

Comparison Results for a Class of Variational Inequalities

M. R. POSTERARO - R. VOLPICELLI

ABSTRACT. In this paper we study a variational inequality related to a linear differential operator of elliptic type. We give a pointwise bound for the rearrangement of the solution u , and an estimate for the L^2 -norm of the gradient of u .

1. INTRODUCTION

Let Ω be an open bounded set of \mathbb{R}^n and let

$$Au = - (a_{ij}(x)u_{x_j})_{x_i} + (b_i(x)u)_{x_i} + c(x)u$$

be a differential operator whose coefficients satisfy:

$$(1.1) \quad \begin{cases} a_{ij}, b_i, c \in L^\infty(\Omega) \\ a_{ij}(x)\xi_i\xi_j \geq |\xi|^2, \\ \sum b_i^2 \leq B^2. \end{cases} \quad \text{for a.e. } x \in \Omega, \xi \in \mathbb{R}^n$$

Let $u \in H_0^1(\Omega)$ be a solution of the related variational inequality

$$(1.2) \quad a(u, U-u) \geq \int_\Omega f(x)(U-u)dx \quad \forall U \in H_0^1(\Omega), U, u \geq 0$$

where

$$a(\varphi, \psi) = \int_{\Omega} (a_{ij}(x)\varphi_{x_i}\psi_{x_j} - b_i(x)\varphi\psi_{x_i} + c(x)\varphi\psi) dx,$$

and $f \in L^2(\Omega)$.

Schwarz symmetrization allows us to obtain sharp estimates for a solution u of (1.2) comparing u with the solution $v \in H_0^1(\Omega^*)$ of the «symmetrized» variational inequality

$$(1.3) \quad \begin{aligned} a^*(v, V-v) &= \int_{\Omega^*} v_{x_i}(V-v)_{x_i} + B \frac{x_i}{|x|} v(V-v)_{x_i} - c^*(x)v(V-v) dx \geq \\ &\geq \int_{\Omega^*} f^*(x)(V-v) dx \quad \forall V \in H_0^1(\Omega^*), \quad V, v \geq 0 \end{aligned}$$

where Ω^* is the n -dimensional ball centered at the origin with measure $|\Omega|$, $f^*(x)$, $c^*(x)$ are the spherically symmetric decreasing rearrangements of $f(x)$ and $c_-(x) = \text{Max}(-c(x), 0)$, respectively (see § 2).

More precisely in this paper we show (see theorem 2.2) that the pointwise estimate

$$(1.4) \quad u^*(x) \leq v(x)$$

holds for all $x \in \Omega^*$. Such an estimate gives an optimal upper bound for the L^p -norm of u and for the measure of the set $\{x: u(x) > 0\}$ or, that is the same, an optimal lower bound for the coincidence set of u .

As consequence of (1.4) we derive an estimate for $a(u, u)$ and for the L^2 -norm of ∇u in terms of $a^*(v, v)$ and the L^2 -norm of ∇v respectively. Finally the previous results are applied to the obstacle problem when the obstacle is in $H_0^1(\Omega)$.

The first results in this direction, in the case $b_i = 0$, $c \geq 0$, are due to Bandle-Mossino [BM] and Maderna-Salsa [MS]. In particular in [BM] the case of a non-linear operator is considered. Subsequently Alvino-Matarasso-Trombetti [AMT] considered a variational inequality of general form taking into account also the influence of the term $c_+(x) = \text{max}(c(x), 0)$, but with constraints on the coefficients different from (1.1). As far as variational parabolic inequalities concern we recall [DM].

The method used was introduced by Talenti [Ta1] for an elliptic equation without lower-order terms and was extended to elliptic equations of more general form by several authors (see i.e. [AT], [ALT2], [Ba], [FP], [GT], [Ta2], [TV]).

2. MAIN RESULTS

If Ω is an open bounded set of \mathbb{R}^n , we will denote by $|\Omega|$ its measure and by Ω^* the ball of \mathbb{R}^n centered at the origin whose measure is $|\Omega|$. Moreover if φ is a measurable function,

$$\mu(t) = |\{x \in \Omega : \varphi(x) > t\}|, \quad t \in \mathbb{R},$$

is the distribution function of φ and

$$\varphi^*(s) = \sup\{t \in \mathbb{R} : \mu(t) > s\}, \quad s \in [0, |\Omega|[,$$

is its decreasing rearrangement. If C_n is the measure of the n-dimensional unit ball,

$$\varphi^\#(x) = \varphi^*(C_n |x|^n), \quad x \in \Omega^*,$$

is the spherically symmetric decreasing rearrangement of $\varphi(x)$. For an exhaustive treatment of rearrangements we refer for example to [ALT1], [Ba], [CR], [HLP], [Ka], [Mo]. Here we just recall the well known Hardy inequality:

$$\int_{\Omega} u(x)v(x)dx \leq \int_0^{|\Omega|} u^*(s)v^*(s)ds$$

where $u(x), v(x)$ are measurable functions.

