finitely generated module that is projective or C-projective is totally C-reflexive. Let $\mathcal{G}_C$ denote the subcategory of totally C-reflexive R-modules and set $\mathcal{G} = \mathcal{G}_R$.

Fact 3.7 The category $\mathcal{P}_C$ is an injective cogenerator for $\mathcal{GP}_C$ by [21, (2.5),(2.13)] and [29, (2.2),(2.9)], and $\mathcal{I}_C$ is a projective generator for $\mathcal{GI}_C$ by [21, (2.6),(2.13)] and results dual to [29, (2.2),(2.9)]. Lemma 1.8 yields the relations $\mathcal{GP}_C \perp \operatorname{res}\widehat{\mathcal{P}_C}$ and $\operatorname{cores}\widehat{\mathcal{I}_C} \perp \mathcal{GI}_C$. From [21, (5.6)] there is an equality $\operatorname{cores}\widetilde{\mathcal{GI}_C} = \mathcal{M}$.

Let M and N be R-modules such that $\mathcal{GP}_C\text{-}\operatorname{pd}_R(M) < \infty$ and $\mathcal{GI}_C\text{-}\operatorname{id}_R(N) < \infty$. The proof of [29, (3.6)] shows that M admits a bounded strict $\mathcal{GP}_C$-resolution such that $G_n = 0$ for each $n > \mathcal{GP}_C\text{-}\operatorname{pd}_R(M)$, and [29, (3.4)] shows that every bounded strict $\mathcal{GP}_C$-resolution of M is $\mathcal{GP}_C$-proper and hence $\mathcal{P}_C$-proper. In particular, every bounded $\mathcal{P}_C$-resolution is $\mathcal{GP}_C$-proper and every $\mathcal{GP}_C$-approximation is $\operatorname{Hom}_R(\mathcal{GP}_C, -)$-exact. Dually, N admits a bounded strict $\mathcal{GI}_C$-coresolution $N \xrightarrow{\simeq} H$ such that $H_{-n} = 0$ for each $n > \mathcal{GI}_C\text{-}\operatorname{id}_R(M)$, every bounded strict $\mathcal{GI}_C$-coresolution is $\mathcal{GI}_C$-proper, and every bounded $\mathcal{I}_C$-coresolution is $\mathcal{GI}_C$-proper.

Assuming that R is noetherian, the equality $\mathcal{G}_C = \mathcal{GP}_C^f$ is by [29, (4.4)], and $\mathcal{P}_C^f$ is an injective cogenerator for $\mathcal{G}_C$ by [29, (2.9),(4.3),(4.4)].

Notation 3.8 We simplify our notation for the relative cohomologies

$$\operatorname{Ext}^n_{\mathcal{P}_C}(-,-) = \operatorname{Ext}^n_{\mathcal{P}_C R}(-,-) \qquad \operatorname{Ext}^n_{\mathcal{GP}_C}(-,-) = \operatorname{Ext}^n_{\mathcal{GP}_C R}(-,-)$$
$$\operatorname{Ext}^n_{\mathcal{I}_C}(-,-) = \operatorname{Ext}^n_{R\,\mathcal{I}_C}(-,-) \qquad \operatorname{Ext}^n_{\mathcal{GI}_C}(-,-) = \operatorname{Ext}^n_{R\,\mathcal{GI}_C}(-,-)$$

and for the various connecting maps from Definition 1.13

$$\begin{aligned} \vartheta^n_{\mathcal{GP}_C\mathcal{P}_C} &= \vartheta^n_{\mathcal{GP}_C\mathcal{P}_C R} & \qquad \vartheta^n_{\mathcal{GI}_C\mathcal{I}_C} &= \vartheta^n_{R\mathcal{GI}_C\mathcal{I}_C} \\ \varkappa^n_{\mathcal{GP}_C} &= \varkappa^n_{\mathcal{GP}_C R} & \varkappa^n_{\mathcal{GI}_C} &= \varkappa^n_{R\mathcal{GI}_C} \\ \varkappa^n_{\mathcal{P}_C} &= \varkappa^n_{\mathcal{P}_C R} & \varkappa^n_{\mathcal{I}_C} &= \varkappa^n_{R\mathcal{I}_C}. \end{aligned}$$

Fact 3.7 implies that each bifunctor $\operatorname{Ext}^n_{\mathcal{GI}_C}(-,-)$ is defined on $\mathcal{M} \times \mathcal{M}$.

The next properties are from [27, (4.1)].

Fact 3.9 Let C, M and N be R-modules with C semidualizing.

(a) If $M \in \operatorname{res}\widetilde{\mathcal{P}_C}$, then there is an isomorphism for each n

$$\operatorname{Ext}^n_{\mathcal{P}_C}(M, N) \cong \operatorname{Ext}^n_R(\operatorname{Hom}_R(C, M), \operatorname{Hom}_R(C, N)).$$

(b) If $N \in \operatorname{cores}\widetilde{\mathcal{I}_C}$, then there is an isomorphism for each n

$$\operatorname{Ext}^n_{\mathcal{I}_C}(M, N) \cong \operatorname{Ext}^n_R(C \otimes_R M, C \otimes_R N).$$

The following is for use in Propositions 6.1 and 6.4.

Lemma 3.10 *Assume that R is noetherian and let C, M and N be finitely generated R-modules with C semidualizing.*

(a) *A $\mathcal{P}_C$-resolution $X \xrightarrow{\simeq} M$ is proper if and only if $\operatorname{Hom}_R(C, X^+)$ is exact.*
(b) *Assume $M \in \operatorname{res}\widetilde{\mathcal{P}_C}$. Then $\operatorname{Ext}^n_{\mathcal{P}_C}(M, N) \cong \operatorname{Ext}^n_{\mathcal{P}^f_C}(M, N)$ is finitely generated and $\operatorname{Supp}(\operatorname{Ext}^n_{\mathcal{P}_C}(M, N)) \subseteq \operatorname{Supp}(M) \cap \operatorname{Supp}(N)$ for each n. Hence, if $\operatorname{Supp}(M) \cap \operatorname{Supp}(N) \subseteq \operatorname{m-Spec}(R)$, then $\operatorname{Ext}^n_{\mathcal{P}_C}(M, N)$ has finite length.*
(c) *Assume $\mathcal{GP}_C\text{-}\operatorname{pd}_R(M) < \infty$. Then $\operatorname{Ext}^n_{\mathcal{GP}_C}(M, N) \cong \operatorname{Ext}^n_{\mathcal{G}_C}(M, N)$ is finitely generated and $\operatorname{Supp}(\operatorname{Ext}^n_{\mathcal{GP}_C}(M, N)) \subseteq \operatorname{Supp}(M) \cap \operatorname{Supp}(N)$ for each n. If $\operatorname{Supp}(M) \cap \operatorname{Supp}(N) \subseteq \operatorname{m-Spec}(R)$, then $\operatorname{Ext}^n_{\mathcal{GP}_C}(M, N)$ has finite length.*

Proof (a) This is immediate from Hom-tensor adjointness.

(b) The proof of [22, (5.3.b)] shows that $\mathcal{P}^f_C$ is precovering for the category of finitely generated R-modules. In other words, there is an R-module homomorphism $\tau : X_0 \to M$ such that $X_0 \in \mathcal{P}^f_C$ and the sequence

$$X_0 \xrightarrow{\tau} M \to 0 \tag{3.10.1}$$

is $\operatorname{Hom}_R(\mathcal{P}^f_C, -)$-exact. In particular, this sequence is $\operatorname{Hom}_R(C, -)$-exact. Since M admits a proper $\mathcal{P}_C$-resolution, the map τ is surjective. It follows from part (a) that the sequence (3.10.1) is $\operatorname{Hom}_R(\mathcal{P}_C, -)$-exact. Using [27, (2.3.a)], we conclude that $\operatorname{Ker}(\tau)$ has a proper $\mathcal{P}_C$-resolution. Inductively, this process yields a $\mathcal{P}^f_C$-resolution $X \xrightarrow{\simeq} M$ that is $\mathcal{P}_C$-proper. This yields the isomorphism $\operatorname{Ext}^n_{\mathcal{P}_C}(M, N) \cong \operatorname{Ext}^n_{\mathcal{P}^f_C}(M, N)$ and the finite generation of $\operatorname{Ext}^n_{\mathcal{P}_C}(M, N)$.

Fix a prime $\mathfrak{p} \in \operatorname{Spec}(R)$. It is straightforward to check that the localization $C_\mathfrak{p}$ is $R_\mathfrak{p}$-semidualizing and, using part (a), that the $\mathcal{P}_{C_\mathfrak{p}}$-resolution $X_\mathfrak{p} \xrightarrow{\simeq} M_\mathfrak{p}$ is proper. This yields an isomorphism $\operatorname{Ext}^n_{\mathcal{P}_C}(M, N)_\mathfrak{p} \cong \operatorname{Ext}^n_{\mathcal{P}_{C_\mathfrak{p}}}(M_\mathfrak{p}, N_\mathfrak{p})$ for each integer n. If $\mathfrak{p} \notin \operatorname{Supp}(M)$, then the complex $X_\mathfrak{p}$ is exact and hence split-exact by [22, Prop. 5.2]; it follows easily that $\operatorname{Ext}^n_{\mathcal{P}_C}(M, N)_\mathfrak{p} \cong \operatorname{Ext}^n_{\mathcal{P}_{C_\mathfrak{p}}}(M_\mathfrak{p}, N_\mathfrak{p}) = 0$. If $\mathfrak{p} \notin \operatorname{Supp}(N)$, then one derives the same vanishing.

(c) Using [29, (4.7)], the assumption $\mathcal{GP}_C\text{-}\operatorname{pd}_R(M) < \infty$ implies that M admits a bounded strict $\mathcal{GP}_C$-resolution G such that G_n is finitely generated for each $n \geqslant 0$. It follows that the localized complex $G_\mathfrak{p}$ is a bounded strict $\mathcal{GP}_{C_\mathfrak{p}}$-resolution of $M_\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec}(R)$, and hence it is a proper $\mathcal{GP}_{C_\mathfrak{p}}$-resolution by Fact 3.7. Furthermore, if $M_\mathfrak{p} = 0$, then $G_\mathfrak{p}$ is a bounded augmented $\mathcal{P}_{C_\mathfrak{p}}$-resolution of $(G_0)_\mathfrak{p}$, and it follows from [22, Prop. 5.2] that $G_\mathfrak{p}$ is split-exact. The proof now concludes as in part (b). □

Over a noetherian ring, the next categories were introduced by Avramov and Foxby [6] when C is dualizing, and by Christensen [11] for arbitrary C. (Note that these works (and others) use the notation $\mathcal{A}_C$ and $\mathcal{B}_C$ for certain categories of complexes, while our categories consist precisely of the modules in these other categories.) In the non-noetherian setting, these definitions are from [22,29].

