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COMPARISON OF STOCHASTIC AND DETERMINISTIC MODELS OF A LINEAR CHEMICAL REACTION WITH DIFFUSION[1]

BY DOUGLAS BLOUNT

Arizona State University

Particles placed in N cells on the unit interval give birth or die according to linear rates. Adjacent cells are coupled by diffusion with a rate proportional to N^2. Cell numbers are divided by a density parameter to represent concentrations, and the resulting space–time Markov process is compared to a corresponding deterministic model, the solution to a partial differential equation. The models are viewed as Hilbert space valued processes and compared by means of a law of large numbers and central limit theorem. New and nearly optimal results are obtained by exploiting the Ornstein–Uhlenbeck type structure of the stochastic model.

Introduction. A chemical reaction with diffusion is often modeled by a partial differential equation describing the evolution of the concentration of the reactant as a function of time and space. In Arnold and Theodosopulu (1980) a stochastic analogue of the usual model is constructed. They divide the unit interval into N cells, allow particles to diffuse between cells by simple random walks with jump rate proportional to N^2 and allow particles to be produced or removed at polynomial rates prescribed by the corresponding deterministic model. Cell numbers are divided by l, a parameter proportional to the initial number of particles in each cell. The rescaled quantities are viewed as concentrations, and the resulting space–time Markov process is compared to the deterministic model, a solution to a partial differential equation. Without diffusion, the stochastic model may be viewed as N independent density dependent birth and death processes of the type considered in Kurtz (1971) and Ethier and Kurtz (1986). By coupling the cell processes through diffusion, the model of Kurtz is extended to the spatially inhomogeneous case.

Arnold and Theodosopulu proved a law of large numbers in $L_2[0, 1]$ (square integrable functions on $[0, 1]$). Their work has been improved and extended by Kotelenez (see references) who has proved laws of large numbers and central limit theorems for linear and nonlinear reactions. He assumes reflecting boundary conditions and has extended the model to the n-dimensional unit cube. For mathematical and notational simplicity, we use a one dimensional "volume," one reactant, and assume periodic boundary conditions. We state

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only the results of Kotelenez that can be directly compared with ours. References to related work can be found in Kotelenez (1987, 1988).

Let $X^N(t)$ and $\psi(t)$ denote the stochastic and deterministic models, respectively, and $\{H_{-\alpha}\}$ the Hilbert distribution spaces (see Section 2) with $H_0 = L_2[0,1]$. For the linear model Kotelenez (1986a) has proved, assuming $N \to \infty$, that $\sup_{[0,T]} \|X^N(t) - \psi(t)\|_{-\alpha} \to_P 0$ if

$$\frac{N^2}{l} \to 0 \quad \text{for } \alpha = 0 \quad \text{and} \quad Nl \to \infty \quad \text{for } \alpha > \tfrac{3}{2}.$$

He shows weak convergence of $\sqrt{Nl}\,(X^N - \psi) \to U$ in $D_{H_{-\alpha}}[0,\infty)$ (Skorohod topology) if $\alpha > \frac{7}{2}$ and $Nl \to \infty$, where $U(t)$ is an infinite dimensional Ornstein–Uhlenbeck process in $C_{H_{-\beta}}[0,\infty)$ for $\beta > \frac{1}{2}$. For the linear model we show $\sup_{[0,T]} \|X^N(t) - \psi(t)\|_{-\alpha} \to_P 0$ if

$$l \to \infty \quad \text{as } N \to \infty \quad \text{for } \alpha = 0,$$

$$\frac{Nl}{N^{1-2\alpha}(\log N)} \to \infty \quad \text{for } \alpha \in \left(0, \tfrac{1}{2}\right),$$

$$\frac{Nl}{(\log N)^2} \to \infty \quad \text{if } \alpha = \tfrac{1}{2},$$

$$Nl \to \infty \quad \text{if } \alpha > \tfrac{1}{2}$$

and

$$\sqrt{Nl}\,(X^N - \psi) \to_{\mathscr{D}} U \quad \text{in } D_{H_{-\alpha}}[0,\infty) \text{ if } \alpha > \tfrac{1}{2} \text{ and } Nl \to \infty.$$

The primary tool in the work of Kotelenez is a maximal inequality for terms of the form $A(t) = \int_0^t U(t,s)\,dM(s)$ where $M(s)$ is a Hilbert space valued martingale and $U(t,s)$ is an evolution operator on H, the Hilbert space. The Doob-like inequality controls $\sup_{[0,T]} \|A(t)\|_H$ in terms of $E\|M(T)\|_H^2$. In our problem $U(t,s)$ is a contraction semigroup $T(t-s)$, and our improvement is due mainly to development of a technique which exploits the smoothing effects of the semigroup on the martingale M.

Note that all theorems, lemmas and numbered equations are considered on the same numbering system.

1. The deterministic model. Following Kotelenez (1982a, 1986a) and Arnold and Theodosopulu (1980) we describe the deterministic model.

For $x \in R$ let $b(x) = b_1 x + b_0$ and $d(x) = d_1 x$ where $b_1, b_0, d_1 \geq 0$. Let

$$c(x) = b(x) - d(x).$$

Δ will denote the Laplacian and D the diffusion coefficient. Let $r \in [0,1]$

denote the space variable and $t \in [0,\infty)$ the time. With periodic boundary conditions the concentration of one reactant is the solution of the reaction diffusion equation:

$$\begin{aligned} &\frac{\partial \psi}{\partial t}(t,r) = D\Delta\psi(t,r) + c(\psi(t,r)), \\ &\psi(t,0) = \psi(t,1) \quad \text{for all } t \geq 0, \\ &0 \leq \psi(0,r) \leq \rho_0 \ (\rho_0 \text{ a finite constant}). \end{aligned} \tag{1.1}$$

We take $D = 1$ for simplicity and assume $\psi(0,\cdot) \in H^2$, the space of real valued functions on $[0,1]$ with square integrable second derivatives in the distributional sense. Then there exists a constant ρ_T such that for $t \in [0,T]$, $0 \leq \psi(t,\cdot) \leq \rho_T$ [Kuiper (1977) and Arnold (1980)]. Letting $\psi(t) = \psi(t,\cdot)$, we view ψ as a function valued process. In particular, $\psi(t) \in L_2[0,1]$.

2. The stochastic model. Divide a circle of circumference 1 into N cells and place an initial distribution of particles in the cells. l is a parameter proportional to the initial average number of particles in a cell. For example, if Nl particles are initially distributed, there is an average of l particles in each cell. The system evolves in time and space as particles diffuse between cells by symmetric random walks and are produced or removed within each cell by reaction. For $1 \leq k \leq N$ and $t \geq 0$, let $n_k^N(t)$ be the number of particles in cell k at time t, and let $\mathscr{F}_t^N = \sigma(n_k^N(s); 1 \leq k \leq N, 0 \leq s \leq t)$ be the history of the process. Let $n^N(t) = (n_1^N(t), n_2^N(t), \dots, n_N^N(t)) \in R^N$. Using the polynomials $b(x)$ and $d(x)$ appearing in the definition of the deterministic model, we define the jump rates for $n^N(t)$ by

$$\begin{aligned} &(n_k, n_{k+1}) \to (n_k - 1, n_{k+1} + 1) \quad \text{at rate } N^2 n_k, \\ &(n_{k-1}, n_k) \to (n_{k-1} + 1, n_k - 1) \quad \text{at rate } N^2 n_k, \\ &n_k \to n_k + 1 \quad \text{at rate } lb(n_k/l), \\ &n_k \to n_k - 1 \quad \text{at rate } ld(n_k/l). \end{aligned} \tag{2.1}$$

For each N and l we have a different system and $n^N(t) = n^{N,l}(t)$, but we suppress the l in the superscript. $n^N(t)$ is an R^N valued $\{\mathscr{F}_t^N\}$ Markov process, which is right continuous with left limits. Let $\Lambda n_k^N(t) = n_k^N(t) - n_k^N(t-)$, where $0- = 0$. Suppose τ is an $\{\mathscr{F}_t^N\}$ stopping time such that

$$\sup_{1\leq k\leq N}\ \sup_{t\geq 0} n_k^N(t\wedge\tau)\cdot I_{\{\tau>0\}} < M(N,l) < \infty.$$

Then $n^N(t)$ has a bounded total jump rate and we have the following lemma, a variant of a similar result of Kotelenez (1982a, 1986a). The proof, based on a result of Kurtz (1971), is found in Blount (1987).

LEMMA 2.2. *For $1 \le k \le N$ the following are $\{\mathscr{F}_t^N\}$ martingales*:

$$\text{(a)}\quad \sum_{s \le t\wedge\tau} \Lambda n_k^N(s) - \int_0^{t\wedge\tau}\big[N^2\big(n_{k-1}^N(s) - 2n_k^N(s) + n_{k+1}^N(s)\big) + lb\big(n_k^N(s)/l\big) + ld\big(n_k^N(s)/l\big)\big]\,ds;$$

$$\text{(b)}\quad \sum_{s \le t\wedge\tau} \big(\Lambda n_k^N(s)\big)^2 - \int_0^{t\wedge\tau}\big[N^2\big(n_{k-1}^N(s) + 2n_k^N(s) + n_{k+1}^N(s)\big) + lb\big(n_k^N(s)/l\big) + ld\big(n_k^N(s)/l\big)\big]\,ds;$$

$$\text{(c)}\quad \sum_{s \le t\wedge\tau} \big(\Lambda n_k^N(s)\big)\big(\Lambda n_{k+1}^N(s)\big) + \int_0^{t\wedge\tau} N^2\big(n_k^N(s) + n_{k+1}^N(s)\big)\,ds.$$

Now we construct a space–time jump Markov process which is the stochastic analogue of (1.1). For $t \ge 0$ and $r \in [0,1)$ let

$$X^{N,l}(t,r) = \frac{n_k^N(t)}{l} \quad \text{for} \quad r \in \left[\frac{k-1}{N}, \frac{k}{N}\right), \qquad 1 \le k \le N. \tag{2.3}$$

We view $\{n_k^N\}$ as a periodic sequence satisfying $n_{k+N}^N = n_k^N$, so $X^{N,l}(t,r)$ is periodic in r with period 1. We suppress the l in the superscript.