The first step to get the comparison result is to obtain a differential inequality involving the decreasing rearrangement of u . In the following we will consider $f_+, f_- \neq 0$. In the other cases proofs are simpler.

Theorem 2.1. *Let u be a solution of the variational inequality (1.2) with conditions (1.1), then*

$$(2.1) \quad \begin{aligned} -u^{*'}(s) &\leq \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_0^s (f^*(\sigma) + c_*(\sigma)u^*(\sigma))d\sigma + \\ &+ \frac{B s^{-1+1/n}}{n C_n^{1/n}} u^*(s) \text{ a.e. in } [0, |u > 0|]. \end{aligned}$$

Proof. Following [BM] we choose as test function U in (1.2) $u \pm \Phi_h$ where

$$\Phi_h(x) = \begin{cases} h & \text{if } u > t + h \\ (u(x) - t) & \text{if } t < u(x) \leq t + h \\ 0 & \text{otherwise} \end{cases}$$

with $h > 0$ and $t \in [0, \sup u[$.

We have

$$\frac{1}{h} a(u, \Phi_h) = \frac{1}{h} \int_{\Omega} f \Phi_h dx$$

and by ellipticity condition, letting h go to 0, we obtain

$$-\frac{d}{dt} \int_{u>t} |\nabla u|^2 dx \leq -\frac{d}{dt} \int_{u>t} b_i(x) u_{x_i} u dx + \int_{u>t} (f(x) - c(x)u) dx.$$

From now on we give a sketch of the proof because the tools are standard. We have (see [ALT2])

$$-\frac{d}{dt} \int_{u>t} b_i(x) u_{x_i} u dx \leq Bt \left(-\frac{d}{dt} \int_{u>t} |\nabla u| dx \right)$$

and by Hardy inequality

$$\int_{u>t} (f(x) - c(x)u(x)) dx \leq \int_0^{\mu(t)} (f^*(\sigma) + c^*(\sigma)u^*(\sigma)) d\sigma.$$

Then

$$(2.2) \quad -\frac{d}{dt} \int_{u>t} |\nabla u|^2 dx - Bt \left(-\frac{d}{dt} \int_{u>t} |\nabla u| dx \right) \leq \int_0^{\mu(t)} (f^*(\sigma) + c^*(\sigma)u^*(\sigma)) d\sigma.$$

Isoperimetric inequality [DG], Fleming-Rishel formula [FR] and Schwarz inequality give (see [ALT2], [Ta2])

$$nC_n^{1/n} \mu(t)^{1-1/n} \leq -\frac{d}{dt} \int_{u>t} |\nabla u| \leq$$

$$(2.3) \quad \leq [-\mu'(t)]^{1/2} \left(-\frac{d}{dt} \int_{u>t} |\nabla u|^2 \right)^{1/2}.$$

Now we observe that the right hand side of (2.2) is non negative. As a matter of fact, letting t go to 0, from (2.2) we get

$$\int_0^{\mu(0)} (f^*(\sigma) + c_-^*(\sigma)u^*(\sigma))d\sigma \geq 0$$

and then

$$(2.4) \quad \int_0^{\mu(t)} (f^*(\sigma) + c_-^*(\sigma)u^*(\sigma))d\sigma \geq 0 \quad \forall t \in [0, \sup u]$$

because $f^*(\sigma) + c_-^*(\sigma)u^*(\sigma)$ is a decreasing function.

Hence by (2.3) and (2.4) we have

$$\begin{aligned} & \int_0^{\mu(t)} (f^*(\sigma) + c_-^*(\sigma)u^*(\sigma))d\sigma \leq \\ & \leq \frac{\mu(t)^{-1+1/n}}{nC_n^{1/n}} [-\mu'(t)]^{1/2} \left(-\frac{d}{dt} \int_{u>t} |\nabla u|^2 \right)^{1/2} \int_0^{\mu(t)} (f^*(\sigma) + \\ & \quad + c_-^*(\sigma)u^*(\sigma))d\sigma. \end{aligned}$$