Definition 3.11 Let C be a semidualizing R-module. The *Auslander class* of C is the subcategory $\mathcal{A}_C$ of R-modules M such that

(1) $\operatorname{Tor}^R_{\geqslant 1}(C, M) = 0 = \operatorname{Ext}^{\geqslant 1}_R(C, C \otimes_R M)$, and
(2) The natural map $M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism.

The *Bass class* of C is the subcategory $\mathcal{B}_C$ of R-modules M such that

(1) $\operatorname{Ext}^{\geqslant 1}_R(C, M) = 0 = \operatorname{Tor}^R_{\geqslant 1}(C, \operatorname{Hom}_R(C, M))$, and
(2) The natural evaluation map $C \otimes_R \operatorname{Hom}_R(C, M) \to M$ is an isomorphism.

Fact 3.12 Let C be a semidualizing R-module, and set $\mathcal{G}(\mathcal{P}_C) = \mathcal{GP}_C \cap \mathcal{B}_C$ and $\mathcal{G}(\mathcal{I}_C) = \mathcal{GI}_C \cap \mathcal{A}_C$. The category $\mathcal{P}_C$ is an injective cogenerator and a projective generator for $\mathcal{G}(\mathcal{P}_C)$ by [26, (5.3)]. Dually, the category $\mathcal{I}_C$ is an injective cogenerator and a projective generator for $\mathcal{G}(\mathcal{I}_C)$ by [26, (5.4)].

Assume that R is noetherian and set $\mathcal{G}(\mathcal{P}^f_C) = \mathcal{G}_C \cap \mathcal{B}_C = \mathcal{G}(\mathcal{P}_C)^f$. The category $\mathcal{P}^f_C$ is an injective cogenerator and a projective generator for $\mathcal{G}(\mathcal{P}^f_C)$ by [26, (5.5)]. If R is Cohen–Macaulay with a dualizing module, then there are containments $\mathcal{GP}_C \subseteq \mathcal{A}_{C^\dagger}$ and $\mathcal{GI}_C \subseteq \mathcal{B}_{C^\dagger}$ by [21, (4.6)], and we conclude $\mathcal{G}(\mathcal{P}^f_C) \subseteq \mathcal{G}(\mathcal{P}_C) \subseteq \mathcal{A}_{C^\dagger} \cap \mathcal{B}_C$ and $\mathcal{G}(\mathcal{I}_C) \subseteq \mathcal{B}_{C^\dagger} \cap \mathcal{A}_C$.

Fact 3.13 Let C be a semidualizing R-module. If any two R-modules in a short exact sequence are in $\mathcal{A}_C$, respectively $\mathcal{B}_C$, then so is the third; see [22, Cor. 6.3]. The class $\mathcal{A}_C$ contains all modules of finite projective dimension and those of finite $\mathcal{I}_C$-injective dimension, and the class $\mathcal{B}_C$ contains all modules of finite injective dimension and those of finite $\mathcal{P}_C$-projective dimension by [22, Cor. 6.1]. If M is in $\mathcal{B}_C$, then M admits a proper $\mathcal{P}_C$-resolution; if M is in $\mathcal{A}_C$, then M admits a proper $\mathcal{I}_C$-injective coresolution; see [27, (2.3),(2.4)].

Using the containment $\mathcal{G}(\mathcal{P}_C) \subseteq \mathcal{B}_C$ and a $\mathcal{GP}_C$-approximation, one checks readily that $\mathcal{G}(\mathcal{P}_C)\text{-}\operatorname{pd}_R(M)$ is finite if and only if $\mathcal{GP}_C\text{-}\operatorname{pd}_R(M)$ is finite and M is in $\mathcal{B}_C$. Consequently, if $\mathcal{G}(\mathcal{P}_C)\text{-}\operatorname{pd}_R(M)$ is finite (e.g., if $\mathcal{P}_C\text{-}\operatorname{pd}_R(M)$ is finite), then M admits a proper $\mathcal{P}_C$-resolution, and $\mathcal{G}(\mathcal{P}_C)\text{-}\operatorname{pd}_R(M) = \mathcal{GP}_C\text{-}\operatorname{pd}_R(M)$.

Dually, $\mathcal{G}(\mathcal{I}_C)\text{-}\operatorname{id}_R(M)$ is finite if and only if $\mathcal{GI}_C\text{-}\operatorname{id}_R(M)$ is finite and M is in $\mathcal{A}_C$. If $\mathcal{G}(\mathcal{I}_C)\text{-}\operatorname{id}_R(M)$ is finite (e.g., if $\mathcal{I}_C\text{-}\operatorname{id}_R(M)$ is finite), then M admits a proper $\mathcal{I}_C$-coresolution, and $\mathcal{G}(\mathcal{I}_C)\text{-}\operatorname{id}_R(M) = \mathcal{GI}_C\text{-}\operatorname{id}_R(M)$.

If R is Cohen–Macaulay with a dualizing module, then Fact 3.12 yields

$$\operatorname{res}\widehat{\mathcal{P}_C} \subseteq \operatorname{res}\widehat{\mathcal{G}(\mathcal{P}_C)} \subseteq \mathcal{A}_{C^\dagger} \cap \mathcal{B}_C \supseteq \operatorname{cores}\widehat{\mathcal{G}(\mathcal{I}_{C^\dagger})} \supseteq \operatorname{cores}\widehat{\mathcal{I}_{C^\dagger}}.$$

The following relations between semidualizing modules are for use in Sect. 6.

Lemma 3.14 *Assume that R is noetherian, and let B and C be semidualizing R-modules. The following conditions are equivalent.*

(i) $\mathcal{GP}_C\text{-}\operatorname{pd}_R(B)$ *is finite.*
(ii) B *is totally C-reflexive.*
(iii) $\operatorname{Ext}^{\geqslant 1}_R(B, C) = 0$ *and* $\operatorname{Hom}_R(B, C)$ *is R-semidualizing.*
(iv) C *is in* $\mathcal{B}_B$.

Proof (i) $\Longrightarrow$ (ii) If $\mathcal{GP}_C\text{-}\operatorname{pd}_R(B)$ is finite then $\mathrm{G}_C\text{-}\dim_R(B) < \infty$ and so [17, (3.1)] provides the equality $\mathrm{G}_C\text{-}\dim_R(B) = 0$ and hence the desired conclusion.

(ii) $\Longrightarrow$ (iii) This is in [11, (2.11)].

(iii) $\Longrightarrow$ (iv) Let $P \xrightarrow{\simeq} B$ and $C \xrightarrow{\simeq} I$ be projective and injective resolutions, respectively. The condition $\operatorname{Ext}^{\geqslant 1}_R(B, C) = 0$ implies that $\operatorname{Hom}_R(P, I)$ is an injective resolution of $\mathrm{Hom}_R(B, C)$. Consider the next commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{X'} & \mathrm{Hom}_R(\mathrm{Hom}_R(P, I), \mathrm{Hom}_R(P, I)) \\ {\scriptstyle X}\downarrow\simeq & & {\scriptstyle \Phi}\downarrow\cong \\ \mathrm{Hom}_R(I, I) & \xrightarrow{\mathrm{Hom}_R(\Omega, I)} & \mathrm{Hom}_R(P \otimes_R \mathrm{Hom}_R(P, I), I), \end{array} \tag{3.14.1}$$

where X and X' are the homothety homomorphisms, Φ is Hom-tensor adjunction, and Ω is tensor-evaluation. Our assumptions imply that X' is a quasiisomorphism, and so the same is true of $\mathrm{Hom}_R(\Omega, I)$. Using [10, (A.8.11)] we conclude that Ω is also a quasiisomorphism; this uses the equality $\mathrm{Supp}_R(C) = \mathrm{Spec}(R)$ which holds because C is semidualizing. In particular, we have

$$\mathrm{Tor}_n^R(B, \mathrm{Hom}_R(B, C)) \cong \mathrm{H}_n(\mathrm{Hom}_R(P, I) \otimes_R P) \cong \mathrm{H}_n(C),$$

which is 0 when $n \geqslant 1$. The isomorphism $\mathrm{H}_0(\Omega)$ is exactly the natural evaluation map $B \otimes_R \mathrm{Hom}_R(B, C) \to C$, and so we have $C \in \mathcal{B}_B$.

(iv) $\Longrightarrow$ (iii) Assume that C is in $\mathcal{B}_B$ and employ the notation from the previous paragraph. It follows that the morphism Ω is a quasiisomorphism, and hence so is X'. This implies that $\mathrm{Hom}_R(B, C)$ is semidualizing. The Bass class conditions then conspire with [17, (3.1.c)] to imply $\mathcal{GP}_C\text{-}\mathrm{pd}_R(B) < \infty$. □

Fact 3.15 If B and C be semidualizing R-modules such that $\mathcal{GP}_C\text{-}\mathrm{pd}_R(B)$ is finite, then there is a containment $\mathcal{P}_B \subseteq \mathcal{GP}_C$, and C admits a proper $\mathcal{P}_B$-resolution by Fact 3.13 and Lemma 3.14. For example, the semidualizing module $B = R$ is always totally C-reflexive; if R is Cohen–Macaulay and C is dualizing, then B is totally C-reflexive. For discussions of methods for generating other nonisomorphic semidualizing modules B and C such that $\mathcal{GP}_C\text{-}\mathrm{pd}_R(B) < \infty$, the interested reader is encouraged to peruse [17,18,25].

Lemma 3.16 *Assume that* $(R, \mathfrak{m}, k)$ *is local and let* B *and* C *be semidualizing* R*-modules with* $\mathcal{GP}_C\text{-}\mathrm{pd}_R(B) < \infty$. *Let* E *be the* R*-injective hull of* k, *and set* $(-)^\vee = \mathrm{Hom}_R(-, E)$. *The following conditions are equivalent.*

(i) $B \cong C$.

(ii) $\mathcal{P}_B\text{-}\mathrm{pd}_R(C) < \infty$.

(iii) $\mathcal{GP}_B\text{-}\mathrm{pd}_R(C) < \infty$.

(iv) $\mathrm{pd}_R(\mathrm{Hom}_R(B, C)) < \infty$.

(v) $\mathcal{P}_C\text{-}\mathrm{pd}_R(B) < \infty$.

(vi) $\mathcal{I}_B\text{-}\mathrm{id}_R(C^\vee) < \infty$.

(vii) $\mathcal{GI}_B\text{-}\mathrm{id}_R(C^\vee) < \infty$.

(viii) $\mathcal{I}_C\text{-}\mathrm{id}_R(B^\vee) < \infty$.

Proof The implication (i) $\Longrightarrow$ (n) is straightforward for $n =$ ii, ..., viii, as are (ii) $\Longrightarrow$ (iii) and (vi) $\Longrightarrow$ (vii). The implication (iii) $\Longrightarrow$ (i) is in [1, (5.3)], and (ii) $\Longleftrightarrow$ (iv) is from Fact 3.4 (a), while (vii) $\Longleftrightarrow$ (iii) is in Lemma 3.3 (c).