Let H^N be the space of step functions on $[0,1)$ which are constant on the intervals $[(k-1)/N, k/N)$. Consider them extended to periodic functions. Define P_N: $L_2[0,1] \to H^N$, an orthogonal projection, by

$$P_N f(r) = N\int_{k/N}^{(k+1)/N} f(x)\,dx \quad \text{for} \quad r \in \left[\frac{k}{N}, \frac{k+1}{N}\right), \qquad 0 \le k \le N-1. \tag{2.4}$$

Similarly, define

$$\begin{aligned} \nabla_N^{\pm} f(r) &= N\big[P_N f(r \pm N^{-1}) - P_N f(r)\big],\\ \Delta_N f(r) &= -\nabla_N^- \nabla_N^+ f(r) = -\nabla_N^+ \nabla_N^- f(r)\\ &= N^2\big[P_N f(r + N^{-1}) - 2P_N f(r) + P_N f(r - N^{-1})\big]. \end{aligned} \tag{2.5}$$

Δ_N is a discrete analogue of the Laplacian. By first applying P_N we consider Δ_N and $\nabla_N^{\pm}$ extended to all of $L_2[0,1]$. Let

$$X^N(t) = X^N(t,\cdot) \in H^N \subset L_2[0,1]. \tag{2.6}$$

X^N is an H^N valued $\{\mathscr{F}_t^N\}$ Markov process, and using (2.3) and Lemma 2.2 we can write

$$X^N(t) = X^N(0) + \int_0^t \Delta_N X^N(s)\,ds + \int_0^t c\big(X^N(s)\big)\,ds + Z^N(t), \tag{2.7}$$

where $Z^N(t \wedge \tau)$ is an H^N valued martingale for τ as in Lemma 2.2.

Now we follow Kotelenez in defining the spaces in which the models are compared. Since $\psi(t)$, $X^N(t) \in L_2[0,1]$ we may compare them in this space and in the following spaces of distributions. For $r \in [0,1]$ and $m \in \{0, 2, 4, \dots\}$, let $\phi_0(r) \equiv 1$, $\phi_m(r) = \sqrt{2}\cos \pi m r$ and $\psi_m(r) = \sqrt{2}\sin \pi m r$ for $m \geq 2$. Let $\beta_m = \pi^2 m^2$. Then $\{\phi_m, \psi_m\}$ form a complete orthonormal system in $L_2[0,1]$ and $\Delta e_m = -\beta_m e_m$ for $e_m = \phi_m$ or ψ_m.

For $f, g \in L_2[0,1]$, let $\langle f, g\rangle$ denote the usual inner product. For $\alpha \geq 0$ and $f \in L_2[0,1]$, let

$$\|f\|_{-\alpha}^2 = \sum_m \frac{\langle f, \phi_m\rangle^2 + \langle f, \psi_m\rangle^2}{(1+\beta_m)^\alpha}$$

and let $H_{-\alpha}$ be the completion of $L_2[0,1]$ in the norm $\|\cdot\|_{-\alpha}$. Note $H_0 = L_2[0,1]$. Define H_α for $\alpha > 0$ by

$$H_\alpha = \left\{ f \in H_0 \colon \|f\|_\alpha^2 = \sum_m \left(\langle f, \phi_m\rangle^2 + \langle f, \psi_m\rangle^2\right)(1+\beta_m)^\alpha < \infty \right\}.$$

For $\alpha \in R$ we can then summarize the definition of H_α as

$$H_\alpha = \left\{ f \colon f = \sum a_m \phi_m + b_m \psi_m, \sum \left(a_m^2 + b_m^2\right)(1+\beta_m)^\alpha < \infty \right\}.$$

If $\alpha \geq 0$ there is a natural duality between $H_{-\alpha}$ and H_α defined by

$$(f, g) = \sum a_m(f) a_m(g) + b_m(f) b_m(g) \quad \text{for } f \in H_{-\alpha}, g \in H_\alpha.$$

We have $|(f,g)| \leq \|f\|_{-\alpha}\|g\|_\alpha$. If $f \in H_0 \cap H_{-\alpha}$ and $g \in H_\alpha \subset H_0$, then $(f,g) = \langle f, g\rangle$. We use $\langle\ ,\ \rangle$ to denote both the inner product in $H_0 = L_2[0,1]$ and the dual pairing between $H_{-\alpha}$ and H_α.

Let $e^{\Delta t}$ denote the contraction semigroup generated by the Laplacian on H_0. Then $e^{\Delta t} e_m = e^{-\beta_m t} e_m$ for $e_m = \phi_m$ or ψ_m, and $e^{\Delta t}$ can be extended to any H_α by letting $e^{\Delta t} f = \sum e^{-\beta_m t}(\langle f, \phi_m\rangle \phi_m + \langle f, \psi_m\rangle \psi_m)$.

Note $H^N \subset H_{-\alpha}$ for $\alpha \geq 0$. To compute the norm $\|\cdot\|_{-\alpha}$ of functions in H^N it is convenient to introduce an alternative but equivalent norm. Assume N is an odd integer. Let $\phi_{0,N}(r) \equiv 1$, and for $m \in \{2, 4, \dots, N-1\}$, $r \in [0,1)$, let

$$\phi_{m,N}(r) = \sqrt{2}\cos\left(\pi m \frac{k}{N}\right) \quad \text{and} \quad \psi_{m,N}(r)\sqrt{2}\sin\left(\pi m \frac{k}{N}\right)$$

for $r \in [k/N, (k+1)/N)$, $0 \leq k \leq N-1$. For $f \in H^N$ and $\alpha \geq 0$ define $\|f\|_{-\alpha,N}$ by

$$\|f\|_{-\alpha,N}^2 = \sum_m \frac{\langle f, \phi_{m,N}\rangle^2 + \langle f, \psi_{m,N}\rangle^2}{(1+\beta_{m,N})^\alpha}, \tag{2.8}$$

where we take $\psi_{0,N} \equiv 0$ in the sum and $\{-\beta_{m,N}\}$ are eigenvalues of Δ_N defined in Lemma 2.9(d). The following lemma is elementary but very useful. We omit the proof, which is computational but straightforward [see Lemma 2.12 of Blount (1987)]. Let $\|f\|_\infty$ denote the essential supremum of $f \in H_0$.

LEMMA 2.9. (a) *If* $v, w \in H^N$, *then* $\langle \nabla_N^- v, w\rangle = \langle v, \nabla_N^+ w\rangle$.

(b) Δ_N: $H^N \to H^N$ *is self-adjoint.*

(c) $\{\phi_{m,N}, \psi_{m,N}\}$ *form an orthonormal basis for* H^N *considered as a subspace of* H_0.

(d) $\Delta_N \phi_{m,N} = -\beta_{m,N}\phi_{m,N}$ *and* $\Delta_N \psi_{m,N} = -\beta_{m,N}\psi_{m,N}$, *where* $-\beta_{m,N} = 2N^2(\cos(\pi m/N) - 1)$.

(e) *There exist constants* $0 < c_1 < c_2 < \infty$ *with* $c_1 < \beta_{m,N}/\beta_m < c_2$ *for* $2 \le m < N$. (*Recall* $\beta_m = \pi^2 m^2$.)

(f) *If* $e_{m,N} = \phi_{m,N}$ *or* $\psi_{m,N}$, *then* $\|\nabla_N^{\pm} e_{m,N}\|_\infty \le \sqrt{2}\,\pi m$.

(g) *There exist constants* $0 < c_1(\alpha) \le c_2(\alpha) < \infty$ *with* $c_1(\alpha)\|f\|_{-\alpha} \le \|f\|_{-\alpha,N} \le c_2(\alpha)\|f\|_{-\alpha}$ *for* $f \in H^N$.

REMARKS. For $h \in H_0$, Kotelenez (1982a) showed $\|h\|_{-\alpha} \le 10\|h\|_{-\alpha,N}$ for $\alpha = 2$. Thus for $h \in H^N$, Lemma 2.9(g) extends this type of inequality to all $\alpha > 0$. Since H^N is finite dimensional, any two norms are equivalent, but the point of the lemma is that constants defining the equivalence may be taken independent of N.

The last result of this section is a variant of a similar result of Kotelenez (1982a, 1986a). For $V \in H^N$ let $Z_V^N(t) = \langle Z^N(t), V\rangle$. Let $|c|(x) = b(x) + d(x)$.

LEMMA 2.10. *Let* $V, W \in H^N$. *Then*

$$\sum_{s \le t\wedge\tau} (\Lambda Z_V^N(s))(\Lambda Z_W^N(s)) - \frac{1}{Nl}\int_0^{t\wedge\tau} \Big[\langle X^N(s), (\nabla_N^+ V)(\nabla_N^+ W) + (\nabla_N^- V)(\nabla_N^- W)\rangle + \langle |c|(X^N(s)), VW\rangle\Big]\, ds$$

is an $\{\mathscr{F}_t^N\}$ *martingale.*

PROOF. Note that $\Lambda Z^N(t) = \Lambda X^N(t)$ by (2.7). The proof is then a direct computation using Lemma 2.2, the definition (2.3) of $X^N(t)$ and the fact that $(\Lambda n_i^N(t))(\Lambda n_j^N(t)) = 0$ for $j \notin \{i-1, i, i+1\}$. □

3. Law of large numbers for the linear model. Recall $b(x) = b_1 x + b_0$ and $d(x) = d_1 x$. We have

$$X^N(t) = X^N(0) + \int_0^t A_N X^N(s)\, ds + b_0 t + Z^N(t), \tag{3.1}$$

$$\psi(t) = \psi(0) + \int_0^t A\psi(s)\, ds + b_0 t, \tag{3.2}$$

where $A_N = \Delta_N + b_1 - d_1$ and $A = \Delta + b_1 - d_1$. In this section we prove the following theorem.

THEOREM 3.3. *Let $\alpha > 0$ and assume*

(i) $N \to \infty$.