Then by (2.2) and (2.3)

$$(2.5) \quad \begin{aligned} & \left(-\frac{d}{dt} \int_{u>t} |\nabla u|^2 dx \right)^{1/2} \leq \\ & \leq Bt [-\mu'(t)]^{1/2} + [-\mu'(t)]^{1/2} \frac{\mu(t)^{-1+1/n}}{nC_n^{1/n}} \int_0^{\mu(t)} (f^*(\sigma) + c_-^*(\sigma)u^*(\sigma))d\sigma. \end{aligned}$$

Using again (2.3), we obtain

$$\begin{aligned} & \frac{nC_n^{1/n} \mu(t)^{1-1/n}}{[-\mu'(t)]^{1/2}} \leq Bt [-\mu'(t)]^{1/2} + \\ & + [-\mu'(t)]^{1/2} \frac{\mu(t)^{-1+1/n}}{nC_n^{1/n}} \int_0^{\mu(t)} (f^*(\sigma) + c_-^*(\sigma)u^*(\sigma))d\sigma. \end{aligned}$$

Making a change of variables we get (2.1). \square

We consider the variational inequality (1.3) whose data are spherically symmetric. If its solution v is spherically symmetric and decreasing, then repeating the above proof, it is easy to verify that all the inequalities used to get (2.1) are equalities. As in [AMT] a way to establish the existence of such a solution is given by the following proposition:

Proposition 2.1. *Let us consider the differential operator*

$$A^*v = -\Delta v - B \left(v \frac{x_i}{|x|} \right)_{x_i} - c_-(x)v.$$

Let us suppose $c_-^* = 0$ or let any one of the following conditions be satisfied:

- (i) *there exists a non negative function $H \not\equiv 0$ such that the Dirichlet problem*

$$\begin{cases} A^*Z = H & \text{in } \Omega^* \\ Z \in H_0^1(\Omega^*) \end{cases}$$

has a non negative solution Z ;

- (ii) *the first eigenvalue of the problem*

$$(2.6) \quad \begin{cases} A^*\varphi = \lambda p(x)\varphi & \text{in } \Omega^* \\ \varphi \in H_0^1(\Omega^*) \end{cases}$$

where $p \in L^{n/2}(\Omega^)$, $p(x) \not\equiv 0$ is a non negative function, is positive;*

- (iii) *there exists $\alpha > 0$ such that*

$$\begin{aligned} \int_{\Omega^*} e^{\beta|x|} (|\nabla \Phi|^2 - B \frac{(n-1)}{|x|} \Phi^2 - c_-^* \Phi^2) dx &\geq \\ &\geq \alpha \int_{\Omega^*} e^{\beta|x|} |\nabla \Phi|^2 dx \quad \forall \Phi \in H_0^1(\Omega^*). \end{aligned}$$

Then (1.3) has a unique solution $v = v^$. Moreover for A^* a maximum principle holds, that is*

$$(2.7) \quad A^*V \geq 0 \text{ in } \Omega^*, \quad V \geq 0 \text{ on } \partial\Omega^* \Rightarrow V \geq 0 \text{ in } \Omega^*.$$

The proof of this proposition will be given in the appendix.

The above arguments yield to state the following

Proposition 2.2. *If one of the conditions of proposition 2.1 is verified then the solution $v = v^*$ of (1.3) satisfies*

$$(2.8) \quad -v^{*'}(s) = \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_0^s (f^*(\sigma) + c^*(\sigma)v^*(\sigma))d\sigma + \frac{Bs^{-1+1/n}}{nC_n^{1/n}} v^*(s) \quad \text{a.e. in } [0, |v > 0|].$$

At this point we are able to prove the comparison result.

Theorem 2.2. *If one of the conditions of proposition 2.1 is verified and u, v are solutions respectively of (1.2), (1.3) with the assumptions (1.1), then*

$$u^*(s) \leq v^*(s) \quad \text{in } [0, |\Omega|].$$

Proof. Starting from (2.1) and (2.8) we show that, setting $w = u^* - v^*$, we will show that

$$(2.9) \quad -w'(s) \leq B \frac{s^{-1+1/n}}{nC_n^{1/n}} w + \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_0^s c^*(\sigma)w(\sigma)d\sigma \quad \text{in } [0, |u > 0|].$$

Obviously (2.9) holds for all $s \in [0, \min\{|u > 0|, |v > 0|\}]$, then it is trivial if $|u > 0| \leq |v > 0|$. We suppose $|u > 0| > |v > 0|$. By regularity results (see [BS]) the solution of (1.3) is in $H^2(\Omega^*)$, then $v^* \in C^1([0, |\Omega|])$ and $v^{*'}(|v > 0|) = 0$.