(v) $\Longrightarrow$ (iii) If $\mathcal{P}_C\text{-}\mathrm{pd}_R(B) < \infty$, then B is in $\mathcal{B}_C$ and so Lemma 3.14 implies $\mathcal{GP}_B\text{-}\mathrm{pd}_R(C) < \infty$.

(viii) $\Longrightarrow$ (v) Assume $\mathcal{I}_C\text{-}\mathrm{id}_R(B^\vee) < \infty$. Hom-evaluation yields an isomorphism

$$\mathrm{Hom}_R(C, B)^\vee \cong C \otimes_R (B^\vee)$$

and hence the first equality in the following sequence

$$\mathrm{id}_R(\mathrm{Hom}_R(C, B)^\vee) = \mathrm{id}_R(C \otimes_R (B^\vee)) = \mathcal{I}_C\text{-}\mathrm{id}_R(B^\vee) < \infty.$$

The second equality is by Fact 3.4 (b). It follows that $\mathrm{Hom}_R(C, B)$ has finite projective dimension and so Fact 3.4 (a) implies $\mathcal{P}_C\text{-}\mathrm{pd}_R(B) < \infty$. □

4 Comparison isomorphisms

The results of this section document situations where different relative cohomology theories agree. The notation for the comparison homomorphisms is given in 3.8. The next result is a more precise version of [27, (4.2)].

Proposition 4.1 *Let C, M and N be R-modules with C semidualizing.*

(a) *If M and N are in $\mathcal{B}_C$, then the natural map*

$$\varkappa^n_{\mathcal{P}_C}(M,N)\colon\ \operatorname{Ext}^n_{\mathcal{P}_C}(M,N)\to\operatorname{Ext}^n_R(M,N)$$

is an isomorphism for each integer n.

(b) *If M and N are in $\mathcal{A}_C$, then the natural map*

$$\varkappa^n_{\mathcal{I}_C}(M,N)\colon\ \operatorname{Ext}^n_{\mathcal{I}_C}(M,N)\to\operatorname{Ext}^n_R(M,N)$$

is an isomorphism for each integer n.

Proof (a) Let $P\xrightarrow{\gamma}\operatorname{Hom}_R(C,M)$ and $P'\xrightarrow{\gamma'}C$ be projective resolutions. Because M is in $\mathcal{B}_C$, we have $\operatorname{Tor}^R_{\geqslant 1}(C,\operatorname{Hom}_R(C,M))=0$ and so the complex $P'\otimes_R P$ is a projective resolution of $C\otimes_R\operatorname{Hom}_R(C,M)\cong M$, and the complex $C\otimes_R P$ is a $\mathcal{P}_C$-resolution of M. The following diagram commutes

$$\begin{array}{ccc} P'\otimes_R P & \xrightarrow{\gamma'\otimes_R P} & C\otimes_R P \\ \simeq\downarrow & & \downarrow\simeq \\ M & \xrightarrow{\operatorname{id}_M} & M \end{array}$$

and so it suffices to show that the induced map

$$\operatorname{Hom}_R(P'\otimes_R P,N)\xrightarrow{\operatorname{Hom}_R(\gamma'\otimes_R P,N)}\operatorname{Hom}_R(C\otimes_R P,N)$$

is a quasiisomorphism. The following standard isomorphisms

$$\begin{aligned}\operatorname{Cone}(\operatorname{Hom}_R(\gamma'\otimes_R P,N)) &\cong \Sigma\operatorname{Hom}_R(\operatorname{Cone}(\gamma'\otimes_R P),N)\\ &\cong \Sigma\operatorname{Hom}_R(\operatorname{Cone}(\gamma')\otimes_R P,N)\end{aligned}$$

imply that it suffices to show that the complex $\operatorname{Hom}_R(\operatorname{Cone}(\gamma')\otimes_R P,N)$ is exact.

Observe that $\operatorname{Cone}(\gamma')$ is exact and bounded below and each module $\operatorname{Cone}(\gamma')_n$ is a direct sum of a projective R-module and a C-projective R-module. Since N is in $\mathcal{B}_C$, we know that $\operatorname{Ext}^{\geqslant 1}_R(C,N)=0$, and it follows that $\operatorname{Ext}^{\geqslant 1}_R(Q\otimes_R C,N)=0$ for each projective R-module Q. Since we also have $\operatorname{Ext}^{\geqslant 1}_R(Q,N)=0$, it follows that $\operatorname{Ext}^{\geqslant 1}_R(\operatorname{Cone}(\gamma')_n,N)=0$ for each n. Breaking up $\operatorname{Cone}(\gamma')$ into short exact sequences and applying $\operatorname{Hom}_R(-,N)$ to each piece yields the desired conclusion.

The proof of (b) is dual. □

The next result compares to [7, (4.2.3)].

Proposition 4.2 *Let C and M be R-modules with C semidualizing.*

(a) *If* $\mathcal{P}_C\text{-}\operatorname{pd}_R(M)$ *is finite, then the following natural transformations are isomorphisms for each* n

$$\vartheta^n_{\mathcal{GP}_C\mathcal{P}_C}(M,-)\colon \operatorname{Ext}^n_{\mathcal{GP}_C}(M,-) \xrightarrow{\cong} \operatorname{Ext}^n_{\mathcal{P}_C}(M,-)$$

and so $\operatorname{Ext}^n_{\mathcal{GP}_C}(M,-)=0$ *for each* $n>\mathcal{P}_C\text{-}\operatorname{pd}(M)$.

(b) *If* $\mathcal{I}_C\text{-}\operatorname{id}_R(M)$ *is finite, then the following natural transformations are isomorphisms for each* n

$$\vartheta^n_{\mathcal{GI}_C\mathcal{I}_C}(-,M)\colon \operatorname{Ext}^n_{\mathcal{GI}_C}(-,M) \xrightarrow{\cong} \operatorname{Ext}^n_{\mathcal{I}_C}(-,M)$$

and so $\operatorname{Ext}^n_{\mathcal{GI}_C}(-,M)=0$ *for each* $n>\mathcal{I}_C\text{-}\operatorname{id}(M)$.

Proof We prove part (a); the proof of (b) is dual. Let $W \xrightarrow{\simeq} M$ be a $\mathcal{P}_C$-resolution such that $W_n=0$ for each $n>\mathcal{P}_C\text{-}\operatorname{pd}_R(M)$. The resolution W is $\mathcal{GP}_C$-proper and $\mathcal{P}_C$-proper by Fact 3.7, so both $\operatorname{Ext}^n_{\mathcal{GP}_C}(M,-)$ and $\operatorname{Ext}^n_{\mathcal{P}_C}(M,-)$ are defined. Further, in the notation of Definition 1.13, we can take $\overline{\operatorname{id}_M}=\operatorname{id}_W$, and so the natural isomorphisms follow from the next equalities

$$\vartheta^n_{\mathcal{XW}R}(M,-)=\operatorname{H}_{-n}(\operatorname{Hom}_R(\operatorname{id}_W,-))=\operatorname{id}_{\operatorname{H}_{-n}(\operatorname{Hom}_R(W,-))}.$$

The vanishing conclusion follows readily since $W_n=0$ for each $n>\mathcal{W}\text{-}\operatorname{pd}(M)$. □

The next lemma is a tool for the proofs of Propositions 4.4 and 4.5. Note that we do not assume that the complexes satisfy any properness conditions.

Lemma 4.3 *Let* C, M, *and* N *be* R*-modules with* C *semidualizing.*

(a) *Let* $\alpha\colon G\to G'$ *be a quasiisomorphism between bounded below complexes in* $\mathcal{GP}_C$. *If* $\mathcal{P}_C\text{-}\operatorname{pd}(N)<\infty$, *then the morphism* $\operatorname{Hom}_R(\alpha,N)\colon \operatorname{Hom}_R(G',N)\to\operatorname{Hom}_R(G,N)$ *is a quasiisomorphism.*

(b) *Let* $\beta\colon H\to H'$ *be a quasiisomorphism between bounded above complexes in* $\mathcal{GI}_C$. *If* $\mathcal{I}_C\text{-}\operatorname{id}(M)<\infty$, *then the morphism* $\operatorname{Hom}_R(M,\beta)\colon \operatorname{Hom}_R(M,H)\to\operatorname{Hom}_R(M,H')$ *is a quasiisomorphism.*

Proof We prove part (a); the proof of part (b) is dual.

It suffices to show that $\operatorname{Cone}(\operatorname{Hom}_R(\alpha,N))$ is exact. From the next isomorphism

$$\operatorname{Cone}(\operatorname{Hom}_R(\alpha,N))\cong\Sigma\operatorname{Hom}_R(\operatorname{Cone}(\alpha),N)$$

we need to show that $\operatorname{Hom}_R(\operatorname{Cone}(\alpha),N)$ is exact. Note that $\operatorname{Cone}(\alpha)$ is an exact, bounded below complex in $\mathcal{GP}_C$. Set $M_j=\operatorname{Ker}(\partial^{\operatorname{Cone}(\alpha)}_j)$ for each integer j, and note $M_{j-1}\in\mathcal{GP}_C$ for $j\ll 0$. Consider the exact sequences

$$0\to M_j\to\operatorname{Cone}(\alpha)_j\to M_{j-1}\to 0. \qquad (*_j)$$

Lemma 1.8 and Fact 3.7 imply $\mathcal{GP}_C\perp N$. Hence, induction on j using Lemma 1.7 (a) implies $\operatorname{Ext}^{\geqslant 1}_R(M_j,N)=0$ for each j and so each sequence $(*_j)$ is $\operatorname{Hom}_R(-,N)$-exact. It follows that $\operatorname{Hom}_R(\operatorname{Cone}(\alpha),N)$ is exact. □

The next two results compare to [7, (4.2.4)]. Notice that the condition M is in $\operatorname{res}\widetilde{\mathcal{GP}_C}\cap\operatorname{res}\widetilde{\mathcal{P}_C}$ of Proposition 4.4 (a) is satisfied when $\mathcal{G}(\mathcal{P}_C)\text{-}\operatorname{pd}_R(M)<\infty$; see Fact 3.12. Also, part (b) uses the equality $\operatorname{cores}\widetilde{\mathcal{GI}_C}=\mathcal{M}$ from Fact 3.7.

Proposition 4.4 *Let C, M, and N be R-modules with C semidualizing.*

(a) *If M is in $\operatorname{res}\widetilde{\mathcal{GP}_C} \cap \operatorname{res}\widetilde{\mathcal{P}_C}$ and N is in $\operatorname{res}\widehat{\mathcal{P}_C}$, then the following natural map is an isomorphism for each n*

$$\vartheta^n_{\mathcal{GP}_C\mathcal{P}_C}(M,N)\colon \operatorname{Ext}^n_{\mathcal{GP}_C}(M,N) \xrightarrow{\cong} \operatorname{Ext}^n_{\mathcal{P}_C}(M,N).$$

(b) *If M is in $\operatorname{cores}\widehat{\mathcal{I}_C}$ and N is in $\operatorname{cores}\widetilde{\mathcal{I}_C}$, then the following natural map is an isomorphism for each n*

$$\vartheta^n_{\mathcal{GI}_C\mathcal{I}_C}(M,N)\colon \operatorname{Ext}^n_{\mathcal{GI}_C}(M,N) \xrightarrow{\cong} \operatorname{Ext}^n_{\mathcal{I}_C}(M,N).$$

Proof We prove part (a); the proof of part (b) is dual.