(ii) $l \to \infty$ *as* $N \to \infty$ *if* $\alpha = 0$,

$$\frac{Nl}{N^{1-2\alpha}\log N} \to \infty \quad \text{if } \alpha \in \left(0, \tfrac{1}{2}\right),$$

$$\frac{Nl}{(\log N)^2} \to \infty \quad \text{if } \alpha = \tfrac{1}{2},$$

$$Nl \to \infty \quad \text{if } \alpha > \tfrac{1}{2}.$$

(iii) $\|X^N(0) - \psi(0)\|_{-\alpha} \to 0$ *in probability.*

Then for all $T > 0$ *and* $\delta > 0$,

$$P\left[\sup_{[0,T]} \|X^N(t) - \psi(t)\|_{-\alpha} > \delta\right] \to 0.$$

REMARK 3.4. Theorem 3.3 eliminates the moment assumption of Kotelenez (1986a) and extends the LLN to $H_{-\alpha}$ for all $\alpha \geq 0$. Both theorems make a high density assumption for $\alpha = 0$, but Theorem 3.3 allows l to grow slowly compared to N. In Theorem 3.3 the density may be low for any $\alpha > \frac{1}{2}$, as long as the initial number of particles approaches infinity. For $\alpha \in (0, \frac{1}{2}]$ the initial number of particles must approach infinity sufficiently fast. The result for $\alpha > \frac{1}{2}$ gives a corresponding improvement in the CLT. Before beginning the proof we give a simple example which indicates the best theorem of this type possible has assumption (ii) replaced by the single condition

$$\text{(ii}')\qquad \lim_{N\to\infty} (Nl)^{-1} \sum_{n=1}^{N} n^{-2\alpha} = 0.$$

This agrees with (ii) for $\alpha = 0$ or $\alpha > \frac{1}{2}$, but eliminates a factor of $\log N$ in the assumptions for $\alpha \in (0, \frac{1}{2}]$. We believe Theorem 3.3 holds with (ii) replaced by (ii′) but cannot prove it.

EXAMPLE 3.5. Let the initial numbers of particles in each cell, $\{n_k^N(0)\}_{k=1}^N$, be distributed as independent Poisson random variables with mean l. Assume $b(x) = d(x) \equiv 0$. The distribution is then stationary, so consider $X^N(t)$ for any fixed t: $X^N(t, k/N) = l^{-1}C_k^N$, $0 \leq k \leq N-1$, where the $\{C_k^N\}$ are independent Poisson with mean l. Note $EX^N(t) \equiv 1$ and set $\psi(0) \equiv 1$. Thus $\psi(t) \equiv 1$.

Let

$$a_m^N = \langle X^N(t) - 1, \varphi_{m,N}\rangle = \frac{1}{Nl}\sum_k (C_k^N - l)\varphi_{m,N}\left(\frac{k}{N}\right),$$

$$b_m^N = \langle X^N(t) - 1, \psi_{m,N}\rangle = \frac{1}{Nl}\sum_k (C_k^N - l)\psi_{m,N}\left(\frac{k}{N}\right).$$

Then $E(a_m^N)^2 = E(b_m^N)^2 = (Nl)^{-1}$, since $\|\varphi_{m,N}\|_0^2 = \|\psi_{m,N^2}\|_0 = 1$. Thus

$$E\|X^N(t) - 1\|_{-\alpha,N}^2 = E\sum_m \frac{(a_m^N)^2 + (b_m^N)^2}{(1+\beta_{m,N})^\alpha}$$
$$= \left(\frac{2}{Nl}\sum_{m>0}(1+\beta_{m,N})^{-\alpha}\right) + \frac{1}{Nl}.$$

Recall $c_1 < \beta_{m,N}/m^2 < c_2$ for $m \neq 0$. Since the distribution is spatially homogeneous, we can obtain convergence by keeping N fixed and letting $l \to \infty$. However, if the initial distribution is not homogeneous, we need $N \to \infty$ to force $\Delta_N \to \Delta$. Assuming $N \to \infty$, we have $E\|X^N(t) - 1\|_{-\alpha,N} \to 0$ if and only if $(1/Nl)\Sigma_1^N n^{-2\alpha} \to 0$ as $N \to \infty$. By Lemma 2.9 the same result holds using $\|\cdot\|_{-\alpha}$. Assuming $\sup_N\|EX^N(0)\|_\infty < \infty$, a similar result can be proved assuming any nondegenerate initial distribution. Since we are only showing convergence for a fixed t, this indicates that (ii') is the weakest possible assumption.

Now we follow Kotelenez and use variation of constants to rewrite (3.1) and (3.2) as

$$X^N(t) = e^{A_N t}X^N(0) + \int_0^t e^{A_N(t-s)}b_0\,ds + \int_0^t e^{A_N(t-s)}\,dZ^N(s), \tag{3.6}$$

$$\psi(t) = e^{At}\psi(0) + \int_0^t e^{A(t-s)}b_0\,ds, \tag{3.7}$$

where $e^{A_N t}$ and e^{At} are the respective semigroups generated on H_0 by A_N and A. As discussed prevously $e^{\Delta t}$ extends to a contraction semigroup on every $H_{-\alpha}$. Using the representation

$$e^{\Delta_N t}f = \sum_m e^{-\beta_{m,N}t}(\langle f, \varphi_{m,N}\rangle\varphi_{m,N} + \langle f, \psi_{m,N}\rangle\psi_{m,N}), \tag{3.8}$$

$e^{\Delta_N t}$ extends to a contraction semigroup on H^N in the norm $\|\cdot\|_{-\alpha,N}$ for $\alpha \geq 0$ and on H_0. Since $e^{\Delta_N t}b_0 = e^{\Delta t}b_0 = b_0$, we have

$$X^N(t) - \psi(t) = e^{A_N t}(X^N(0) - \psi(0)) + (e^{A_N t} - e^{At})\psi(0) + Y^N(t), \tag{3.9}$$

where $Y^N(t) = \int_0^t e^{A_N(t-s)}\,dZ^N(s)$. Choose $\delta \in (0,1)$ and let

$$\tau = \tau(N, l, \alpha, \delta) = \inf\{t : \|X^N(t) - \psi(t)\|_{-\alpha} \geq \delta\}.$$

It suffices to consider $X^N(t\wedge\tau) - \psi(t\wedge\tau)$. Recall $\sup_{[0,T]}\|\psi(t)\|_\infty \leq \rho_T$. By definition of the possible transitions for $X^N(t)$,

$$\|X^N(t) - X^N(t-)\|_{-\alpha,N} \leq \frac{C}{Nl}\left(\sum_{m=1}^N m^{-2\alpha}\right)^{1/2} \to 0$$

under the assumptions of Theorem 3.3. Thus, using the equivalence of $\|\cdot\|_{-\alpha,N}$ and $\|\cdot\|_{-\alpha}$ on H^N and taking ρ_T larger if necessary, we may assume

$$\sup_{[0,T]}\left\|1_{\{\tau>0\}}X^N(t\wedge\tau)\right\|_{-\alpha} \leq \rho_T \quad\text{and}\quad \rho_T \geq 1. \tag{3.10}$$

Since X^N is a finite dimensional process and $\|\varphi_{m,N}\|_\infty \le \sqrt{2}$, $\|\psi_{m,N}\|_\infty \le \sqrt{2}$, we have $\sup_{[0,T]}\|1_{\{\tau>0\}}X^N(t\wedge\tau)\|_\infty \le c(T,N,l) < \infty$. Thus $X^N(t\wedge\tau)$ has a bounded total jump rate for $t\in[0,T]$, $T<\infty$.

Using (3.9) we can write

(3.11)
$$X^N(t\wedge\tau) - \psi(t\wedge\tau) = e^{A_N(t\wedge\tau)}(X^N(0)-\psi(0)) + (e^{A_N(t\wedge\tau)} - e^{A(t\wedge\tau)})\psi(0) + Y^N(t\wedge T).$$

By Trotter–Kato, $\lim_{N\to\infty}\sup_{[0,T]}\|(e^{A_N(t\wedge\tau)} - e^{A(t\wedge\tau)})\psi(0)\|_0 = 0$. Using $\|P_N f - f\|_0 \to 0$ for $f\in H_0$ and the equivalence of $\|\cdot\|_{-\alpha}$ and $\|\cdot\|_{-\alpha,N}$ shows $\sup_{[0,T]}\|e^{A_N(t\wedge\tau)}(X^N(0)-\psi(0))\|_{-\alpha} \to_P 0$. Thus, to prove Theorem 3.3 it remains to show $\sup_{[0,T]}\|Y^N(t\wedge\tau)\|_{-\alpha}\to_P 0$.

Let $-\theta_{m,N} = b_1 - d_1 - \beta_{m,N}$. Then $A_N e_{m,N} = -\theta_{m,N}e_{m,N}$ for $e_{m,N} = \varphi_{m,N}$ or $\psi_{m,N}$. By definition of Y^N we have, for $e_{m,N} = \varphi_{m,N}$ or $\psi_{m,N}$,

$$\langle Y^N(t), e_{m,N}\rangle = \int_0^t e^{-\theta_{m,N}(t-s)}\,d\langle Z^N(s), e_{m,N}\rangle.$$

We use $Y_m^N(t)$, $Z_m^N(t)$ to denote either of the respective pairs $\langle Y^N(t), \varphi_{m,N}\rangle$, $\langle Z^N(t), \varphi_{m,N}\rangle$ or $\langle Y^N(t), \psi_{m,N}\rangle$, $\langle Z^N(t), \psi_{m,N}\rangle$. Thus we have

(3.12)
$$Y_m^N(t) = \int_0^t e^{-\theta_{m,N}(t-s)}\,dZ_m^N(s),$$
$$\|Y^N(t)\|^2_{-\alpha,N} = \sum \frac{(Y_m^N(t))^2}{(1+\beta_{m,N})^\alpha}.$$

Note that by Lemma 2.9 at most a finite number of $\{-\theta_{m,N}\}$ are nonnegative, and for $N\ge N_0$ there are m_0, C_1 and C_2 which may be taken independent of N, such that for

(3.13)
$$m\ge m_0,\qquad 0 < C_1 m^2 < \theta_{m,N} < C_2 m^2.$$

REMARK 3.14. The results of Kotelenez were proved by applying a very general maximal inequality [Kotelenez (1982b)] to $Y^N(t)$. Using only Doob's inequality and the spectral properties of Δ_N, the following lemma gives a straightforward proof of a special case of this inequality relating to our problem.

