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Comparison of Riemann and Lebesgue Sampling for First Order Stochastic Systems

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Abstract

The normal approach to digital control is to sample periodically in time. Using an analog of integration theory we can call this Riemann sampling. Lebesgue sampling or event based sampling, is an alternative to Riemann sampling. It means that signals are sampled only when measurements pass certain limits. In this paper it is shown that Lebesgue sampling gives better performance for some simple systems.

1. Introduction

The traditional way to design digital control systems is to sample the signals equidistant in time, see Åström and Wittenmark (1997). A nice feature of this approach is that analysis and design becomes very simple. For linear time-invariant processes the closed loop system become linear and periodic. It is often sufficient to describe the behavior of the the closed loop system at times synchronized with the the sampling instants. The time-varying nature of sampled systems then disappears and the system can be described by difference equations with constant coefficients. The standard sampled theory also matches the time-triggered model of real time software, see Kopetz (2002).

There are several alternatives to periodic sampling. One possibility is to sample the system when the output has changed with a specified amount. Such a scheme has many conceptual advantages. Control is not executed unless it is required, control by exception, see Kopetz (1993). This type of sampling is natural when using many digital sensors such as encoders. A disadvantage is that analysis and design are complicated. This type of sampling can be called Lebesgue sampling. Referring to to integration theory in mathematics we can also call conventional sampling Riemann sampling and Lebesgue sampling Lebesgue sampling. Much work on systems of this type was done in the period 1960-1980. The interest vaned but there has been a resurgence in connection with the strong development of hybrid systems, Branicky *et al.* (1998), DiBenedetto and Sangiovanni-Vincentelli (2001) Some of the early work is reviewed in Section 2. In Section 3 we will analyze an integrator with with random disturbances and Lebesgue sampling. The system can be viewed as a simple model of an accelerometer with pulse feedback. In this case it is possible to formulate and solve sensible control problems, which makes it possible to compare Riemann and Lebesgue sampling. The control strategy is very simple, it just resets the state with a given control pulse whenever the output exceeds the limits. The analysis indicates clearly that there are situations where it is advantageous with Lebesgue sampling. The mathematics used to deal with the problem is based on classical results on diffusion processes, Feller (1952), Feller (1954a), Feller (1954b). An interesting conclusion is that the steady state probability distribution of the control error is non-Gaussian even if the disturbances are Gaussian. There are many interesting extensions of the problem discussed in the paper. Extensions to systems of higher order and output feedback are examples of natural extensions. An interesting property of systems with Lebesgue sampling is that the control strategy is an interesting mix of feedback and feed-forward control that often occurs in biological systems, see Hobbie (1997).

2. Examples

Because of their simplicity Lebesgue sampling was used in many of early feedback systems. An accelerometer with pulse feedback is a typical example, see Draper *et al.* (1960). A pendulum was provided with pulse generators that moved the pendulum towards the center position as soon as a deviation was detected. Since all correcting impulses had the same form the velocity could be obtained simply by adding pulses.

Lebesgue sampling occurs naturally in many context. A common case is in motion control where angles and positions are sensed by encoders that give a pulse whenever a position or an angle has changed by a specific amount. Lebesgue sampling is also a natural approach when actuators with on-off characteristic are used. Satellite control by thrusters is a typical example, Dodds (1981). Systems with pulse frequency modulation, Polak (1968), Pavlidis and Jury (1965), Skoog (1968), Noges and Frank (1975), Frank (1979) and Sira-Ramirez (1989) are other examples. In this case the control signal is restricted to be a positive or negative pulse of given size. The control actions decide when the pulses should be applied and what sign they should have. Other examples are analog or real neurons whose outputs are pulse trains, see Mead (1989) and DeWeerth *et al.* (1990).

Systems with relay feedback are yet other examples which can be regarded as special cases of Lebesgue sampling, see Tsypkin (1984). The sigma delta modulator or the one-bit AD converter, Norsworthy *et al.* (1996), which is commonly used in audio and mobile telephone system is one example. It is interesting to note that in spite of their wide spread there does not exist a good theory for design of systems with sigma delta modulators.