The module M has a proper $\mathcal{P}_C$-resolution $\gamma\colon W \to M$ and a proper $\mathcal{GP}_C$-resolution $\gamma'\colon G \to M$. Lemma 1.9 (a) yields a quasiisomorphism $\overline{\mathrm{id}_M}\colon W \to G$ such that $\gamma = \gamma'\overline{\mathrm{id}_M}$, and Lemma 4.3 (a) implies that $\operatorname{Hom}_R(\overline{\mathrm{id}_M}, N)$ is a quasiisomorphism. The result now follows from the definition of $\vartheta^n_{\mathcal{GP}_C\mathcal{P}_C}(M,N)$. □

The next result is proved like Proposition 4.4 using the containment $\mathcal{P} \subseteq \mathcal{GP}_C$.

Proposition 4.5 *Let C, M, and N be R-modules with C semidualizing.*

(a) *If M is in $\operatorname{res}\widetilde{\mathcal{GP}_C}$ and N is in $\operatorname{res}\widehat{\mathcal{P}_C}$, then the following natural map is an isomorphism for each n*

$$\varkappa^n_{\mathcal{GP}_C}(M,N)\colon \operatorname{Ext}^n_{\mathcal{GP}_C}(M,N) \xrightarrow{\cong} \operatorname{Ext}^n_R(M,N).$$

(b) *If M is in $\operatorname{cores}\widehat{\mathcal{I}_C}$, then the next natural map is an isomorphism for each n*

$$\varkappa^n_{\mathcal{GI}_C}(M,N)\colon \operatorname{Ext}^n_{\mathcal{GI}_C}(M,N) \xrightarrow{\cong} \operatorname{Ext}^n_R(M,N).$$

The next four lemmata are tools for the proofs of Propositions 4.10 and 4.11 and for Theorem 5.7. Part (a) of the first one is a consequence of Proposition 4.1; parts (b) and (c) follow from part (a). Note that Fact 3.13 gives conditions guaranteeing that $M, N \in \mathcal{B}_C \cap \mathcal{A}_{C^\dagger}$.

Lemma 4.6 *Let R be a Cohen–Macaulay ring with dualizing module, and let C, M and N be R-modules with C semidualizing.*

(a) *If $M, N \in \mathcal{B}_C \cap \mathcal{A}_{C^\dagger}$, then the natural map*

$$\operatorname{Ext}^n_{\mathcal{P}_C}(M,N) \xrightarrow[\cong]{\varkappa^n_{\mathcal{P}_C}(M,N)} \operatorname{Ext}^n_R(M,N) \xleftarrow[\cong]{\varkappa^n_{\mathcal{I}_{C^\dagger}}(M,N)} \operatorname{Ext}^n_{\mathcal{I}_{C^\dagger}}(M,N)$$

is an isomorphism for each n.

(b) *If $M \in \mathcal{B}_C \cap \mathcal{A}_{C^\dagger}$ and $N \in \mathcal{I}_{C^\dagger}$, then $\operatorname{Ext}^{\geqslant 1}_{\mathcal{P}_C}(M,N) = 0 = \operatorname{Ext}^{\geqslant 1}_R(M,N)$.*

(c) *If $M \in \mathcal{P}_C$ and $N \in \mathcal{B}_C \cap \mathcal{A}_{C^\dagger}$, then $\operatorname{Ext}^{\geqslant 1}_{\mathcal{I}_{C^\dagger}}(M,N) = 0 = \operatorname{Ext}^{\geqslant 1}_R(M,N)$.*

Lemma 4.7 *Let C be a semidualizing R-module. One has $\mathcal{A}_C \perp \mathcal{I}_C$ and $\mathcal{P}_C \perp \mathcal{B}_C$. If R is Cohen–Macaulay and has a dualizing module, then $\mathcal{GP}_C \perp \mathcal{I}_{C^\dagger}$ and $\mathcal{P}_C \perp \mathcal{GI}_{C^\dagger}$, and so $\mathcal{P}_C \perp \mathcal{I}_{C^\dagger}$.*

Proof We verify the first orthogonality condition; the second one is verified similarly, and the others follow immediately from the containments $\mathcal{P}_C \subseteq \mathcal{GP}_C \subseteq \mathcal{A}_{C^\dagger}$ and $\mathcal{I}_{C^\dagger} \subseteq \mathcal{GI}_{C^\dagger} \subseteq \mathcal{B}_C$; see Fact 3.13. Let $M \in \mathcal{A}_C$ and $N \in \mathcal{I}_C \subseteq \mathcal{A}_C$. For each $n \geqslant 1$, the isomorphism in the following sequence is in Proposition 4.1 (b)

$$\operatorname{Ext}^n_R(M, N) \cong \operatorname{Ext}^n_{\mathcal{I}_C}(M, N) = 0$$

and the vanishing holds because $N \in \mathcal{I}_C$. □

Lemma 4.8 *If C is a semidualizing R-module, then one has $\mathcal{GP}_C \perp \operatorname{cores}\widehat{\mathcal{I}}$ and $\operatorname{res}\widehat{\mathcal{P}} \perp \mathcal{GI}_C$.*

Proof We verify the first orthogonality condition; the second one is verified similarly. Fix modules $G_0 \in \mathcal{GP}_C$ and $N \in \operatorname{cores}\widehat{\mathcal{I}}$ and set $j = \operatorname{id}_R(N) < \infty$. For each $n \geqslant 0$ use the fact that $\mathcal{P}_C$ is a cogenerator for $\mathcal{GP}_C$ to find exact sequences

$$0 \to G_n \to W_n \to G_{n+1} \to 0 \tag{$*_n$}$$

with $G_{n+1} \in \mathcal{GP}_C$ and $W_n \in \mathcal{P}_C$; see Fact 3.7. From Fact 3.13 we know $N \in \mathcal{B}_C$ and so Lemma 4.7 implies $\mathcal{P}_C \perp N$. Hence, for $i > 0$ the long exact sequences in $\operatorname{Ext}_R(-, N)$ associated to $(*_n)$ yield the isomorphism in the following sequence

$$\operatorname{Ext}^i_R(G_0, N) \cong \operatorname{Ext}^{i+j}_R(G_j, N) = 0$$

while the vanishing holds because $i + j > j = \operatorname{id}_R(N)$. □

Lemma 4.9 *Let C, M, and N be R-modules with C semidualizing. Assume that* $\operatorname{id}_R(N) < \infty$ *and* $\operatorname{pd}_R(M) < \infty$.

(a) *If $\alpha\colon G \to G'$ is a quasiisomorphism between bounded below complexes in $\mathcal{GP}_C$, then $\operatorname{Hom}_R(\alpha, N)\colon \operatorname{Hom}_R(G', N) \to \operatorname{Hom}_R(G, N)$ is a quasiisomorphism.*
(b) *If $\beta\colon H \to H'$ is a quasiisomorphism between bounded above complexes in $\mathcal{GI}_C$, then $\operatorname{Hom}_R(M, \beta)\colon \operatorname{Hom}_R(M, H) \to \operatorname{Hom}_R(M, H')$ is a quasiisomorphism.*

Proof We prove part (a); the proof of part (b) is dual. Set $M_j = \operatorname{Ker}(\partial^{\operatorname{Cone}(\alpha)}_j)$ for each j and consider the following exact sequences

$$0 \to M_j \to \operatorname{Cone}(\alpha)_j \to M_{j-1} \to 0. \tag{4.9.1}$$

Because $\operatorname{Cone}(\alpha)$ is an exact bounded below complex in $\mathcal{GP}_C$, we know $M_j \in \mathcal{GP}_C$ for $j \ll 0$. From [29, (2.8)] we know that $\mathcal{GP}_C$ is closed under kernels of epimorphisms, so an induction argument using (4.9.1) implies $M_j \in \mathcal{GP}_C$ for all j. Thus, Lemma 4.8 yields $M_j \perp N$ and $\operatorname{Cone}(\alpha)_j \perp N$ for all j. The long exact sequence in $\operatorname{Ext}_R(-, N)$ shows that (4.9.1) is $\operatorname{Hom}_R(-, N)$-exact, and the conclusion now follows as in the proof of Lemma 4.3. □

The next two results follow from Lemma 4.9 in the same way that Propositions 4.4 and 4.5 follow from Lemma 4.3.

Proposition 4.10 *Let C, M, and N be R-modules with C semidualizing.*

(a) *If M is in $\operatorname{res}\widetilde{\mathcal{GP}_C} \cap \operatorname{res}\widetilde{\mathcal{P}_C}$ and N is in $\operatorname{cores}\widehat{\mathcal{I}}$, then the following natural map is an isomorphism for each n*

$$\vartheta^n_{\mathcal{GP}_C\mathcal{P}_C}(M, N)\colon \operatorname{Ext}^n_{\mathcal{GP}_C}(M, N) \xrightarrow{\cong} \operatorname{Ext}^n_{\mathcal{P}_C}(M, N).$$

(b) *If M is in* $\operatorname{res}\widehat{\mathcal{P}}$ *and N is in* $\operatorname{cores}\widetilde{\mathcal{I}_C}$*, then the following natural map is an isomorphism for each n*

$$\vartheta^n_{\mathcal{GI}_C\mathcal{I}_C}(M,N)\colon\ \operatorname{Ext}^n_{\mathcal{GI}_C}(M,N)\xrightarrow{\cong}\operatorname{Ext}^n_{\mathcal{I}_C}(M,N).$$

Proposition 4.11 *Let C, M, and N be R-modules with C semidualizing.*

(a) *If M is in* $\operatorname{res}\widetilde{\mathcal{GP}_C}$ *and N is in* $\operatorname{cores}\widehat{\mathcal{I}}$*, then the following natural map is an isomorphism for each n*

$$\varkappa^n_{\mathcal{GP}_C}(M,N)\colon\ \operatorname{Ext}^n_{\mathcal{GP}_C}(M,N)\xrightarrow{\cong}\operatorname{Ext}^n_R(M,N).$$

(b) *If M is in* $\operatorname{res}\widehat{\mathcal{P}}$*, then the following natural map is an isomorphism for each n*

$$\varkappa^n_{\mathcal{GI}_C}(M,N)\colon\ \operatorname{Ext}^n_{\mathcal{GI}_C}(M,N)\xrightarrow{\cong}\operatorname{Ext}^n_R(M,N).$$

5 Balance for relative cohomology

This section focuses on balance for the functors $\operatorname{Ext}^n_{\mathcal{P}_C}(-,-)$ and $\operatorname{Ext}^n_{\mathcal{I}_B}(-,-)$, and for $\operatorname{Ext}^n_{\mathcal{GP}_C}(-,-)$ and $\operatorname{Ext}^n_{\mathcal{GI}_B}(-,-)$.