LEMMA 3.15. $(Y_m^N(t\wedge\tau))^2 \le B_m^N(t)$, *where $B_m^N(t)$ is a submartingale with*
$$EB_m^N(t) \le c(t)(m^2+1)(Nl)^{-1}.$$

PROOF. From (3.12) we have
$$dY_m^N(t) = -\theta_{m,N}Y_m^N(t)\,dt + dZ_m^N(t),\qquad Y_m^N(0) = 0.$$
Let $[Z_m^N](t) = \sum_{s\le t}(\Lambda Z_m^N(s))^2$ be the quadratic variation of $Z_m^N(t)$. By Itô's

formula

$$\left(Y_m^N(t \wedge \tau)\right)^2 = -2\theta_{m,N} \int_0^{t\wedge\tau} \left(Y_m^N(s)\right)^2 ds + \int_0^{t\wedge\tau} 2Y_m^N(s-)\, dZ_m^N(s) + \left[Z_m^N\right](t \wedge \tau).$$

Let

$$B_m^N(t) = \left(Y_m^N(t \wedge \tau)\right)^2 \quad \text{if } -\theta_{m,N} \geq 0,$$

$$B_m^N(t) = \int_0^{t\wedge\tau} 2Y_m^N(s-)\, dZ_m^N(s) + \left[Z_m^N\right](t\wedge\tau) \quad \text{if } -\theta_{m,N} < 0.$$

Recall $m \geq m_0$ implies $-\theta_{m,N} < 0$, so $EB_m^N(t) \leq e^{2m_0 t} E[Z_m^N](t \wedge \tau)$. By Lemmas 2.9 and 2.10 and (3.10),

$$E\left[Z_m^N\right](t \wedge \tau) \leq C\rho_T(m^2+1)(Nl)^{-1},$$

which proves the result. □

The proof of Lemma 3.15 shows $\|Y^N(t \wedge \tau)\|_{-\alpha,N}^2 \leq B^N(t)$, where $B^N(t)$ is a nonnegative submartingale satisfying

$$EB^N(t) \leq e^{2m_0 t} E\|Z^N(t\wedge\tau)\|_{-\alpha,N}^2 \leq C(t)(Nl)^{-1} \sum_{n=1}^{N} \frac{n^2}{n^{2\alpha}}.$$

Using the equivalence of $\|\cdot\|_{-\alpha}$ and $\|\cdot\|_{-\alpha,N}$ and applying Doob's inequality shows

$$P\left[\sup_{[0,T]} \|Y^N(t\wedge\tau)\|_{-\alpha}^2 > \varepsilon^2\right] \leq \frac{C(T)}{Nl\varepsilon^2} \sum_{n=1}^{N} n^{2(1-\alpha)}. \tag{3.16}$$

For $\alpha \in \{0\} \cup (\frac{3}{2}, \infty)$ a similar inequality was used in Kotelenez (1986a). By (3.16) and the discussion after (3.11) we have extended the LLN in that paper to the following result.

THEOREM 3.17. *Let $\alpha \geq 0$ and assume*

(i) $N \to \infty$.

(ii) $N^{2\alpha}l/N^2 \to \infty$ *if* $\alpha \in [0, \frac{3}{2})$, $Nl/\log N \to \infty$ *if* $\alpha = \frac{3}{2}$ *and* $Nl \to \infty$ *if* $\alpha > \frac{3}{2}$.

(iii) $\|X^N(0) - \psi(0)\|_{-\alpha} \to_P 0$.

Then for all $T > 0$ and $\delta > 0$, $P[\sup_{[0,T]}\|X^N(t) - \psi(t)\|_{-\alpha} > \delta] \to 0$.

REMARK 3.18. The proof of (3.16) is based on the fact that Δ_N has nonpositive eigenvalues, but does not fully exploit this fact. The following lemma will be used in the proof of the subsequent CLT. We prove it here since the method can be generalized to give a proof of Theorem 3.3 which more fully exploits the spectral properties of Δ_N and $e^{\Delta_N t}$.

LEMMA 3.19. *Let* $V(t) = \int_0^t e^{-\beta(t-s)}\sqrt{2\gamma(s)}\,dW(s)$, *where* $\beta > 0$, $0 < \Gamma = \sup_{[0,T]}\gamma(s) < \infty$ *and* $W(s)$ *is a standard Brownian motion. Then*

$$P\Big[\sup_{[0,T]} V^2(t) \ge A\Big] \le \frac{T}{f(A)},$$

where

$$f(A) = \beta^{-1}\int_0^{(\beta A/\Gamma)^{1/2}} e^{s^2/2}\int_0^s e^{-r^2/2}\,dr\,ds.$$

PROOF. Let $Dg(u) = g'(u)(-2\beta u + 2\Gamma) + 4\Gamma u g''(u)$ for $g \in C^2(R)$. It is easily shown that $Df(u) \equiv 1$, $f(0) = 0$ and $f'(u), f''(u) > 0$ for $u > 0$. We have $dV^2(t) = -2\beta V^2(t)\,dt + 2V(t)\sqrt{2\gamma(t)}\,dW(t) + 2\gamma(t)\,dt$. By Itô's formula, for $0 \le t \le T$,

$$\begin{aligned} f(V^2(t)) &= \int_0^t \big[f'(V^2(s))(-2\beta V^2(s) + 2\gamma(s)) \\ &\qquad + 4f''(V^2(s))V^2(s)\gamma(s)\big]\,ds + R(t) \\ &\le \int_0^t \big[f'(V^2(s))(-2\beta V^2(s) + 2\Gamma) + 4f''(V^2(s))V^2(s)\Gamma\big]\,ds + R(t) \\ &= \int_0^t Df(V^2(s))\,ds + R(t) = t + R(t), \end{aligned}$$

where $R(t)$ is a martingale with $ER(t) = 0$. Let $\tau_A = \inf\{t\colon V^2(t) = A\}$ and note $f(u)$ is strictly increasing on $[0,\infty)$. Thus

$$P\Big[\sup_{[0,T]} V^2(t) \ge A\Big] \le \frac{E\big(f(V^2(T \wedge \tau_A))\big)}{f(A)} \le \frac{E(T \wedge \tau_A)}{f(A)} \le \frac{T}{f(A)}. \qquad \square$$

The proof of Lemma 3.15 shows that $Y^N(t)$ can be decomposed into what resembles a system of Ornstein–Uhlenbeck processes. We can exploit this structure by applying the method used in Lemma 3.19 to $Y^N(t)$. First we need a simple technical lemma.

LEMMA 3.20. (a) *The function*

$$f(v) = \frac{1}{a}\sum_{n=1}^{\infty}\frac{[(a/c)(v+e)]^n}{\prod_{j=0}^{n-1}((b+ae)/c + j)}$$

solves $f'(v)(b - av) + f''(v)c(v+e) = 1$.

(b) *If* $a, b, c, e > 0$ *and* $0 \le \Delta v < c/a$, *then for* $v \ge 0$,

$$\frac{f''(v+\Delta v)}{f''(v)} \le \frac{1}{1 - a\Delta v/c}.$$

PROOF. (a) can be checked directly. A term by term comparison of the power series for f'' and f''' shows $f'''(v) \le (a/c)f''(v)$ for $v > 0$. Thus, for some $\bar{v} \in [v, v + \Delta v]$,

$$\begin{aligned} f''(v + \Delta v) &= f''(v) + f'''(\bar{v})\Delta v \\ &\le f''(v) + f'''(v + \Delta v)\Delta v \\ &\le f''(v) + \frac{a}{c} f''(v + \Delta v)\Delta v. \qquad \square \end{aligned}$$

The following lemma provides the estimates needed to prove Theorem 3.3. For its proof, the assumptions of Theorem 3.3 are in effect and we are stopping in the sense defined after (3.9). $|B|$ denotes cardinality of a set.

LEMMA 3.21. (a) *Suppose we are stopping in the norm* $\|\cdot\|_{-\alpha}$ *for* $\alpha \ge 0$. *Then, for* $m \ge m_0$ *and* $A > 0$,

$$\begin{aligned} P\Big[\sup_{[0,T]} \big(Y_m^N(t \wedge \tau)\big)^2 \ge A\Big] &\le C(T) m^2 \left[\sum_{n=1}^{\infty} \frac{(rNlA)^n}{(\beta\rho_T)^n (n+1)!}\right]^{-1} \\ &= C(T) m^2 u (e^u - u - 1)^{-1}, \end{aligned}$$

where $u = rNlA/\beta\rho_T$, $r > 0$ *does not depend on* N *or* m, *and* $\beta = \max(2, 2\sqrt{A})$.

(b) *Suppose we are stopping in the norm* $\|\cdot\|_0$. *Let* $[\cdot]$ *denote the greatest integer function. Suppose* $B \subset \{0, 2, \dots, N-1\} \cap \{m \ge m_0\}$ *is such that* $\overline{m} < C\underline{m}$, *where* $\overline{m}$ *and* $\underline{m}$ *are the maximal and minimal elements of* B *and* $C = C(B)$ *is a constant not dependent on* N. *Let* $|B| \le L = [(\log N)^2]$. *Then for* $0 < q \le 1$,

$$P\Big[\sup_{[0,T]} \sum_B \big(Y_m^N(t \wedge \tau)\big)^2 > \frac{qL}{N}\Big] \le C(T) N^2 L^{1/2} \left(\frac{rlq}{\rho_T}\right)^{-[L^{1/2}]},$$

where $r > 0$ *is proportional to* $C^{-1}(B)$.