Analysis of systems with Lebesgue sampling are related to general work on discontinuous systems, Utkin (1981), Utkin (1987), Tsypkin (1984) and to work on impulse control, see Bensoussan and Lions (1984) and Aubin (1999). It is also relevant in situations where control complexity has to be weighted against execution time. It also raises other issues such as complexity of control. Control of production processes with buffers is another application area. It is highly desirable to run the processes at constant rates and make as few changes as possible to make sure that buffers are not empty and do not overflow, see Pettersson (1969). Another example is where limited communication resources put hard restrictions on the number of measurement and control actions that can be transmitted.

All sampled systems run open loop between the sampling instants, for systems with Riemann sampling the control signal is typically constant or affine. Systems with Lebesgue sampling can have a much richer behavior which offers interesting possibilities. Consider, for example, a regulation problem where it is desired to keep the state of the system in a given region in the state space. When the state leaves that region an control signal is generated. This open loop control signal is typically such that the state will move into the origin and stay at a desired point. Under ideal circumstances of no disturbances the state will then reach the origin and stay there until new disturbances occur. There are thus two issues in the design, to find a suitable switching boundary and to specify the behavior of the control signal after the switch. The system can thus be regarded as a special version of a hybrid system where the system runs open loop between the regions.

The design of a system with Lebesgue sampling involves selection of an appropriate region and design of the control signal to be used when the state leaves the region. It is also possible to use several different control regions, for example, one can deal with normal disturbances and another larger region can deal with extreme disturbances. The design of such systems lead to many interesting new problems.

3. Integrator Dynamics

We will first consider a simple case where all calculations can be performed analytically. For this purpose it is assumed that the system to be controlled is described by the equation

$$dx = udt + dv,$$

where the disturbance $v(t)$ is a Wiener process with unit incremental variance and u the control signal. The problem of controlling the system so that the state is close to the origin will be discussed. Conventional periodic sampling will be compared with Lebesgue sampling where control actions are taken only when the output is outside the interval $-d < x < d$. We will compare the distribution of $x(t)$ and the variances of the outputs for both sampling schemes.

Periodic Sampling

First consider the case of periodic sampling with period h. The output variance is then minimized by the minimum variance controller, see Åström (1970). The sampled system becomes

$$x(t+h) = x(t) + u(t) + e(t)$$

and the mean variance over one sampling period is

$$\begin{aligned} V &= \frac{1}{h}\int_0^h Ex^2(t)\,dt = \frac{1}{h}J_e(h) \\ &+ \frac{1}{h}\left(Ex^TQ_1(h)x + 2x^TQ_{12}(h)u + u^TQ_2(h)u\right) \\ &= \frac{1}{h}(R_1(h)S(h) + J_e(h)), \end{aligned} \tag{1}$$

where $Q_1(h) = h$, $Q_{12}(h) = h^2/2$, $Q_2(h) = h^3/3$, $R_1(h) = h$ and

$$J_e(h) = \int_0^h Q_{1c}\int_0^t R_{1c}\,d\tau\,dt = h^2/2. \tag{2}$$

The Riccati equation for the minimum variance strategy gives $S(h) = \sqrt{3}h/6$, and the control law becomes

$$u = -\frac{1}{h}\frac{3+\sqrt{3}}{2+\sqrt{3}}x$$

and the variance of the output is

$$V_R = \frac{3+\sqrt{3}}{6}h. \tag{3}$$

Lebesgue Sampling

In this case it is natural to choose the control region as an interval that contains the region. It is also very easy to find good control signals when the system reaches the region. One possibility is simply to apply an impulse that drives the state of the system to the origin, another is to use an pulse of finite width. The case of impulse control Bensoussan and Lions (1984), is the simplest so we will start with that. Control actions are taken thus only taken when $|x(t_k)| = d$. When this happens, an impulse control that makes $x(t_k + 0) = 0$ is applied to the system. With this control law the closed loop system becomes a Markovian diffusion process of the type investigated in Feller (1954a).

Let $T_{\pm d}$ denote the exit time i.e. the first time the process reaches the boundary $|x(t_k)| = d$ when it starts from the origin. The mean exit time can be computed from the fact that $t - x_t^2$ is a martingale between two impulses and hence

$$h_L := E(T_{\pm d}) = E(x^2_{T_{\pm d}}) = d^2.$$

The average sampling period thus equals $h_L = d^2$.