Definition 5.1 Fix subcategories $\mathcal{X}'\subseteq\operatorname{res}\widetilde{\mathcal{X}}$ and $\mathcal{Y}'\subseteq\operatorname{cores}\widetilde{\mathcal{Y}}$. We say that $\operatorname{Ext}_{\mathcal{X}R}$ and $\operatorname{Ext}_{R\mathcal{Y}}$ are *balanced* on $\mathcal{X}'\times\mathcal{Y}'$ when the following condition holds: for each object M in $\mathcal{X}'$ and N in $\mathcal{Y}'$, if $X\xrightarrow{\simeq}M$ is a proper $\mathcal{X}$-resolution, and $N\xrightarrow{\simeq}Y$ a proper $\mathcal{Y}$-coresolution, then the induced morphisms of complexes

$$\operatorname{Hom}_R(M,Y)\to\operatorname{Hom}_R(X,Y)\leftarrow\operatorname{Hom}_R(X,N)$$

are quasiisomorphisms.

Remark 5.2 When $\operatorname{Ext}_{\mathcal{X}R}$ and $\operatorname{Ext}_{R\mathcal{Y}}$ are balanced on $\mathcal{X}'\times\mathcal{Y}'$, there are isomorphisms $\operatorname{Ext}^n_{\mathcal{X}R}(M,N)\cong\operatorname{Ext}^n_{R\mathcal{Y}}(M,N)$ for all $M\in\mathcal{X}'$ and $N\in\mathcal{Y}'$ and $n\in\mathbb{Z}$.

The next example shows that the naive version of balance for relative cohomology does not hold: when D is dualizing, one can have $\operatorname{Ext}^n_{\mathcal{P}_D}(M,N)\ncong\operatorname{Ext}^n_{\mathcal{I}_D}(M,N)$ and $\operatorname{Ext}^n_{\mathcal{GP}_D}(M,N)\ncong\operatorname{Ext}^n_{\mathcal{GI}_D}(M,N)$, even if $\mathcal{P}_D\text{-}\operatorname{pd}_R(M)<\infty$ and $\mathcal{I}_D\text{-}\operatorname{id}_R(N)<\infty$.

Example 5.3 Let $(R,\mathfrak{m},k)$ be a local non-Gorenstein Cohen–Macaulay ring of dimension $d>0$ with dualizing module D. Assume that R is Gorenstein on the punctured spectrum. Fact 3.4 (b) implies that $\mathcal{I}_D\text{-}\operatorname{id}_R(R)=\operatorname{id}_R(D)=d$, and Proposition 4.2 provides isomorphisms

$$\operatorname{Ext}^n_{\mathcal{GP}_D}(D,R)\cong\operatorname{Ext}^n_{\mathcal{P}_D}(D,R)\qquad\operatorname{Ext}^n_{\mathcal{I}_D}(D,R)\cong\operatorname{Ext}^n_{\mathcal{GI}_D}(D,R)$$

for each n. One has $\mathcal{P}_D\text{-}\operatorname{pd}_R(D)=0$ because D is in $\mathcal{P}_D$, and so $\operatorname{Ext}^{\geqslant 1}_{\mathcal{P}_D}(D,R)=0$. Fact 3.9 (b) yields an isomorphism

$$\operatorname{Ext}^n_{\mathcal{I}_D}(D,R)\cong\operatorname{Ext}^n_R(D\otimes_R D,D)$$

for each integer n. To complete the example, we verify

$$\operatorname{Ext}^n_{\mathcal{P}_D}(D,R)\ncong\operatorname{Ext}^n_{\mathcal{I}_D}(D,R)$$

for some $n \geqslant 1$. From the vanishing $\mathrm{Ext}^{\geqslant 1}_{\mathcal{P}_D}(D, R) = 0$, it suffices to find an integer $n \geqslant 1$ such that $\mathrm{Ext}^n_R(D \otimes_R D, D) \neq 0$. We utilize the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_R(\mathrm{Tor}^R_q(D, D), D) \implies \mathrm{Ext}^{p+q}_R(D, R).$$

For each prime ideal $\mathfrak{p} \subsetneq \mathfrak{m}$, the module $D_\mathfrak{p}$ is dualizing for the Gorenstein ring $R_\mathfrak{p}$, and so $D_\mathfrak{p} \cong R_\mathfrak{p}$; see [8, (3.3.7.a)]. It follows that $\mathrm{Tor}^R_q(D, D)$ has finite length for each $q > 0$, and so $E_2^{p,q} = 0$ if $p \neq d$ and $q > 0$. For each $n \leq d$, this yields

$$\mathrm{Ext}^n_R(D \otimes_R D, D) \cong \mathrm{Ext}^n_R(D, R).$$

Because R is not Gorenstein, we deduce from [5, (2.1)] that the displayed modules are nonzero for some integer n such that $1 \leq n \leq d$, as desired.

The following result contains part of Theorem A from the introduction.

Proposition 5.4 *Assume that R is Cohen–Macaulay ring and admits a dualizing module, and let C be a semidualizing R-module. Then $\mathrm{Ext}_{\mathcal{P}_C}$ and $\mathrm{Ext}_{\mathcal{I}_{C^\dagger}}$ are balanced on $\mathrm{res}\,\widehat{\mathcal{P}_C} \times \mathrm{cores}\,\widehat{\mathcal{I}_{C^\dagger}}$. In particular, if $\mathcal{P}_C\text{-}\mathrm{pd}_R(M) < \infty$ and $\mathcal{I}_{C^\dagger}\text{-}\mathrm{id}_R(N) < \infty$, then there are isomorphisms for each integer n*

$$\mathrm{Ext}^n_{\mathcal{P}_C}(M, N) \cong \mathrm{Ext}^n_{\mathcal{I}_{C^\dagger}}(M, N).$$

Proof Lemma 4.6 implies $\mathrm{Ext}^{\geqslant 1}_{\mathcal{P}_C}(\mathrm{res}\,\widehat{\mathcal{P}_C}, \mathcal{I}_{C^\dagger}) = 0 = \mathrm{Ext}^{\geqslant 1}_{\mathcal{I}_C}(\mathcal{P}_C, \mathrm{cores}\,\widehat{\mathcal{I}_{C^\dagger}})$ and so the desired conclusion follows from [14, (8.2.14)]. □

The next two lemmata are the primary tools for Theorem 5.7.

Lemma 5.5 *Let R be a Cohen–Macaulay ring with dualizing module. Let C, M and N be R-modules with C semidualizing.*

(a) *If $N \in \mathcal{GI}_{C^\dagger}$ and $\mathcal{P}_C\text{-}\mathrm{pd}_R(M) < \infty$, then $\mathrm{Ext}^{\geqslant 1}_{\mathcal{P}_C}(M, N) = 0$.*
(b) *If $M \in \mathcal{GP}_C$ and $\mathcal{I}_{C^\dagger}\text{-}\mathrm{id}_R(N) < \infty$, then $\mathrm{Ext}^{\geqslant 1}_{\mathcal{I}_{C^\dagger}}(M, N) = 0$.*

Proof We prove part (a); part (b) is dual. Set $N_0 = N$, and for each $n \geqslant 0$ use the fact that $\mathcal{I}_{C^\dagger}$ is a generator for $\mathcal{GI}_{C^\dagger}$ to find exact sequences

$$0 \to N_{n+1} \to V_n \to N_n \to 0 \tag{5.5.1}$$

with V_n in $\mathcal{I}_{C^\dagger}$ and N_{n+1} in $\mathcal{GI}_{C^\dagger}$; see Fact 3.7. Lemma 4.7 implies $\mathcal{P}_C \perp \mathcal{GI}_{C^\dagger}$, and so the long-exact sequence in $\mathrm{Ext}_R(\mathcal{P}_C, -)$ shows that (5.5.1) is $\mathrm{Hom}_R(\mathcal{P}_C, -)$-exact. Fix an integer $j \geqslant 1$ and set $p = \mathcal{P}_C\text{-}\mathrm{pd}(M)$. Lemma 4.6 (b) implies $\mathrm{Ext}^{\geqslant 1}_{\mathcal{P}_C}(M, V_n) = 0$ for each n. Lemma 1.15 (a) and Remark 1.12 yield the isomorphism in the next sequence

$$\mathrm{Ext}^j_{\mathcal{P}_C}(M, N) = \mathrm{Ext}^j_{\mathcal{P}_C}(M, N_0) \cong \mathrm{Ext}^{j+p}_{\mathcal{P}_C}(M, N_p) = 0,$$

where the vanishing is from the (in)equalities $j + p > p = \mathcal{P}_C\text{-}\mathrm{pd}_R(M)$. □

Lemma 5.6 *Let R be a Cohen–Macaulay ring admitting a dualizing module. Let C, M and N be R-modules with C semidualizing.*

(a) *Assume that $\mathcal{GP}_C\text{-}\mathrm{pd}_R(M)$ is finite and let $G \xrightarrow{\alpha} M$ be a proper $\mathcal{GP}_C$-resolution. If Y is a bounded above complex of objects in $\mathcal{GI}_{C^\dagger}$, then the induced map $\mathrm{Hom}_R(M, Y) \to \mathrm{Hom}_R(G, Y)$ is a quasiisomorphism.*

(b) *Assume that $\mathcal{GI}_{C^\dagger}\text{-}\operatorname{id}_R(N)$ is finite and let $N \xrightarrow{\beta} H$ be a proper $\mathcal{GI}_{C^\dagger}$-resolution. If X is a bounded below complex of objects in $\mathcal{GP}_C$, then the induced map $\operatorname{Hom}_R(X, N) \to \operatorname{Hom}_R(X, H)$ is a quasiisomorphism.*

Proof We proof part (a); the proof of (b) is dual. To show that the induced map $\operatorname{Hom}_R(\alpha, Y)\colon \operatorname{Hom}_R(M, Y) \to \operatorname{Hom}_R(G, Y)$ is a quasiisomorphism, it suffices to show that $\operatorname{Cone}(\operatorname{Hom}_R(\alpha, Y))$ is exact. From the isomorphisms of complexes

$$\operatorname{Cone}(\operatorname{Hom}_R(\alpha, Y)) \cong \Sigma \operatorname{Hom}_R(\operatorname{Cone}(\alpha), Y) \cong \Sigma \operatorname{Hom}_R(G^+, Y)$$

and a standard argument, it suffices to show that $\operatorname{Hom}_R(G^+, Y_j)$ is exact for each j.

The module M has a bounded strict $\mathcal{GP}_C$-resolution $G' \xrightarrow{\alpha'} M$ by Fact 3.7. From Lemma 1.9 (a) we conclude that G^+ and $(G')^+$ are homotopy equivalent, and so the complex $\operatorname{Hom}_R(G^+, Y_j)$ is exact if and only if $\operatorname{Hom}_R((G')^+, Y_j)$ is exact. Thus, we may replace G with G' to assume that G is strict.