PROOF. We will suppress the N in the superscript on Y_m^N, Z_m^N and X^N for the duration of the proof. Recall $Y_m(t) = \int_0^t \exp(-\theta_{m,N}(t-s))\, dZ_m(s)$ and that for $m \ge m_0$, $0 < C_1 \le \theta_{m,N}/m^2 \le C_2 < \infty$ for C_1, C_2 not dependent on N. We have

$$\begin{aligned} dY_m(t) &= -\theta_{m,N} Y_m(t)\, dt + dZ_m(t), \qquad Y_m(0) = 0, \\ dY_m^2(t) &= -2\theta_{m,N}^2 Y_m^2(t)\, dt + 2Y_m(t-)\, dZ_m(t) + d[Z_m](t). \end{aligned} \tag{3.22}$$

Let B be a set of indices as described in the assumptions of the lemma. B may have only one element, which is the case in Lemma 3.21(a). Suppose $f(u)$ is a C^3 function such that $f^{(i)}(u) > 0$ for $u > 0$ and $i \le 3$. Let $\tau_A = \inf\{t\colon \sum Y_m^2(t \wedge \tau) \ge A\}$ and let $\delta = \tau \wedge \tau_A$. Summing over $m \in B$ and using change of variables for functions of bounded variation,

$$
\begin{aligned}
f\left(\sum Y_m^2(t\wedge\delta)\right) &= f(0) + \int_0^{t\wedge\delta} f'\left(\sum Y_m^2(s-)\right) d\left(\sum Y_m^2(s)\right)\\
&\quad + \sum_{s\le t\wedge\delta}\left(f\left(\sum Y_m^2(s)\right) - f\left(\sum Y_m^2(s-)\right)\right)\\
&\quad - \sum_{s\le t\wedge\delta} f'\left(\sum Y_m^2(s-)\right)\Lambda\left(\sum Y_m^2(s)\right)\\
&= f(0) + \int_0^{t\wedge\delta} f'\left(\sum Y_m^2(s-)\right) d\left(\sum Y_m^2(s)\right)\\
&\quad + \tfrac{1}{2}\sum_{s\le t\wedge\delta} f''(E(s))\left(\Lambda\sum Y_m^2(s)\right)^2,
\end{aligned}
\tag{3.23}
$$

where $E(s)$ is between $\sum Y_m^2(s-)$ and $\sum Y_m^2(s)$. From 3.22,

$$\Lambda Y_m^2(s) = (\Lambda Y_m(s))^2 + 2Y_m(s-)\Lambda Y_m(s) = (\Lambda Z_m(s))^2 + 2Y_m(s-)\Lambda Z_m(s).$$

By the definition of $Z_m(t)$ and the possible transitions for $X^N(t)$, $|\Lambda Z_m(t)| \le 2\sqrt{2}/Nl$. Thus

$$\left|\Lambda\sum Y_m^2(s)\right| \le \frac{16L}{N^2l^2} + \frac{4\sqrt{2}\left(2L\sum Y_m^2(s-)\right)^{1/2}}{Nl},$$

using Jensen's inequality and the fact that $Y_m(t)$ represents $\langle Y^N, \phi_{m,N}\rangle$ and $\langle Y^N, \psi_{m,N}\rangle$ in the sum. For $s \le \tau_A$, $(\sum Y_m^2(s-))^{1/2} \le \sqrt{A}$ and

$$
\begin{aligned}
\left(\Lambda\sum Y_m^2(s)\right)^2 &= \left[\sum(\Lambda Z_m(s))^2 + 2Y_m(s-)\Lambda Z_m(s)\right]^2\\
&\le 4\left(\sum Y_m(s-)\Lambda Z_m(s)\right)^2 + \varepsilon\sum(\Lambda Z_m(s))^2,
\end{aligned}
$$

$$\text{for } \varepsilon = \frac{16L}{N^2l^2} + \frac{8\sqrt{LA}}{Nl}.$$

We also have $|\mathbf{L}\sum Y_m^2(s)| \le \varepsilon$. Let $\gamma_f = \sup\{(f''(u+\varepsilon))/(f''(u)) : u \ge 0\}$. Since $f'''(u) > 0$ for $u > 0$, we have, from (3.23) and the last estimate,

$$
\begin{aligned}
&f\left(\sum Y_m^2(t\wedge\delta)\right)\\
&\quad\le f(0) + \int_0^{t\wedge\delta} f'\left(\sum Y_m^2(s-)\right)\\
&\qquad\times\left[-2\sum\theta_{m,N}Y_m^2(s)\,ds + 2\sum Y_m(s-)\,dZ_m(s) + \sum d[Z_m](s)\right]\\
&\qquad + 2\gamma_f\sum_{s\le t\wedge\delta} f''\left(\sum Y_m^2(s-)\right)\\
&\qquad\quad\times\left[\left(\sum Y_m(s-)\Lambda Z_m(s)\right)^2 + \varepsilon\sum(\Lambda Z_m(s))^2\right].
\end{aligned}
$$

Assume $\gamma_f < \infty$, as it will be for the appropriate f and let V_m denote $\varphi_{m,N}$ or

$\psi_{m,N}$. By Lemma 2.10 we have

$$
\begin{aligned}
f\big(\textstyle\sum Y_m^2(t\wedge\delta)\big) &\le f(0) + \int_0^{t\wedge\delta} f'\big(\textstyle\sum Y_m^2(s)\big) \\
&\quad\times\left[\begin{array}{l} -2\sum \theta_{m,N} Y_m^2(s) + (Nl)^{-1}\langle |c|(X^N(s)), \sum V_m^2\rangle \\ +(Nl)^{-1}\Big\langle X^N(s), \sum \big[(\nabla_N^+ V_m)^2 + (\nabla_N^- V_m)^2\big]\Big\rangle \end{array}\right] ds \\
&\quad + 2\gamma_f (Nl)^{-1}\int_0^{t\wedge\delta} f''\big(\textstyle\sum Y_m^2(s)\big) \\
&\quad\times\left[\begin{array}{l} \langle |c|(X^N(s)), \big(\sum Y_m(s)V_m\big)^2\rangle \\ +\langle X^N(s), \big(\sum Y_m(s)\nabla_N^+ V_m\big)^2 \\ +\big(\sum Y_m(s)\nabla_N^- V_m\big)^2\rangle \\ +\varepsilon\langle |c|(X^N(s)), \sum V_m^2\rangle \\ +\varepsilon\Big\langle X^N(s), \sum\big[(\nabla_N^+ V_m)^2 + (\nabla_N^- V_m)^2\big]\Big\rangle \end{array}\right] ds + R(t),
\end{aligned} \tag{3.24}
$$

where $R(t)$ is a mean zero martingale.

We consider the cases $|B| = 1$ and $1 < |B| \le L = [(\log N)^2]$ separately. By (3.10), the bounds on $\theta_{m,N}/m^2$ for $m \ge m_0$ and $\|\nabla_N^{\pm} V_m\|_\infty \le \sqrt{2}\,\pi m$, we have

$$
\begin{aligned}
&f\big(Y_m^2(t\wedge\delta)\big) \\
&\quad \le f(0) + \int_0^{t\wedge\delta} f'\big(Y_m^2(s)\big)\big[-C_1 m^2 Y_m^2(s) + C_2\rho_t m^2 (Nl)^{-1}\big]\,ds \\
&\qquad + \int_0^{t\wedge\delta}\big[f''\big(Y_m^2(s)\big)\gamma_f C_2 \rho_t (Nl)^{-1} m^2 \big(Y_m^2(s) + \varepsilon\big)\big]\,ds + R(t),
\end{aligned} \tag{3.25}
$$

where $0 < C_1 \le C_2 < \infty$ and C_1, C_2 are not dependent on N or m.

Now suppose $|B| > 1$. In this case we are stopping in the norm $\|\cdot\|_0$ and need a bound on

$$
\mathbf{1}_{\{\tau>0\}} \sup_{[0,t\wedge\tau]} \left[\begin{array}{l} \langle X^N(s), \big(\sum Y_m(s)\nabla_N^+ V_m\big)^2 + \big(\sum Y_m(s)\nabla_N^- V_m\big)^2\rangle \\ +\langle |c|(X^N(s)), \big(\sum Y_m(s) V_m\big)^2\rangle \end{array}\right].
$$

For $s \le t\wedge\tau$ consider

$$
\begin{aligned}
&\mathbf{1}_{\{\tau>0\}}\langle X^N(s), \big(\textstyle\sum Y_m(s)\nabla_N^+ V_m\big)^2\rangle \\
&\quad \le \big(\textstyle\sum |Y_m(s)|\,\|\nabla_N^+ V_m\|_\infty\big)\mathbf{1}_{\{\tau>0\}}\|X^N(s)\|_0 \|\textstyle\sum Y_m(s)\nabla_N^+ V_m\|_0 \\
&\quad \le C\rho_t\sqrt{L}\big(\textstyle\sum m^2 Y_m^2(s)\big)^{1/2}\big(\textstyle\sum Y_m^2(s)\|\nabla_N^+ V_m\|_0^2\big)^{1/2} \\
&\quad \le C\rho_t\sqrt{L}\,(\underline{m})^2\big(\textstyle\sum Y_m^2(s)\big),
\end{aligned}
$$

where we have used the fact that $\{\nabla_N^+ V_m\}$ are orthogonal in $L_2[0,1]$ and

$$\langle \nabla_N^+ V_m, \nabla_N^+ V_m \rangle = \langle -\Delta_N V_m, V_m \rangle = \beta_{m,N} \le C(\underline{m})^2$$

by the assumptions on B. The same calculations hold for ∇_N^+ replaced by ∇_N^-, and similarly

$$\mathbf{1}_{\{\tau>0\}} \langle |c|(X^N(s)), \left(\sum Y_m(s) V_m\right)^2 \rangle \le C \rho_t \sqrt{L} \sum Y_m^2(s).$$

The remaining terms in (3.24) can be estimated as

$$-\sum \theta_{m,N} Y_m^2(s) \le -C_1(\underline{m})^2 \sum Y_m^2(s)$$

by the assumptions on B and the bounds on $\theta_{m,N}/m^2$. For $s \le t \wedge \tau$ and $\tau > 0$, $\tau > 0$,

$$\langle X^N(s), \sum \left[(\nabla_N^+ V_m)^2 + (\nabla_N^- V_m)^2 \right] \rangle + \langle |c|(X^N(s)), \sum V_m^2 \rangle \le C_2 L \rho_t (\underline{m})^2$$

using the assumptions on B and the same argument as in (3.25). Thus, we have

$$\begin{aligned} f\left(\sum Y_m^2(T \wedge \delta)\right) \le f(0) &+ \int_0^{T\wedge\delta} f'\left(\sum Y_m^2(s)\right) \\ &\times \left(-C_1(\underline{m})^2 \sum Y_m^2(s) + C_2 \rho_T L (\underline{m})^2 (Nl)^{-1}\right) ds \\ &+ \int_0^{T\wedge\delta} \gamma_f C_2 \rho_T L^{1/2} (\underline{m})^2 (Nl)^{-1} f''\left(\sum Y_m^2(s)\right) \\ &\times \left(\sum Y_m^2(s) + \bar{\varepsilon}\right) ds + R(T), \end{aligned} \tag{3.26}$$

where $R(\cdot)$ is a mean 0 martingale, $\bar{\varepsilon} = L^{1/2}\varepsilon$ and $C_1, C_2 > 0$ do not depend on N. They do depend on $\overline{m}/\underline{m}$, but the blocks are chosen so as to keep the ratio bounded.