The stationary probability distribution of x is given by the stationary solution to the Kolmogorov forward equation for the Markov process, i.e.

$$0 = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x) - \frac{1}{2}\frac{\partial f}{\partial x}(d)\delta_x + \frac{1}{2}\frac{\partial f}{\partial x}(-d)\delta_x.$$

with $f(-d) = f(d) = 0$ This ordinary differential equation has the solution

$$f(x) = (d - |x|)/d^2 \tag{4}$$

The distribution is thus symmetric and triangular with the support $-d \leq x \leq d$. The steady state variance is

$$V_L = \frac{d^2}{6} = \frac{h_L}{6}.$$

Comparison

To compare the results obtained with the different sampling schemes it is natural to assume that the average sampling rates are the same in both cases, i.e. $h_L = h$. This implies that $d^2 = h$ and it follows from equations (3) and (3) that

$$\frac{V_R}{V_L} = 3 + \sqrt{3} = 4.7.$$

Another way to say this is that one must sample 4.7 times faster with Riemann sampling to get the same mean error variance.

Notice that we have compared Lebesgue sampling with impulse control with periodic sampling with conventional sampling and hold. A natural question is if the improvement is due to the impulse nature of control or to the sampling scheme. To get some insight into this we observe that periodic sampling with impulse control gives and error which is a Wiener process which is periodically reset to zero. The average variance of such a process is

$$V_R' = Ex^2 = \frac{1}{h}E\int_0^h e^2(t)dt = \frac{1}{h}\int_0^h tdt = \frac{h}{2} \tag{5}$$

Periodic sampling with impulse control thus gives

$$\frac{V_R'}{V_L} = 3 \tag{6}$$

The major part of the improvement is thus due to the sampling scheme.

Approximate Lebesgue Sampling

In the analysis it has been assumed that sampling is instantaneous. It is perhaps more realistic to assume that that sampling is made at a high fast rate but that no control action is taken if $x(t) < d$. The variance then becomes

$$V_{AL} = d^2\left(\frac{1}{6} + \frac{5}{6}\frac{h_a}{h_a + d^2}\right).$$

The second term is negligible when $h_a << d^2 = h_L$. Approximate Lebesgue sampling is hence good as long as d is relatively large.

The results are illustrated with the simulation in Figure 1. The simulation was made by rapid sampling (h=0.001). The parameter values used were $d = 0.1$, $h_R = 0.012$ and $\sigma_e = \sqrt{d}$. In the particular realization shown in the Figure there were 83 switches with Riemann sampling and 73 switches with Lebesgue sampling. Notice also the clearly visible decrease in output variance.

4. A First Order System

Consider now the first order system

$$dx = axdt + udt + dv. \tag{7}$$

Periodic Sampling

Sampling the system (7) with period h gives

$$x(t+h) = e^{ah}x(t) + \frac{1}{a}(e^{ah} - 1)u(t) + e(t) \tag{8}$$

where the variance of e is given by

$$J_e(h) = \int_0^h \int_0^t e^{2a\tau}d\tau dt = \left(\frac{e^{2ah} - 1}{2a}\right)^2 = R_1 \tag{9}$$

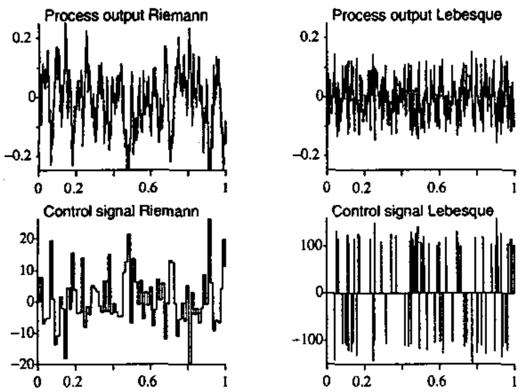

Figure 1 Simulation of an integrator with Riemann (left) and Lebesgue sampling (right).