For each n, set $M_n = \operatorname{Coker}(\partial^G_{n+2})$ and note that $M_{-1} \cong M$. For each $n \geqslant 0$, we have $\mathcal{P}_C\text{-}\operatorname{pd}(M_n) < \infty$ and we consider the following exact sequences

$$0 \to M_n \xrightarrow{\gamma_i} G_n \xrightarrow{\delta_n} M_{n-1} \to 0. \tag{5.6.1}$$

It suffices to show that each of these sequences is $\operatorname{Hom}_R(-, Y_j)$-exact, that is, that the following map is surjective.

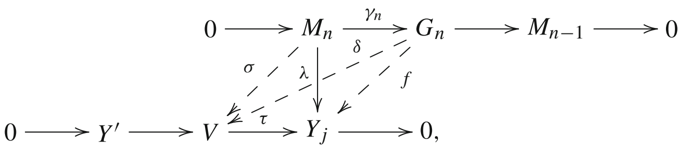
$$\operatorname{Hom}_R(\gamma_n, Y_j)\colon\ \operatorname{Hom}_R(G_n, Y_j) \to \operatorname{Hom}_R(M_n, Y_j).$$

Use the fact that $\mathcal{I}_{C^\dagger}$ is a generator for $\mathcal{GI}_{C^\dagger}$ to find an exact sequence

$$0 \to Y' \to V \xrightarrow{\tau} Y_j \to 0 \tag{5.6.2}$$

such that Y' is in $\mathcal{GI}_{C^\dagger}$ and V is in $\mathcal{I}_{C^\dagger}$; see Fact 3.7. Lemma 4.7 implies $\mathcal{P}_C \perp \mathcal{GI}_{C^\dagger}$ and so Lemma 1.7 (b) guarantees that this sequence is $\operatorname{Hom}_R(\mathcal{P}_C, -)$-exact.

Fix an element $\lambda \in \operatorname{Hom}_R(M_n, Y_j)$. The proof will be complete once we find an element $f \in \operatorname{Hom}_R(G_n, Y_j)$ such that $\lambda = f\gamma_n$. The following diagram is our guide



wherein the top row is (5.6.1) and the bottom row is (5.6.2).

Since the sequence (5.6.2) is $\operatorname{Hom}_R(\mathcal{P}_C, -)$-exact, it gives rise to a long exact sequence in $\operatorname{Ext}_{\mathcal{P}_C}(M_n, -)$. The vanishing of $\operatorname{Ext}^1_{\mathcal{P}_C}(M_n, Y')$ from Lemma 5.5 (a) implies that this long exact sequence has the form

$$0 \to \operatorname{Hom}_R(M_n, Y') \to \operatorname{Hom}_R(M_n, V) \xrightarrow{\operatorname{Hom}_R(M_n,\tau)} \operatorname{Hom}_R(M_n, Y_j) \to 0.$$

Hence, there exists $\sigma \in \operatorname{Hom}_R(M_n, V)$ such that $\lambda = \tau\sigma$.

Proposition 4.1 (b) implies $\operatorname{Ext}^1_R(M_{n-1}, V) \cong \operatorname{Ext}^1_{\mathcal{I}_{C^\dagger}}(M, N) = 0$, so the long exact sequence in $\operatorname{Ext}_R(-, V)$ associated to (5.6.1) has the form

$$0 \to \operatorname{Hom}_R(M_{n-1}, V) \to \operatorname{Hom}_R(G_n, V) \xrightarrow{\operatorname{Hom}_R(\gamma_n,V)} \operatorname{Hom}_R(M_n, V) \to 0.$$

Hence, there exists $\delta \in \operatorname{Hom}_R(G_n, V)$ such that $\sigma = \delta\gamma_n$. It follows that we have $(\tau\delta)\gamma_n = \tau\sigma = \lambda$ and so $f = \tau\delta \in \operatorname{Hom}_R(G_n, Y_j)$ has the desired property. □

Our main balance result for relative cohomology now follows. It contains part of Theorem A from the introduction.

Theorem 5.7 *Assume that R is Cohen–Macaulay and admits dualizing module, and let C be a semidualizing R-module. The functors $\operatorname{Ext}_{\mathcal{GP}_C}$ and $\operatorname{Ext}_{\mathcal{GI}_{C^\dagger}}$ are balanced on $\operatorname{res}\widehat{\mathcal{GP}_C} \times \operatorname{cores}\widehat{\mathcal{GI}_{C^\dagger}}$. In particular, if $\mathcal{GP}_C\text{-}\operatorname{pd}_R(M) < \infty$ and $\mathcal{GI}_{C^\dagger}\text{-}\operatorname{id}_R(N) < \infty$, then there are isomorphisms for each integer n*

$$\operatorname{Ext}^n_{\mathcal{GP}_C}(M, N) \cong \operatorname{Ext}^n_{\mathcal{GI}_{C^\dagger}}(M, N).$$

Proof From Fact 3.7 we obtain a bounded proper $\mathcal{GP}_C$-resolution $\alpha\colon G \xrightarrow{\simeq} M$ and a bounded proper $\mathcal{GI}_{C^\dagger}$-coresolution $\beta\colon N \xrightarrow{\simeq} Y$. By Lemma 5.6, the morphisms

$$\operatorname{Hom}_R(M, Y) \xrightarrow{\operatorname{Hom}_R(\alpha, Y)} \operatorname{Hom}_R(G, Y) \xleftarrow{\operatorname{Hom}_R(X, \beta)} \operatorname{Hom}_R(X, N)$$

are quasiisomorphisms, as desired. □

6 Distinguishing between relative cohomology theories

From Sects. 4 and 5 we see that there are numerous situations where different relative cohomology theories agree. The purpose of this section is to show that these theories are almost never *identically* equal. Part (a) of the next result shows $\operatorname{Ext}^n_{\mathcal{P}_C}(-,-) \ncong \operatorname{Ext}^n_R(-,-) \ncong \operatorname{Ext}^n_{\mathcal{GP}_C}(-,-)$. Part (b) shows again $\operatorname{Ext}^n_R(-,-) \ncong \operatorname{Ext}^n_{\mathcal{GP}_C}(-,-)$. Part (c) shows $\operatorname{Ext}^n_{\mathcal{P}_C}(-,-) \ncong \operatorname{Ext}^n_{\mathcal{GP}_C}(-,-)$. Lemma 6.2 shows how one can construct modules satisfying the hypotheses of part (c).

Proposition 6.1 *Let $(R, \mathfrak{m}, k)$ be a local ring and C a semidualizing R-module such that $C \ncong R$.*

(a) *If $n \geqslant 1$, then $\operatorname{Ext}^n_{\mathcal{P}_C}(C, k) = 0 = \operatorname{Ext}^n_{\mathcal{GP}_C}(C, k)$ and $\operatorname{Ext}^n_R(C, k) \neq 0$.*
(b) *Assume $\operatorname{depth}(R) \geqslant 1$ and fix an R-regular element $x \in \mathfrak{m}$. The exact sequence $\zeta = (0 \to R \xrightarrow{x} R \to R/xR \to 0)$ is not $\operatorname{Hom}_R(\mathcal{GP}_C, -)$-exact. Hence, the natural map*

$$\varkappa^1_{\mathcal{GP}_C}(R/xR, R)\colon \operatorname{Ext}^1_{\mathcal{GP}_C}(R/xR, R) \hookrightarrow \operatorname{Ext}^1_R(R/xR, R)$$

is not surjective. If $\dim(R) = 1$, then $\operatorname{Ext}^1_{\mathcal{GP}_C}(R/xR, R) \ncong \operatorname{Ext}^1_R(R/xR, R)$.
(c) *If M admits a proper $\mathcal{P}_C$-resolution and $\mathcal{GP}_C\text{-}\operatorname{pd}_R(M) < \infty = \mathcal{P}_C\text{-}\operatorname{pd}_R(M)$, then $\operatorname{Ext}^n_{\mathcal{GP}_C}(M, -) = 0 \ncong \operatorname{Ext}^n_{\mathcal{P}_C}(M, -)$ for each $n > \mathcal{GP}_C\text{-}\operatorname{pd}_R(M)$.*

Proof (a) Since C is in $\mathcal{P}_C$, we have $\operatorname{Ext}^n_{\mathcal{GP}_C}(C, k) \cong \operatorname{Ext}^n_{\mathcal{P}_C}(C, k) = 0$ for each $n \geqslant 1$ by Proposition 4.2 (a). On the other hand, we have $\operatorname{Ext}^n_R(C, k) \neq 0$ because C is a finitely generated module of infinite projective dimension by Lemma 3.16 using $B = R$.

(b) Suppose that the sequence ζ is $\operatorname{Hom}_R(\mathcal{GP}_C, -)$-exact. It follows that ζ is an augmented proper $\mathcal{GP}_C$-resolution of R/xR. Lemma 1.9 (a) combines with [29, (4.10)] to show that ζ has an exact sequence of the following form as a summand

$$0 \to C^n \to G \to R/xR \to 0.$$

where $n \geqslant 1$. It follows that C is a summand of R. Because R is local, the R-module R is indecomposable, so this implies $C \cong R$, a contradiction.

Theorem 2.3 (a) now implies that the natural inclusion $\varkappa^1_{\mathcal{GP}_C}(R/xR, R)$ is not surjective. If $\dim(R) = 1$, then R/xR has finite length. It follows from Lemma 3.10 (c) that the module $\operatorname{Ext}^n_{\mathcal{GP}_C}(R/xR, R)$ has finite length, as does $\operatorname{Ext}^n_R(R/xR, R)$. Because the inclusion $\varkappa^1_{\mathcal{GP}_C}(R/xR, R)$ is not surjective, one has

$$\operatorname{length}_R(\operatorname{Ext}^n_{\mathcal{GP}_C}(R/xR, R)) < \operatorname{length}_R(\operatorname{Ext}^n_R(R/xR, R))$$

and so $\operatorname{Ext}^n_{\mathcal{GP}_C}(R/xR, R) \ncong \operatorname{Ext}^n_R(R/xR, R)$.

(c) The vanishing $\operatorname{Ext}^n_{\mathcal{GP}_C}(M, -) = 0$ when $n > \mathcal{GP}_C\text{-}\operatorname{pd}_R(M)$ follows from Fact 3.7. The nonvanishing $\operatorname{Ext}^n_{\mathcal{P}_C}(M, -) \neq 0$ is in [27, (3.2)]. □

Notice that the following lemma does not assume any relation between the semidualizing modules B and C. Also, the cases $B = R$ and $B = C$ imply $\operatorname{pd}_R(M) = \infty$ and $\mathcal{P}_C\text{-}\operatorname{pd}_R(M) = \infty$.