Now we choose an appropriate f. Let $f(v)$ be the solution of

$$\begin{aligned} &f'(v)\left(-C_1 \underline{m}^2 v + C_2 \rho_T L \underline{m}^2 (Nl)^{-1}\right) \\ &\quad + \beta C_2 \rho_T (Nl)^{-1} L^{1/2} (\underline{m})^2 f''(v)(v + \bar{\varepsilon}) = 1 \end{aligned}$$

given by Lemma 3.20 with

$$a = C_1 \underline{m}^2, \qquad b = C_2 \rho_T L \underline{m}^2 (Nl)^{-1},$$
$$e = \bar{\varepsilon}, \qquad c = \beta C_2 \rho_T (Nl)^{-1} L^{1/2} (\underline{m})^2,$$

where β is a constant we need to choose appropriately. Recall

$$\gamma_f = \sup\left\{ \frac{f''(v+\varepsilon)}{f''(v)} : v \ge 0 \right\} \quad \text{and} \quad \varepsilon = \frac{16L}{N^2 l^2} + \frac{8\sqrt{LA}}{Nl}.$$

By Lemma 3.20(b), $\gamma_f \le (1 - a\varepsilon/c)^{-1}$.

In Lemma 3.21(a) we are assuming $L = 1$ and $Nl \to \infty$, and in Lemma 3.21(b) we are assuming $L \le (\log N)^2$, $N \to \infty$, $l \to \infty$ as $N \to \infty$. Also, $\rho_T \ge 1$ and we can assume $C_2 \ge 9C_1$.

Let $\beta = \max(2, 2\sqrt{A})$. A straightforward estimate shows this gives $\gamma_f \le 2 \le \beta$. This choice of f and (3.26) gives

$$Ef\left(\sum Y_m^2(T\wedge\delta)\right) \le Ef(0) + E(T\wedge\delta) \le Ef(0) + T. \tag{3.27}$$

Recall

$$f(v) = a^{-1}\sum_{n=1}^{\infty}\frac{[a/c(v+e)]^n}{n\prod_{j=0}^{n-1}((b+ae)/c+j)}.$$

The observations in bounding γ_f show $f(0) < C(T) < \infty$ in case (a) of the proof and $f(0) \to 0$ as $N \to \infty$ in case (b). Thus, $Ef(0) + T \le C(T)$. Also, $(b + ae)/c \le \sqrt{L}$. In case (a) we have

$$f(A) > \left(C_1 m^2\right)^{-1}\sum_{n=1}^{\infty}\frac{(rNlA/\beta\rho_T)^n}{\prod_0^{n-1}(1+j)} \quad \text{where } r = \frac{C_1}{C_2}. \tag{3.28}$$

In case (b) $A = qL/N \to 0$ as $N \to \infty$, so we can take $\beta = 2$. Using only the term for $n = [L^{1/2}]$ in the power series for f shows

$$f\left(\frac{qL}{N}\right) > \left(C_1 N^2 L^{1/2}\right)^{-1}\left(\frac{rlq}{\rho_T}\right)^{[L^{1/2}]} \quad \text{where } r = \frac{C_1}{4C_2}. \tag{3.29}$$

Finally, note $f(v)$, $f'(v) > 0$ for $v > 0$. Thus

$$P\left[\sup_{[0,T]}\sum Y_m^2(t\wedge\tau) > A\right] \le \frac{Ef(\sum Y_m^2(T\wedge\delta))}{f(A)} \le \frac{C(T)}{f(A)}$$

by (3.27) and the observations after it. With (3.28) and (3.29) this proves the lemma. □

We can now complete the proof of Theorem 3.3.

Proof of Theorem 3.3. By the equivalence of $\|\cdot\|_{-\alpha}$ and $\|\cdot\|_{-\alpha,N}$ and the discussion after (3.11), it remains to show that

$$\sup_{[0,T]}\left\|Y^N(t\wedge\tau)\right\|_{-\alpha,N}^2 = \sup_{[0,T]}\sum\frac{Y_m^2(t\wedge\tau)}{(1+\beta_{m,N})^{\alpha}} \to_P 0.$$

First consider the case $\alpha = 0$. By Lemma 3.15, for any $A > 0$,

$$P\left[\sup_{[0,T]}\sum_{m\le(\log N)^2}Y_m^2(t\wedge\tau) > A\right] \le C(T)\rho_T(\log N)^4(NlA)^{-1} \to 0$$

since we are assuming $l \to \infty$ as $N \to \infty$. Thus we need only sum over $B = \{m\colon (\log N)^2 < m \le N-1, m \text{ even}\}$. But we can let $B = \bigcup_{i=1}^M B_i$ where $M \le N/[(\log N)^2]$, $|B_i| \le (\log N)^2$ and B_i satisfies the assumptions of Lemma 3.21(b) where $\sup\{C(B_i)\colon 1 \le i \le M\}$ is bounded independently of N. By

Lemma 3.21(b), for any $q \in (0, 1]$,

$$P\left[\sup_{[0,T]} \sum_{B} Y_m^2(t \wedge \tau) \geq q\right] \leq \sum_i P\left[\sup_{[0,T]} \sum_{B_i} Y_m^2(t \wedge \tau) \geq q\left[(\log N)^2\right] N^{-1}\right]$$

$$\leq N(\log N)^{-1} C(T) N^2 \left(\frac{rlq}{\rho_T}\right)^{-\{[(\log N)^2]\}^{1/2}} \to 0$$

if $l \to \infty$ as $N \to \infty$.

Now let $\alpha > 0$. If k is fixed, then by Lemma 3.15,

$$\sup_{[0,T]} \sum_{m \leq k} Y_m^2(t \wedge \tau) \to_P 0.$$

Thus it suffices to show that, given $\varepsilon > 0$,

$$P\left[\sup_{[0,T]} \sum_{m > k(\varepsilon)} m^{-2\alpha} Y_m^2(t \wedge \tau) > \varepsilon\right] < \varepsilon$$

for all large N. If $\alpha \in (0, \frac{1}{2})$, then $C(\alpha) N^{2\alpha - 1} \sum_{n=1}^{N} n^{-2\alpha} \leq 1$ for all large N and $C(\log N)^{-1} \sum_{n=1}^{N} n^{-2\alpha} \leq 1$ for $\alpha = \frac{1}{2}$. Fix $k > m_0$. If $\alpha \in (0, \frac{1}{2})$, we have

$$P\left[\sup_{[0,T]} \sum_{m > k} m^{-2\alpha} Y_m^2(t \wedge \tau) > \varepsilon\right]$$

$$\leq P\left[\sup_{[0,T]} \sum_{m > k} m^{-2\alpha} Y_m^2(t \wedge \tau) > \varepsilon C(\alpha) N^{2\alpha - 1} \sum_{n=1}^{N} n^{-2\alpha}\right]$$

$$\leq \sum_{m > k} P\left[\sup_{[0,T]} Y_m^2(t \wedge \tau) > \varepsilon C(\alpha) N^{2\alpha - 1}\right]$$

$$\leq C(T) N^{3 + 2\alpha} l \exp\left(-C_1(T) N^{2\alpha} l \varepsilon\right) \to 0 \quad \text{if } N^{2\alpha} l (\log N)^{-1} \to \infty.$$

The last inequality follows from Lemma 3.21(a) with $A = \varepsilon C(\alpha) N^{2\alpha - 1}$. In this case k does not depend on ε. For $\alpha = \frac{1}{2}$, a similar calculation shows

$$P\left[\sup_{[0,T]} \sum_{m > k} m^{-2\alpha} Y_m^2(t \wedge \tau) > \varepsilon\right]$$

$$\leq C(T) N^4 l (\log N)^{-1} \exp\left(-C_1(T) N l \varepsilon (\log N)^{-1}\right) \to 0$$

if $Nl(\log N)^{-2} \to \infty$. This proves the theorem for $\alpha \in (0, \frac{1}{2}]$.