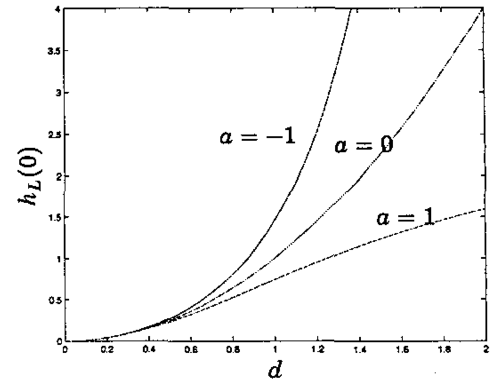

Figure 3 Mean exit time $E(T_{\pm d}) = h_L(0)$ as a function of d for $a = -1, 0, 1$ for a system with Lebesgue sampling.

The sampled loss function is characterized by

$$Q_1 = \frac{e^{2ah} - 1}{2a}$$
$$Q_{12} = \frac{e^{ah}ah - e^{ah} + 1}{a^2}$$
$$Q_2 = \frac{h^3}{3}$$

The minimum variance control law is obtained by solving a Riccati equation for $S(h)$. The formula which is complicated is omitted. The variance of the output is shown in Figure 2 for different values of the parameter a. Notice that the increase of the variance with the sampling period increases much faster for unstable systems $a > 0$.

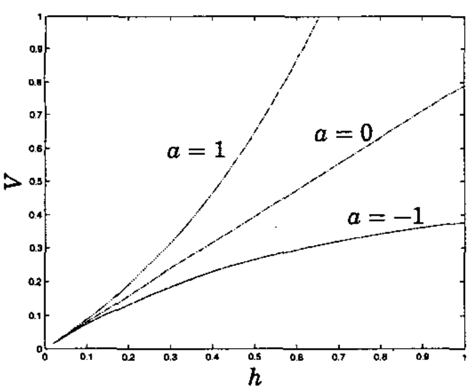

Figure 2 Variance $V_R(h)$ as a function of sampling time for $a = -1, 0, 1$ for a system with Riemann sampling.

Lebesgue Sampling

For Lebesgue sampling we assume as in Section 2 that the variable x is reset to zero when $|x(t_k)| = d$. The closed loop system obtained is then a diffusion process. The average sampling period is the mean exit time when the process starts at $x = 0$. This can be computed from the following result in Feller (1954a).

THEOREM 1

Consider the differential equation $dx = b(x)dt + \sigma(x)dv$ and introduce the backward Kolmogorov operator

$$(\mathcal{A}h)(x) \triangleq \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}(x)\frac{\partial^2 h(x)}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(x)\frac{\partial h(x)}{\partial x_i}, \tag{10}$$

where $h \in C^2(R^n)$ and $a_{ik} = [\sigma\sigma^T]_{ik}$. The mean exit time from $[-d, d]$, starting in x, is given by $h_L(x)$, where

$$\mathcal{A}h_L = -1$$

with $h_L(d) = h_L(-d) = 0$.

In our case the Kolmogorov backward equation becomes

$$\frac{1}{2}\frac{\partial^2 h_L}{\partial x^2} + ax\frac{\partial h_L}{\partial x} = -1$$

with $h_L(d) = h_L(-d) = 0$. The solution is given by

$$h_L(x) = 2\int_x^d \int_0^y e^{-a(y^2-t^2)}\, dt\, dy,$$

which gives

$$\begin{aligned} h_L(0) &= \sum_{k=1}^{\infty} 2^{2k-1}(-a)^{k-1}(k-1)!\, d^{2k}/(2k)! \\ &= d^2 - \frac{a}{3}d^4 + \frac{4a^2}{45}d^6 + O(d^8)). \end{aligned}$$

Figure 3 shows $h_L(0)$ for $a = -1, 0, 1$.

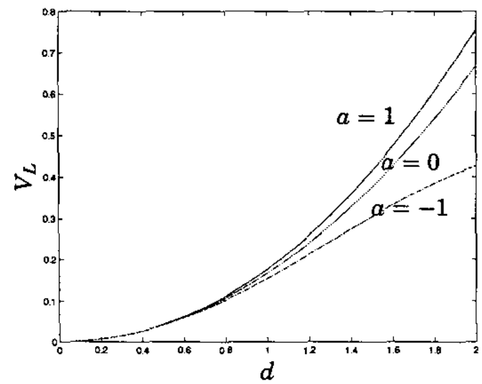

Figure 4 Variance as a function of level d for $a = -1, 0, 1$, for a system with Lebesgue sampling.