Lemma 6.2 *Let $(R, \mathfrak{m})$ be a local ring and let B and C be semidualizing R-modules. Assume that there exist elements $y, z \in \mathfrak{m}$ such that $\operatorname{Ann}_R(y) = zR$ and $\operatorname{Ann}_R(z) = yR$, and set $M = C/yC$. Then $M \in \mathcal{G}(\mathcal{P}_C) = \mathcal{GP}_C \cap \mathcal{B}_C$ and so M admits a proper $\mathcal{P}_C$-resolution. Also, one has $\mathcal{P}_B\text{-}\operatorname{pd}_R(M) = \infty$.*

Proof Consider the chain complex

$$Z = \cdots \xrightarrow{y} C \xrightarrow{z} C \xrightarrow{y} C \xrightarrow{z} \cdots.$$

We shall show that this complex is exact and that it is $\operatorname{Hom}_R(\mathcal{P}_C, -)$-exact and $\operatorname{Hom}_R(-, \mathcal{P}_C)$-exact. Once this is done, we will conclude from [26, (5.2)] that M is in $\mathcal{G}(\mathcal{P}_C)$. Furthermore, we will know that the truncated complex

$$\cdots \xrightarrow{y} C \xrightarrow{z} C \xrightarrow{y} C \to 0$$

is a proper $\mathcal{P}_C$-resolution of M.

To see that the complex Z is exact, we first show $\operatorname{Ann}_R(zC) \subseteq yR$: If $w \in \operatorname{Ann}_R(zC)$, we have $wz \in \operatorname{Ann}_R(C) = 0$ and so $w \in \operatorname{Ann}_R(z) = yR$. From this the obvious containment $\operatorname{Ann}_R(zC) \supseteq yR$ implies $\operatorname{Ann}_R(zC) = yR$, and by symmetry we have $\operatorname{Ann}_R(yC) = zR$ and the desired exactness.

Next, we show that the complex Z is $\operatorname{Hom}_R(\mathcal{P}_C, -)$-exact and $\operatorname{Hom}_R(-, \mathcal{P}_C)$-exact. The isomorphism $\operatorname{Hom}_R(C, C) \cong R$ shows that an application of either $\operatorname{Hom}_R(C, -)$ or $\operatorname{Hom}_R(-, C)$ yields the complex

$$\cdots \xrightarrow{y} R \xrightarrow{z} R \xrightarrow{y} R \xrightarrow{z} \cdots, \tag{6.2.1}$$

which is exact because of the assumptions $\operatorname{Ann}_R(y) = zR$ and $\operatorname{Ann}_R(z) = yR$. Hom-tensor adjointness implies that Z is $\operatorname{Hom}_R(\mathcal{P}_C, -)$-exact. On the other hand, the natural isomorphism $\operatorname{Hom}_R(C, C \otimes_R P) \cong \operatorname{Hom}_R(C, C) \otimes_R P$ from [29, (1.11)] implies that $\operatorname{Hom}_R(Z, C \otimes_R P) \cong \operatorname{Hom}_R(Z, C) \otimes_R P$, so Z is $\operatorname{Hom}_R(-, \mathcal{P}_C)$-exact.

The fact that M is in $\mathcal{GP}^f_C$ yields the first two equalities in the next sequence

$$0 = \mathcal{GP}_C\text{-}\operatorname{pd}_R(M) = \operatorname{G}_C\text{-}\dim_R(M) = \operatorname{depth}(R) - \operatorname{depth}_R(M)$$

while the third one is from [11, (3.14)]. Now, suppose $\mathcal{P}_B\text{-}\operatorname{pd}_R(M) < \infty$. Via the next sequence, the previous display works with [29, (4.6)] to show that M is in $\mathcal{P}^f_B$:

$$\mathcal{P}_B\text{-}\operatorname{pd}_R(M) = \operatorname{depth}(R) - \operatorname{depth}_R(M) = 0.$$

This implies $0 \neq M \cong B^m$ for some m, and so $\mathrm{Ann}_R(M) = \mathrm{Ann}_R(B) = 0$. The membership $0 \neq y \in \mathrm{Ann}_R(M)$ contradicts this, and so $\mathcal{P}_B\text{-}\mathrm{pd}_R(M) = \infty$. □

We follow-up with an example where the assumptions of Lemma 6.2 are satisfied.

Example 6.3 Let Q be a local ring with semidualizing module A. Set $R = Q[\![X]\!]/(X^2)$ or $R = Q[\![Y, Z]\!]/(YZ)$. The R-module $C = R \otimes_Q A$ is semidualizing by [11, (5.6)], and the residues $y = \overline{X} = z$ or $y = \overline{Y}$ and $z = \overline{Z}$ satisfy the hypotheses of Lemma 6.2.

We now contrast the relative cohomology theories arising from distinct semidualizing modules B and C. With Proposition 6.1 (a), part (a) of the next result shows $\mathrm{Ext}^n_{\mathcal{P}_C}(-,-) \ncong \mathrm{Ext}^n_{\mathcal{P}_B}(-,-) \ncong \mathrm{Ext}^n_{\mathcal{GP}_C}(-,-)$. Part (b) shows $\mathrm{Ext}^n_{\mathcal{GP}_B}(-,-) \ncong \mathrm{Ext}^n_{\mathcal{GP}_C}(-,-)$ and again $\mathrm{Ext}^n_{\mathcal{P}_B}(-,-) \ncong \mathrm{Ext}^n_{\mathcal{GP}_C}(-,-)$ and part (c) shows that $\mathrm{Ext}^n_{\mathcal{GP}_B}(-,-) \ncong \mathrm{Ext}^n_{\mathcal{P}_C}(-,-)$. Note that Lemmas 3.14 and 3.16 contain analyses of the conditions $\mathcal{GP}_C\text{-}\mathrm{pd}_R(B) < \infty$ and $C \ncong B$.

Proposition 6.4 *Let $(R, \mathfrak{m}, k)$ be a local ring and let B and C be semidualizing R-modules such that $\mathcal{GP}_C\text{-}\mathrm{pd}_R(B) < \infty$ and $C \ncong B$.*

(a) *If $n \geqslant 0$, then $\mathrm{Ext}^n_{\mathcal{P}_B}(C, k) \neq 0$.*
(b) *Assume $\mathrm{depth}(R) \geqslant 1$ and fix an R-regular element $x \in \mathfrak{m}$. The exact sequence $\zeta = (0 \to B \xrightarrow{x} B \to B/xB \to 0)$ is not $\mathrm{Hom}_R(\mathcal{GP}_C, -)$-exact, and so the natural inclusions*

$$\vartheta^1_{\mathcal{GP}_C\mathcal{P}_B}(B/xB, B) : \mathrm{Ext}^1_{\mathcal{GP}_C}(B/xB, B) \hookrightarrow \mathrm{Ext}^1_{\mathcal{P}_B}(B/xB, B)$$
$$\vartheta^1_{\mathcal{GP}_C\mathcal{GP}_B}(B/xB, B) : \mathrm{Ext}^1_{\mathcal{GP}_C}(B/xB, B) \hookrightarrow \mathrm{Ext}^1_{\mathcal{GP}_B}(B/xB, B)$$

are not onto. If $\dim(R) = 1$, then $\mathrm{Ext}^1_{\mathcal{GP}_C}(B/xB, B) \ncong \mathrm{Ext}^1_{\mathcal{P}_B}(B/xB, B)$ and $\mathrm{Ext}^1_{\mathcal{GP}_C}(B/xB, B) \ncong \mathrm{Ext}^1_{\mathcal{GP}_B}(B/xB, B)$.
(c) *If C admits a proper $\mathcal{GP}_B$-resolution, then $\mathrm{Ext}^n_{\mathcal{P}_C}(C, -) = 0 \neq \mathrm{Ext}^n_{\mathcal{GP}_B}(C, -)$ for each $n \geqslant 1$.*

Proof (a) Because $\mathrm{pd}_R(\mathrm{Hom}_R(B, C)) = \infty$ by Lemma 3.16, the nth Betti number $\beta^R_n(\mathrm{Hom}_R(B, C))$ is nonzero for each $n \geqslant 0$. Using Fact 3.9 (a), the membership $C \in \mathcal{B}_B$ from Lemma 3.14 yields the first isomorphism in the following sequence

$$\begin{aligned}\mathrm{Ext}^n_{\mathcal{P}_B}(C, k) &\cong \mathrm{Ext}^n_R(\mathrm{Hom}_R(B, C), \mathrm{Hom}_R(B, k)) \\ &\cong \mathrm{Ext}^n_R(\mathrm{Hom}_R(B, C), k^{\beta^R_0(B)}) \cong k^{\beta^R_n(\mathrm{Hom}_R(B,C))\beta^R_0(B)} \neq 0\end{aligned}$$

while the others are standard.

(b) The sequence ζ is $\mathcal{P}_B$-proper and $\mathcal{GP}_B$-proper by Fact 3.7. Suppose that ζ is $\mathrm{Hom}_R(\mathcal{GP}_C, -)$-exact. Because B is totally C-reflexive by Lemma 3.14, this sequence is an augmented proper $\mathcal{GP}_C$-resolution of B/xB. Lemma 1.9 (a) combines with [29, (4.10)] to show that ζ has an exact sequence of the following form as a direct summand

$$0 \to C^n \to G \to B/xB \to 0,$$

where $n \geqslant 1$. It follows that C is a summand of B. Because R is local, the R-module R is irreducible, so this implies $C \cong R$, a contradiction. The remainder of (b) is verified as in the proof of Proposition 6.1 (b).

(c) Since C is in $\mathcal{P}_C$, we have $\mathrm{Ext}^n_{\mathcal{P}_C}(C, -) = 0$ for each $n \geqslant 1$. Lemma 3.16 implies $\mathcal{GP}_B\text{-}\mathrm{pd}_R(C) = \infty$; arguing as in [7, (4.2.2.a)], we conclude $\mathrm{Ext}^n_{\mathcal{GP}_B}(C, -) \neq 0$ for each $n \geqslant 0$. □

Remark 6.5 In light of the hypothesis "C admits a proper $\mathcal{GP}_B$-resolution" in Proposition 6.4 (c), we note that this condition is satisfied when R admits a dualizing complex and $B = R$ by [23, (2.11)]. As of the writing of this paper, the authors do not know if this condition holds in general.

We conclude this paper with dual versions of the above results in this section.