If $\alpha > \frac{1}{2}$, choose $\gamma \in (\frac{1}{2}, \alpha)$. Then

$$P\left[\sup_{[0,T]} \sum_{m > k} m^{-2\alpha} Y_m^2(t \wedge \tau) > 2(Nl)^{-1} \sum_{n=1}^{N} n^{-2\gamma}\right]$$

$$\leq \sum_{m > k} P\left[\sup_{[0,T]} Y_m^2(t \wedge \tau) > m^{2(\alpha - \gamma)} (Nl)^{-1}\right]$$

$$\leq \sum_{m > k} a_m, \quad \text{where } a_m = C(T) m^2 \left[\sum_{n=1}^{\infty} \frac{(r m^{\alpha - \gamma})^n}{\rho_T^n (n+1)!}\right]^{-1}$$

is determined by setting $A = m^{2(\alpha-\gamma)}(Nl)^{-1}$ in Lemma 3.21(a). Since $\sum_{m>m_0}^{\infty} a_m < \infty$ and $(Nl)^{-1}\sum_{n=1}^{\infty} n^{-2\alpha} \to 0$ as $(Nl) \to \infty$, we can choose $k(\varepsilon)$ large enough to obtain

$$P\left[\sup_{[0,T]} \sum_{m>k(\varepsilon)} m^{-2\alpha} Y_m^2(t\Lambda\tau) > \varepsilon\right] < \varepsilon. \qquad \square$$

4. The central limit theorem for the linear model. Again consider

$$X^N(t) - \psi(t) = e^{A_N t}(X^N(0) - \psi(0)) \tag{4.1}$$
$$+ \int_0^t e^{A_N(t-s)}\, dZ^N(s) + (e^{A_N t} - e^{At})\psi(0),$$

with the same notation as Section 3. As discussed in Kotelenez (1986a), there is a unique (in distribution) $H_{-\beta}$ (for $\beta > \frac{3}{2}$) valued Gaussian process on some probability space with independent increments, continuous sample paths and characteristic functional

$$E\exp(i\langle M(t), \varphi\rangle) = \exp\left(-\tfrac{1}{2}\int_0^t \left[\langle \psi(s), 2(\dot\varphi)^2\rangle + \langle |c|(\psi(s)), \varphi^2\rangle\right] ds\right) \tag{4.2}$$

for $\varphi \in H_\beta$.

Let $U^N(t) = \sqrt{Nl}\,(X^N(t) - \psi(t))$.

In this section we prove the following result.

THEOREM 4.3. *Assume $\alpha > \frac{1}{2}$ and*

(a) $\dfrac{l}{N} \to 0$,

(b) $Nl \to \infty$,

(c) $U^N(0) \to U_0$ *in distribution on $H_{-\alpha}$, where U_0 is independent of M.*

Then

(i) $\sqrt{Nl}\,(X^N - \psi) = U^N \to U$ *in distribution on $D_{H-\alpha}[0,\infty)$, where*

$$U(t) = e^{At}U_0 + \int_0^t e^{A(t-s)}\, dM(s) \tag{4.4}$$

is the mild solution of the stochastic partial differential equation

$$dU(t) = AU(t)\, dt + dM(t), \qquad U(0) = U_0. \tag{4.5}$$

(ii) *If $\beta > \frac{3}{2}$, then $(U^N(0), M^N) \to_{\mathscr{D}} (U_0, M)$ on $H_{-\alpha} \times D_{H-\beta}[0,\infty)$.*

(iii) $U \in C_{H-\alpha}[0,\infty)$ *a.s.*

REMARKS. In Kotelenez (1986a), $U \in C_{H-\alpha}[0,\infty) \subset D_{H-\alpha}[0,\infty)$, but convergence $U_N \to_{\mathscr{D}} U$ takes place in $D_{H-\alpha-3}[0,\infty)$. We now have $U_N \to_{\mathscr{D}} U$ in $D_{H-\alpha}[0,\infty)$ and have eliminated the asymptotic independence assumptions of Kotelenez by proving (ii). In Kotelenez (1986a) the proof that $U \in C_{H-\alpha}[0,T]$ a.s. was based on a generalization of a theorem of Dawson (1972), which was an application of a theorem of Newell (1963). We have reproved it here using Lemma 3.19, a direct consequence of Itô's formula. We note this since the estimates of Section 3 are based on generalizing the proof of Lemma 3.19. The proof of Theorem 4.3 parallels the proof of the CLT given in Kotelenez (1986a). The improvement is due to the estimates developed in Section 3 with additional technical improvements. The proof is a series of lemmas.

For the remainder of this section the assumptions of Theorem 4.3 are in effect. In particular, we are assuming

$$N \to \infty,$$
$$Nl \to \infty, \tag{4.6}$$
$$\alpha > \tfrac{1}{2} \quad \text{and} \quad \|X^N(0) - \psi(0)\|_{-\alpha} \to_P 0.$$

For a fixed $\delta \in (0,1)$, let $\tau = \inf\{t: \|X^N(t) - \psi(t)\|_{-\alpha} \geq \delta\}$. By Theorem 3.3, $\sup_{[0,T]}\|X^N(t) - \psi(t)\|_{-\alpha} \to_P 0$ and $P[\tau \leq T] \to 0$ for all $T > 0$. Let

$$M^N(t) = \sqrt{Nl}\, Z^N(t \wedge \tau). \tag{4.7}$$

LEMMA 4.8. *Suppose $\beta > \frac{3}{2}$. Then $\{M^N(t)\}$ is relatively compact in $H_{-\beta}$.*

PROOF. Proofs are given in Kotelenez (1986a) and Blount (1987). □

LEMMA 4.9. *Let $\varphi \in H_\beta$, $\beta > \frac{3}{2}$. Then for $0 \leq s \leq t$,*

$$\begin{aligned} &E\left[\exp(i\langle M^N(t) - M^N(s), \varphi\rangle | \mathscr{F}_s^N\right] \\ &\quad \to_P \exp\left(-\tfrac{1}{2}\int_s^t \left[\langle 2\psi(u), (\dot{\varphi})^2\rangle + \langle |c|(\psi(u)), \varphi^2\rangle\right] du\right) \\ &\quad = E\left[\exp(i\langle M(t) - M(s), \varphi\rangle)\right]. \end{aligned}$$

PROOF. In Kotelenez (1986a) it was shown that

$$E\left[\exp(i\langle M^N(t), \varphi\rangle)\right] \to E\left[\exp(i\langle M(t), \varphi\rangle)\right].$$

His proof can be generalized to prove Lemma 4.9. Details are in Lemma 4.4 of Blount (1987). □

LEMMA 4.10. *Let $0 < s \leq s_0 < 1$, $0 \leq t \leq T$ and $\beta > \frac{3}{2}$. Then*

$$E\left[\|M^N(t+s) - M^N(t)\|_{-\beta}^2 \middle| \mathscr{F}_t^N\right] \leq C(T)s.$$

PROOF. Proofs are given in Kotelenez (1986a) and Blount (1987). □

Let

$$V^N(t) = \int_0^t e^{A_N(t-s)}\, dM^N(s) \quad \text{and} \quad V(t) = \int_0^t e^{A(t-s)}\, dM(s).$$

LEMMA 4.11. (a) *For* $m \geq m_0$, $e_{m,N} = \varphi_{m,N}$ *or* $\psi_{m,N}$ *and* $\gamma \in (\frac{1}{2}, \alpha)$,

$$P\Big[\sup_{[0,T]} m^{-2\alpha} \langle V^N(t), e_{m,N}\rangle^2 > m^{-2\gamma}\Big] < a_m(\alpha, T) \quad \text{where } \sum a_m < \infty.$$

(b) *For* $m \geq m_0$, $e_m = \varphi_m$ *or* ψ_m *and* $\gamma \in (\frac{1}{2}, \alpha)$,

$$P\Big[\sup_{[0,T]} m^{-2\alpha} \langle V(t), e_m\rangle^2 > m^{-2\gamma}\Big] < b_m(\alpha, T) \quad \text{where } \sum b_m < \infty.$$

(c) $V \in C_{H_{-\alpha}}[0, \infty)$ *a.s.*

PROOF. Recall [see (3.13)] that for $m \geq m_0$, $0 < C_1 m^2 < \theta_{m,N} < C_2 m^2$ and the same holds for $\theta_m = \pi^2 m^2 - (b_1 - d_1)$. Consider

$$\langle V^N(t), e_{m,N}\rangle = \int_0^t e^{-\theta_{m,N}(t-s)} d\langle M^N(s), e_{m,N}\rangle.$$

For $m \geq m_0$, $\langle V^N(t), e_{m,N}\rangle^2 \leq Nl\langle Y^N(t \wedge \tau), e_{m,N}\rangle^2$. But then Lemma 4.11(a) was shown in the proof of Theorem 3.3 using Lemma 3.21(a).

For $m \geq m_0$, consider

$$\langle V(t), e_m\rangle = \int_0^t e^{-\theta_m(t-s)} d\langle M(s), e_m\rangle = \int_0^t e^{-\theta_m(t-s)} \big(g(e_m, s)\big)^{1/2} dW_m(s),$$

where $g(e_m, s) = \langle 2\psi(s), (\dot{e}_m)^2\rangle + \langle |c|(\psi(s)), e_m^2\rangle$ and W_m is a standard Brownian motion. Since $\sup_{[0,T]} g(e_m, s) \leq C(T)m^2$, it follows from Lemma 3.19 that, for $\frac{1}{2} < \gamma < \alpha$,

$$P\Big[\sup_{[0,T]} m^{-2\alpha} \langle V(t), e_m\rangle^2 > m^{-2\gamma}\Big]$$
$$\leq b_m(\alpha, T) = C(T)m^2 \left[\int_0^{C_1(T)m^{(\alpha-\gamma)}} \exp\left(\frac{s^2}{2}\right) \int_0^s \exp\left(-\frac{r^2}{2}\right) dr\, ds\right]^{-1}.$$

This proves (b).