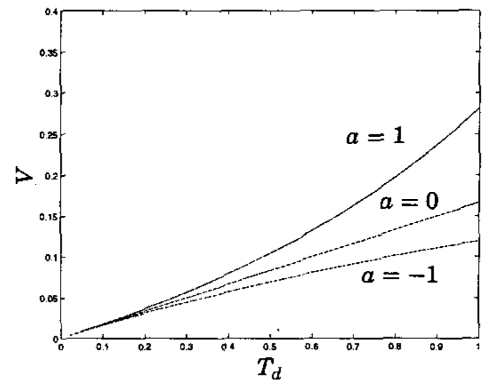

Figure 5 Variance as a function of mean exit time $T_{\pm d}$ for $a = -1, 0, 1$, for a system with Lebesgue sampling.

The stationary distribution of x is given by the forward Kolmogorov equation

$$0 = \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial f}{\partial x} - axf\right) - \left(\frac{1}{2}\frac{\partial f}{\partial x} - axf\right)_{x=d}\delta_x + \left(\frac{1}{2}\frac{\partial f}{\partial x} - axf\right)_{x=-d}\delta_x. \tag{11}$$

To solve this equation we observe that the equation

$$0 = \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial f}{\partial x} - axf\right) \tag{12}$$

has the solutions

$$f(x) = c_1 e^{ax^2} + c_2 \int_0^x e^{a(x^2-t^2)}dt.$$

The even function

$$f(x) = c_1 e^{ax^2} + c_2 \operatorname{sign}(x) \int_0^x e^{a(x^2-t^2)}dt,$$

then satisfies (11) also at $x = 0$. The constants c_1, c_2 are determined by the equations

$$\int_{-d}^{d} f(x)dx = 1, \tag{13}$$

$$f(d) = 0, \tag{14}$$

which gives a linear equation system to determine c_1, c_2.

Having obtained the stationary distribution of x we can now compute the variance of the output

$$V_L = \int_{-d}^{d} x^2 f(x)dx.$$

The variance V_L is plotted as a function of d in Figure 4 for $a = -1, 0, 1$, and as a function of mean exit time h_L in Figure 5.

Comparison

The ratio V_R/V_L as a function of h is plotted in Figure 6 for $a = -1, 0, 1$. The figure shows that Lebesgue sampling gives substantially smaller variances for the same average sampling rates. For short sampling periods there are small differences between stable and unstable system as can be expected. The improvement of Lebesgue sampling is larger for unstable systems and large sampling periods.

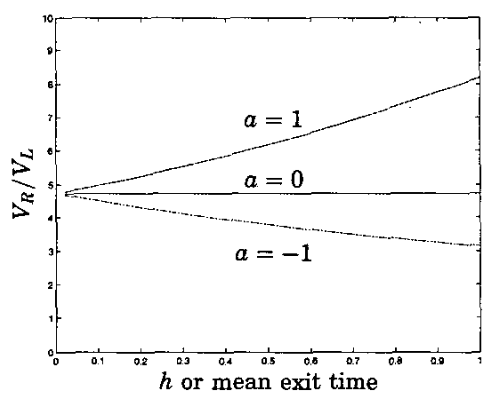

Figure 6 Comparison of V_L and V_R for $a = -1, 0, 1$. Note that the performance gain of using Lebesgue sampling is larger for unstable systems with slow sampling.

Note that the results for other a can be obtained from these plots since the transformation $(x, t, a, v) \to (\alpha^{1/2}x, \alpha t, \alpha^{-1}a, \alpha^{1/2}v)$ for $\alpha > 0$ leaves the problem invariant.

5. Conclusions

The standard approach to digital control using constant sampling rates has been very successful. There are, however, an increasing number of applications where the assumption of constant sampling rate is no longer valid, typical examples are multi-rate sampling and networked systems. Lebesgue sampling may be a useful alternative. The simple problems solved in this paper indicate that Lebesgue sampling may be worth while to pursue. The field of Lebesgue sampling is still in its infancy. There are many problems that may be worth while to pursue. The signal representation which is a mixture of analog and discrete is interesting, it is a good model for signals in biological systems. It would be very attractive to have a system theory similar to the one for periodic sampling. Particularly since many sensors that are commonly used today have this character. The design problem in the general case is still largely an unsolved problem. Implementation of controller of the type discussed in this paper can be made using programmable logic arrays without any need for AD and DA converters. There are many generalizations of the specific problems discussed in this paper that are worthy of further studies for example higher order systems and systems with output feedback.