Proposition 6.6 *Let $(R, \mathfrak{m}, k)$ be a local ring and C a semidualizing R-module such that $C \ncong R$. Let E denote the R-injective hull of k.*

(a) *If $n \geqslant 1$, then* $\mathrm{Ext}^n_{\mathcal{I}_C}(-, \mathrm{Hom}_R(C, E)) = 0 = \mathrm{Ext}^n_{\mathcal{GI}_C}(-, \mathrm{Hom}_R(C, E))$ *and* $\mathrm{Ext}^n_R(-, \mathrm{Hom}_R(C, E)) \neq 0$.
(b) *Assume that R is complete and that* $\mathrm{depth}(R) \geqslant 1$. *Fix an R-regular element $x \in \mathfrak{m}$, and set* $K = \mathrm{Ker}(E \xrightarrow{x} E) \cong \mathrm{Hom}_R(R/xR, E)$. *Then the exact sequence* $\zeta = (0 \to K \xrightarrow{\iota} E \xrightarrow{x} E \to 0)$ *is not* $\mathrm{Hom}_R(-, \mathcal{GI}_C)$*-exact, and so the map*

$$\varkappa^1_{\mathcal{GI}_C}(E, K)\colon\ \mathrm{Ext}^1_{\mathcal{GI}_C}(E, K) \hookrightarrow \mathrm{Ext}^1_R(E, K)$$

is not surjective.
(c) *If M admits a proper $\mathcal{I}_C$-coresolution and* $\mathcal{GI}_C\text{-}\mathrm{id}_R(M) < \infty = \mathcal{I}_C\text{-}\mathrm{id}_R(M)$, *then* $\mathrm{Ext}^n_{\mathcal{GI}_C}(-, M) = 0 \neq \mathrm{Ext}^n_{\mathcal{I}_C}(-, M)$ *for each* $n > \mathcal{GI}_C\text{-}\mathrm{id}_R(M)$.

Proof The proofs of (a) and (c) are dual to the parallel parts of Proposition 6.1.

(b) Because E is injective, it is divisible, so the sequence ζ is exact. As in the proof of Proposition 6.1 (b) it suffices to show that ζ is not $\mathrm{Hom}_R(-, \mathcal{GI}_C)$-exact.

Because R is complete, there is an isomorphism $\mathrm{Hom}_R(E, E) \cong R$. Applying the exact functor $\mathrm{Hom}_R(-, E)$ to ζ yields the exact sequence

$$0 \to R \xrightarrow{x} R \xrightarrow{\pi} R/xR \to 0. \tag{6.6.1}$$

Proposition 6.1 (b) shows that there exists a module $G \in \mathcal{GP}_C$ such that the following sequence is not exact

$$0 \to \mathrm{Hom}_R(G, R) \xrightarrow{x} \mathrm{Hom}_R(G, R) \xrightarrow{\mathrm{Hom}_R(G,\pi)} \mathrm{Hom}_R(G, R/xR) \to 0. \tag{6.6.2}$$

Lemma 3.3 (b) implies that the R-module $G^\vee = \mathrm{Hom}_R(G, E)$ is in $\mathcal{GI}_C$. To complete the proof, we show that the complex

$$0 \to \mathrm{Hom}_R(E, G^\vee) \xrightarrow{x} \mathrm{Hom}_R(E, G^\vee) \xrightarrow{\mathrm{Hom}_R(\iota,G^\vee)} \mathrm{Hom}_R(K, G^\vee) \to 0$$

is not exact. The "swap" isomorphism $\mathrm{Hom}_R(-, G^\vee) \cong \mathrm{Hom}_R(G, (-)^\vee)$ shows that this sequence is isomorphic to (6.6.2), which is not exact. □

Lemma 6.7 *Let $(R, \mathfrak{m})$ be a local ring and C a semidualizing R-module. Let E denote the R-injective hull of k. Assume that there exist elements $y, z \in \mathfrak{m}$ such that* $\mathrm{Ann}_R(y) = zR$ *and* $\mathrm{Ann}_R(z) = yR$, *and set* $M = \mathrm{Hom}_R(C, E)/y\,\mathrm{Hom}_R(C, E)$. *Then* $M \in \mathcal{G}(\mathcal{I}_C) = \mathcal{GI}_C \cap \mathcal{A}_C$ *and so M admits a proper $\mathcal{I}_C$-coresolution. Also, one has* $\mathcal{I}_C\text{-}\mathrm{id}_R(M) = \infty$.

Proof The isomorphisms $\mathrm{Hom}_R(E, E) \cong \widehat{R}$ and $C \otimes_R \mathrm{Hom}_R(C, E) \cong E$ yield the following containments

$$0 \subseteq \mathrm{Ann}_R(\mathrm{Hom}_R(C, E)) \subseteq \mathrm{Ann}_R(E) \subseteq \mathrm{Ann}_R(\widehat{R}) = 0$$

and so $\mathrm{Ann}_R(\mathrm{Hom}_R(C, E)) = 0$. As in the proof of Lemma 6.2, it follows that the following complex is exact

$$\cdots \xrightarrow{y} \mathrm{Hom}_R(C, E) \xrightarrow{z} \mathrm{Hom}_R(C, E) \xrightarrow{y} \mathrm{Hom}_R(C, E) \xrightarrow{z} \cdots . \tag{6.7.1}$$

We claim that this complex is also $\mathrm{Hom}_R(-, \mathcal{I}_C)$-exact and $\mathrm{Hom}_R(\mathcal{I}_C, -)$-exact. From this it will follow that the complex

$$0 \to \mathrm{Hom}_R(C, E) \xrightarrow{z} \mathrm{Hom}_R(C, E) \xrightarrow{y} \mathrm{Hom}_R(C, E) \xrightarrow{z} \cdots \tag{6.7.2}$$

is a proper $\mathcal{I}_C$-coresolution of M and $M \in \mathcal{G}(\mathcal{I}_C) = \mathcal{GI}_C \cap \mathcal{A}_C$. For injective R-modules I and J, the module $\mathrm{Hom}_R(I, J)$ is R-flat. The first isomorphism in the following sequence is Hom-tensor adjunction

$$\mathrm{Hom}_R(\mathrm{Hom}_R(C, I), \mathrm{Hom}_R(C, J)) \cong \mathrm{Hom}_R(C \otimes_R \mathrm{Hom}_R(C, I), J) \cong \mathrm{Hom}_R(I, J)$$

and the second one follows from the membership $I \in \mathcal{B}_C$. Let X denote the exact complex (6.2.1) from the proof of Lemma 6.2. The displayed isomorphisms show that an application of the functor $\mathrm{Hom}_R(\mathrm{Hom}_R(C, I), -)$ to the complex (6.7.1) yields the complex $X \otimes_R \mathrm{Hom}_R(I, E)$. As X is exact and $\mathrm{Hom}_R(I, E)$ is flat, the complex $X \otimes_R \mathrm{Hom}_R(I, E)$ is exact, and so (6.7.1) is $\mathrm{Hom}_R(\mathcal{I}_C, -)$-exact. Similarly, it is $\mathrm{Hom}_R(-, \mathcal{I}_C)$-exact, as desired.

We conclude by showing $\mathcal{I}_C\text{-}\mathrm{id}_R(M) = \infty$. Applying the functor $C \otimes_R -$ to the complex (6.7.2) yields an injective resolution of $C \otimes_R M$

$$0 \to E \xrightarrow{z} E \xrightarrow{y} E \xrightarrow{z} \cdots .$$

This uses the memberships $M, \mathrm{Hom}_R(C, E) \in \mathcal{A}_C$. The fact that this resolution is minimal and nonterminating provides the first equality in the following sequence

$$\infty = \mathrm{id}_R(C \otimes_R M) = \mathcal{I}_C\text{-}\mathrm{id}_R(M)$$

while the second equality is from Fact 3.4 (b). □

Proposition 6.8 *Let $(R, \mathfrak{m}, k)$ be a local ring and let B and C be semidualizing R-modules such that $\mathcal{GP}_C\text{-}\mathrm{pd}_R(B) < \infty$ and $C \not\cong B$. Let E denote the R-injective hull of k, and set $(-)^\vee = \mathrm{Hom}_R(-, E)$.*

(a) *If $n \geqslant 0$, then $\mathrm{Ext}^n_{\mathcal{I}_B}(-, C^\vee) \neq 0$.*
(b) *Assume that R is complete and $\mathrm{depth}(R) \geqslant 1$. Fix an R-regular element $x \in \mathfrak{m}$, and set $K = (B/xB)^\vee$. The sequence $\zeta = (0 \to K \to B^\vee \xrightarrow{x} B^\vee \to 0)$ is exact but not $\mathrm{Hom}_R(-, \mathcal{GI}_C)$-exact, and so the natural inclusions*

$$\vartheta^1_{\mathcal{GI}_C \mathcal{I}_B}(K, B^\vee) : \mathrm{Ext}^1_{\mathcal{GI}_C}(K, B^\vee) \hookrightarrow \mathrm{Ext}^1_{\mathcal{I}_B}(K, B^\vee)$$

$$\vartheta^1_{\mathcal{GI}_C \mathcal{GI}_B}(K, B^\vee) : \mathrm{Ext}^1_{\mathcal{GI}_C}(K, B^\vee) \hookrightarrow \mathrm{Ext}^1_{\mathcal{GI}_B}(K, B^\vee)$$

are not surjective.
(c) *One has $\mathrm{Ext}^n_{\mathcal{I}_C}(-, C^\vee) = 0 \neq \mathrm{Ext}^n_{\mathcal{GI}_B}(-, C^\vee)$ for each $n \geqslant 1$.*

Proof (a) This follows from Lemma 3.16 using [27, (3.2.b)].

(b) Consider the exact sequence

$$0 \to B \xrightarrow{x} B \xrightarrow{\pi} B/xB \to 0. \tag{6.8.1}$$

Apply the exact functor $(-)^\vee$ to show that ζ is exact. The module $B^\vee$ is in $\mathcal{I}_B$, so ζ is an augmented $\mathcal{I}_B$-coresolution of K. As in the proof of Proposition 6.1 (b) it suffices to show that ζ is not $\mathrm{Hom}_R(-, \mathcal{GI}_C)$-exact.

Proposition 6.4 (b) shows that there exists a module $G \in \mathcal{GP}_C$ such that the following sequence is not exact:

$$0 \to \mathrm{Hom}_R(G, B) \xrightarrow{x} \mathrm{Hom}_R(G, B) \xrightarrow{\mathrm{Hom}_R(G,\pi)} \mathrm{Hom}_R(G, B/xB) \to 0. \quad (6.8.2)$$

Lemma 3.3 (b) implies $G^\vee \in \mathcal{GI}_C$. To complete the proof, we show that the complex

$$0 \to \mathrm{Hom}_R(B^\vee, G^\vee) \xrightarrow{x} \mathrm{Hom}_R(B^\vee, G^\vee) \xrightarrow{\mathrm{Hom}_R(\pi^\vee,G^\vee)} \mathrm{Hom}_R(K, G^\vee) \to 0 \quad (6.8.3)$$

is not exact. Because R is complete, the following natural isomorphisms are valid on the category of finitely generated R-modules

$$\mathrm{Hom}_R((-)^\vee, G^\vee) \cong \mathrm{Hom}_R(G, (-)^{\vee\vee}) \cong \mathrm{Hom}_R(G, -)$$

and so the sequence (6.8.3) is isomorphic to (6.8.2), which is not exact.

(c) As in the proof of Proposition 6.4 (c), it suffices to observe that Lemma 3.16 implies $\mathcal{GI}_B\text{-}\mathrm{id}_R(C^\vee) = \infty$. □

Acknowledgments The authors are grateful to Hans-Bjørn Foxby, Anders Frankild and Amelia Taylor for sharing Lemma 3.14 and to Ryo Takahashi for sharing Example 5.3.

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