Let $R_n(t) = \sum_{m \geq n} \langle V(t), \varphi_m\rangle \varphi_m + \langle V(t), \psi_m\rangle \psi_m$ and choose $\gamma \in (\frac{1}{2}, \alpha)$. By Lemma 4.11(b) and Borel–Cantelli,

$$P\Big[\sup_{[0,T]} m^{-2\alpha} \langle V(t), e_m\rangle^2 > m^{-2\gamma} \text{ infinitely often}\Big] = 0.$$

Since $\sum m^{-2\gamma} < \infty$, this implies $\lim_{n\to\infty} \sup_{[0,T]} \|R_n(t)\|_{-\alpha}^2 = 0$ a.s. for all $T > 0$. Since $V(\cdot) - R_n(\cdot) \in C_{H_{-\alpha}}[0, \infty)$ a.s., this proves (c). □

LEMMA 4.12. *Assume* $U^N(0) \to_{\mathscr{D}} U_0$ *on* $H_{-\alpha}$, *and* $\beta > \frac{3}{2}$. *Then*

(a) $$(U^N(0), M^N) \to_{\mathscr{D}} (U_0, M) \quad \text{on } H_{-\alpha} \times D_{H_{-\beta}}[0, \infty).$$

(b) $$e^{At}U^N(0) + \int_0^t e^{A_N(t-s)} dM^N(s) \to_{\mathscr{D}} e^{At}U_0 + \int_0^t e^{A(t-s)} dM(s) \quad \text{on } D_{H_{-\alpha}}[0, \infty).$$

PROOF. Let $t_1 < t_2 < \cdots < t_k$, $f_0 \in H_\alpha$ and $f_r \in H_\beta$, $1 \le r \le k$. The assumption on $U^N(0)$ and Lemma 4.8 imply $(U^N(0), M^N(t_1), \dots, M^N(t_k))$ is relatively compact in $H_{-\alpha} \times (H_{-\beta})^k$. By Lemma 4.9,

$$E\left[\exp\left(i\langle U^N(0), f_0\rangle + i\sum_{r=1}^{k} \langle M^N(t_r), f_r\rangle\right)\right]$$
$$\to E\exp(i\langle U(0), f_0\rangle) \cdot E\exp\left(i\sum_{r=1}^{k} \langle M(t_r), f_r\rangle\right).$$

Since the Borel σ-algebra in $H_{-\alpha} \times (H_{-\beta})^k$ is generated by the finite dimensional sets, this shows $(U^N(0), M^N(t_1), \dots, M^N(t_k)) \to_{\mathscr{D}} (U_0, M(t_1), \dots, M(t_k))$. (a) then follows by Lemma 4.10, together with Theorems 7.8 and 8.6 of Ethier and Kurtz.

Let

$$\bar{V}^N(t) = \int_0^t e^{A(t-s)}\, dM^N(s),$$
$$\bar{V}_n^N(t) = \sum_{m \le n} \langle \bar{V}^N(t), \varphi_m\rangle \varphi_m + \langle \bar{V}^N(t), \psi_m\rangle \psi_m,$$
$$V_n(t) = \sum_{m \le n} \langle V(t), \varphi_m\rangle \varphi_m + \langle V(t), \psi_m\rangle \psi_m.$$

We claim that

$$V^N(t) = \bar{V}_n^N(t) + \tilde{R}_n^N(t) \quad \text{and} \quad V(t) = V_n(t) + R_n(t), \tag{4.13}$$

where $P[\sup_{[0,T]}\|R_n(t)\|_{-\alpha} > \varepsilon] < \varepsilon$ and $P[\sup_{[0,T]}\|\tilde{R}_n^N(t)\|_{-\alpha} > \varepsilon] < \varepsilon$ for $n \ge n(\varepsilon)$ and $N \ge N(\epsilon, n)$. Assume for now the claim is true. It was shown in Lemma 3.1 of Kotelenez (1986a) that $\bar{V}_n^N = \Phi(M^N)$ and $V_n = \Phi(M)$ where $\Phi\colon D_{H-\beta}[0,\infty) \to D_{H-\alpha}[0,\infty)$ is continuous. The map $f \to e^{At}f$ defines a continuous map from $H_{-\alpha} \to C_{H-\alpha}[0,\infty)$. In Kotelenez (1982a) it was shown that addition defines a continuous map from $C_{H-\alpha}[0,\infty) \times D_{H-\alpha}[0,\infty)$ into $D_{H-\alpha}[0,\infty)$. Thus $e^{At}U_N(0) + \bar{V}_n^N \to_{\mathscr{D}} e^{At}U_0 + V_n$ by Lemma 4.12(a).

By the claim (4.13) and an argument using the Prohorov metric given in Lemma 3.1 of Kotelenez (1986a), it follows that

$$e^{At}U_N(0) + V^N(t) \to_{\mathscr{D}} e^{At}U_0 + V(t) \quad \text{in } D_{H-\alpha}[0,\infty).$$

This proves (b), assuming (4.13), which we now prove.

The claim (4.13) for $V(t)$ was shown in the proof of Lemma 4.11(c). Consider

$$V^N(t) = \sum_{m \le n} \langle V^N(t), \varphi_{m,N}\rangle \varphi_{m,N} + \langle V^N(t), \psi_{m,N}\rangle \psi_{m,N}$$
$$+ \sum_{m > n} \langle V^N(t), \varphi_{m,N}\rangle \varphi_{m,N} + \langle V^N(t), \psi_{m,N}\rangle \psi_{m,N}$$
$$= V_n^N(t) + R_n^N(t).$$

If $h = \varphi_m$, ψ_m, $\varphi_{m,N}$ or $\psi_{m,N}$ for $m \le n$, then $E\langle M^N(T), h\rangle^2 \le C(T)n^2$.

Doob's inequality shows $\sup_{[0,T]}\|V_n^N(t) - \bar{V}_n^N(t)\|_\infty \to_P 0$ as $N \to \infty$ for n fixed. By Lemma 4.11(a), for $n \geq m_0$ and $\frac{1}{2} < \gamma < \alpha$,

$$P\left[\sup_{[0,T]} \| R_n^N(t) \|_{-\alpha}^2 > \sum_{m>n} m^{-2\gamma}\right] < \sum_{m>n} a_m \quad \text{where } \sum_{m_0}^{\infty} a_m < \infty.$$

Now let $\tilde{R}_n^N(t) = V_n^N(t) - \bar{V}_n^N(t) + R_n^N(t)$. □

Before proving Theorem 4.3, we need a final technical lemma.

LEMMA 4.14. (a) *If* $K \subset H_{-\alpha}$ *is compact, then*

$$\sup_{[0,T]} \sup_{f \in H^N \cap K} \|(e^{\Delta_N t} - e^{\Delta t}) f\|_{-\alpha} \to 0 \quad \text{as } N \to \infty.$$

(b) *If* $f \in H_2$ *then* $\sup_{[0,T]}\|(e^{\Delta_N t} - e^{\Delta t}) f\|_0 \leq C(T, f)N^{-1}$.

PROOF. See Lemmas 4.19 and 4.20 in Blount (1987). □

PROOF OF THEOREM 4.3. Parts (ii) and (iii) of the theorem follow from Lemmas 4.12 and 4.11, respectively. Consider

$$U^N(t) = e^{At}U^N(0) + \int_0^t e^{A_N(t-s)}\, dM^N(s) + \varepsilon_N(t)$$

where

$$\varepsilon_N(t) = (e^{A_N t} - e^{At})U^N(0) + \int_0^t e^{A_N(t-s)} d\big(\sqrt{Nl}\, Z^N(s) - M^N(s)\big)$$
$$+ \sqrt{Nl}\,(e^{A_N t} - e^{At})\psi(0).$$

By Lemma 4.12(b) it suffices to show $\sup_{[0,T]}\|\varepsilon_N(t)\|_{-\alpha} \to_P 0$. But $\psi(0) \in H_2$, so it has a continuous derivative in the space variable, implying $\|\psi(0) - P_N\psi(0)\|_\infty \leq CN^{-1}$. Thus $\sup_{[0,T]}\|(e^{At} - e^{A_N t})U^N(0)\|_{-\alpha} \to_P 0$ by Lemma 4.14(a) and relative compactness of $\{U_N(0)\}$ in $H_{-\alpha}$. We have

$$\sup_{[0,T]} \sqrt{Nl}\, \|(e^{A_N t} - e^{At})\psi(0)\|_0 \leq C(T)(l/N)^{1/2}$$

by Lemma 4.14(b). By 4.7 and the sentence preceding it,

$$P\big(\sqrt{Nl}\, Z^N(t) - M^N(t) \neq 0 \text{ in } [0,T]\big) \leq P[\tau \leq T] \to 0. \qquad \square$$

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REFERENCES

ARNOLD, L. and THEODOSOPULU, M. (1980). Deterministic limit of the stochastic model of chemical reactions with diffusion. *Adv. in Appl. Probab.* **12** 367–379.

BLOUNT, D. J. (1987). Comparison of a stochastic model of a chemical reaction with diffusion and the deterministic model. Ph.D. dissertation, Univ. Wisconsin-Madison.

DAWSON, D. A. (1972). Stochastic evolution equations. *Math. Biosci.* **15** 287–316.

ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes, Characterization and Convergence.* Wiley, New York.

KOTELENEZ, P. (1982a). Ph.D. dissertation, Report 8, Universität Bremen Forschungsschwerpunkt Dynamische Systems.

KOTELENEZ, P. (1982b). A submartingale type inequality with applications to stochastic evolution equations. *Stochastics* **8** 139–151.

KOTELENEZ, P. (1986a). Law of large numbers and central limit theorem for linear chemical reactions with diffusion. *Ann. Probab.* **14** 173–193.

KOTELENEZ, P. (1986b). Gaussian approximation to the nonlinear reaction-diffusion equation. Report 146, Universität Bremen Forschungsschwerpunkt Dynamische Systems.

KOTELENEZ, P. (1987). Fluctuations near homogeneous states of chemical reactions with diffusion. *Adv. in Appl. Probab.* **19** 352–370.

KOTELENEZ, P. (1988). High density limit theorems for nonlinear chemical reactions with diffusion. *Probab. Theory Related Fields* **78** 11–37.

KUIPER, H. J. (1977). Existence and comparison theorems for nonlinear diffusion systems. *J. Math. Anal. Appl.* **60** 166–181.

KURTZ, T. G. (1971). Limit theorems for sequences of jump Markov processes approximating ordinary differential processes. *J. Appl. Probab.* **9** 344–356.

NEWELL, G. F. (1963). Asymptotic extreme value distributions for one-dimensional diffusion processes. *J. Math. Mech.* **11** 481–496.

DEPARTMENT OF MATHEMATICS
ARIZONA STATE UNIVERSITY
TEMPE, ARIZONA 85287