6. References

Åström, K. J. (1970): *Introduction to Stochastic Control Theory.* Academic Press, New York.

Åström, K. J. and B. Wittenmark (1997): *Computer-Controlled Systems,* third edition. Prentice Hall.

Aubin, J. (1999): *Impulse differential inclusions and hybrid system: A viability approach.* Lecture notes, University of California, Berkeley.

Bensoussan, A. and J.-L. Lions (1984): *Impulse control and quasi-variational inequalities.* Gauthier-Villars, Paris.

Branicky, M. S., V. S. Borkar, and S. Mitter (1998): "A unified framework for hybrid control." *IEEE Transactions on Automatic Control,* **43:1**, pp. 31–45.

DeWeerth, S., L. Nielsen, C. Mead, and K. J. Åström (1990): "A neuron-based pulse servo for motion control." In *IEEE Int. Conference on Robotics and Automation.* Cincinnati, Ohio.

DiBenedetto, M. D. and A. Sangiovanni-Vincentelli (2001): *Hybrid systems in computation and control.* Springer, New York.

Dodds, S. J. (1981): "Adaptive, high precision, satellite attitude control for microprocessor implementation." *Automatica,* **17:4**, pp. 563–573.

Draper, S. S., W. Wrigley, and J. Hovorka (1960): *Inertial Guidance.* Pergamon Press, Oxford.

Feller, W. (1952): "The parabolic differential equations and the associated semi-groups of transformations." *Ann. of Math.*, **55**, pp. 468–519.

Feller, W. (1954a): "Diffusion processes in one dimension." *Trans. Am. Math. Soc.*, **55**, pp. 1–31.

Feller, W. (1954b): "The general diffusion operator and positivity preserving semi-groups in one dimension." *Ann. of Math.*, **60**, pp. 417–436.

Frank, P. M. (1979): "A continuous-time model for a pfm-controller." *IEEE Transactions on Automatic Control,* **AC-25:5**, pp. 782–784.

Hobbie, R. K. (1997): *Intermediate Physics for Medicine and Biology.* Springer.

Kopetz, H. (1993): "Should responsive systems be event triggered or time triggered?" *IEICE Trans. on Information and Systems,* **E76-D:10**, pp. 1525–1532.

Kopetz, H. (2002): "Time-triggered real-time systems." In *Proc 15th World Congress of IFAC.* Barcelona, Spain.

Mead, C. A. (1989): *Analog VLSI and Neural Systems.* Addison-Wesley, Reading, Massachusetts.

Noges, E. and P. M. Frank (1975): *Pulsfrequenzmodulierte Regelungssysteme.* R. Oldenbourg, München.

Norsworthy, S. R., R. Schreier, and G. Temes (1996): *Delta-Sigma Data Converters.* IEEE Press, New York.

Pavlidis, T. and E. J. Jury (1965): "Analysis of a new class of pulse frequency modulated control systems." *IEEE Transactions on Automatic Control,* **AC-10**, pp. 35–43.

Pettersson, B. (1969): "Production control of a complex integrated pulp and paper mill." *Tappi,* **52:11**, pp. 2155–2159.

Polak, E. (1968): "Stability and graphical analysis of first order of pulse-width-modulated sampled-data regulator systems." *IRE Trans. Automatic Control,* **AC-6:3**, pp. 276–282.

Sira-Ramirez, H. (1989): "A geometric approach to pulse-width modulated control in nonlinear dynamical systems." *IEEE Transactions on Automatic Control,* **AC-34:2**, pp. 184–187.

Skoog, R. A. (1968): "On the stability of pulse-width-modulated feedback systems." *IEEE Transactions on Automatic Control,* **AC-13:5**, pp. 532–538.

Tsypkin, Ya. Z. (1984): *Relay Control Systems.* Cambridge University Press, Cambridge, UK.

Utkin, V. (1981): *Sliding modes and their applications in variable structure systems.* MIR, Moscow.

Utkin, V. I. (1987): "Discontinuous control systems: State of the art in theory and applications." In *Preprints 10th IFAC World Congress.* Munich, Germany.