Hence by (2.8)

$$(2.10) \quad \int_0^{|v > 0|} f^*(\sigma)d\sigma + \int_0^{|v > 0|} c^*(\sigma)v^*(\sigma)d\sigma = 0.$$

By (2.1), taking into account that $w(s) = u^*(s)$ for $|v > 0| \leq s \leq |u > 0|$ and (2.10), we get

$$-w'(s) \leq B \frac{s^{-1+1/n}}{nC_n^{1/n}} w + \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_0^s c_-^*(\sigma) w(\sigma) d\sigma + \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_{|v > 0|}^s f^*(\sigma) d\sigma$$

for $|v > 0| \leq s \leq |u > 0|$.

Then we obtain (2.9) observing that by (2.10)

$$\int_0^{|v > 0|} f^*(\sigma) d\sigma \leq 0,$$

which implies $f^*(s) < 0$ in $[|v > 0|, |\Omega|]$ because f^* is a decreasing function.

To get the thesis we first suppose $c_-^* > 0$. Setting $V = \int_0^s c_-^*(\sigma) w(\sigma) d\sigma$, (2.9) can be written as

$$(2.11) \quad \begin{cases} -\left(\exp\left(\frac{Bs^{1/n}}{C_n^{1/n}}\right) \frac{V'}{c_-^*}\right)' - \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \exp\left(\frac{Bs^{1/n}}{C_n^{1/n}}\right) V \leq 0 \text{ in } [0, |u > 0|] \\ V(0) = 0 \\ V'(|u > 0|) \leq 0. \end{cases}$$

We will show that $V \leq 0$. If we suppose *ab absurdo* $V_+ \neq 0$, then there exists \bar{s} such that $V(\bar{s}) > 0, V'(\bar{s}) = 0$ and $V_+ \neq 0$ in $[0, \bar{s}]$. We denote by \bar{B} the ball centered at the origin whose measure is \bar{s} and we consider the eigenvalue problem

$$(2.12) \quad \begin{cases} A^* \varphi = \lambda c_-^*(x) \varphi & \text{in } \bar{B} \\ \varphi \in H_0^1(\bar{B}). \end{cases}$$

If we observe that

$$A^* \varphi = \lambda c_-^*(x) \varphi \Leftrightarrow -(e^{B|x|} \varphi_{x_i})_{x_i} - e^{B|x|} B \frac{(n-1)}{|x|} \varphi - e^{B|x|} c_-^*(x) \varphi = e^{B|x|} \lambda c_-^*(x) \varphi$$

then the first eigenvalue of (2.12) can be characterized by

$$\lambda_1 = \min_{\varphi \in H_0^1(\bar{B})} \frac{\int_{\bar{B}} e^{B|x|} (|\nabla \Phi|^2 - B \frac{(n-1)}{|x|} \Phi^2 - c_-^*(x) \Phi^2) dx}{\int_{\bar{B}} e^{B|x|} c_-^*(\bar{x}) \Phi^2 dx}$$

Since $\tilde{B} \subset \Omega^\#$, by (ii) of proposition 2.1 we find $\lambda_1 > 0$. On the other hand if φ_1 is the eigenfunction corresponding to λ_1 , we have

$$\begin{aligned}
 -\varphi_1'(s) - B \frac{s^{-1+1/n}}{nC_n^{1/n}} \varphi_1(s) - \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_0^s c_-^*(\sigma) \varphi_1(\sigma) d\sigma = \\
 = \lambda_1 \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \int_0^s c_-^*(\sigma) \varphi_1(\sigma) d\sigma \quad \text{a.e. in } [0, \bar{s}].
 \end{aligned}$$

Hence setting $\Phi = \int_0^s c_-^*(\sigma) \varphi_1(\sigma) d\sigma$, it is easy to verify that λ_1 is the same as the first eigenvalue of the problem

$$(2.13) \quad \left\{ \begin{aligned}
 & - \left(\exp \left(\frac{Bs^{1/n}}{C_n^{1/n}} \right) \frac{\Phi'}{c_-^*} \right)' - \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \exp \left(\frac{Bs^{1/n}}{C_n^{1/n}} \right) \Phi = \\
 & = \lambda_1 \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \exp \left(\frac{Bs^{1/n}}{C_n^{1/n}} \right) \Phi \\
 & \Phi(0) = \Phi'(\bar{s}) = 0.
 \end{aligned} \right.$$

By (2.13), using the variational characterization of λ_1 and (2.11), we get

$$\lambda_1 \leq \frac{\int_0^{\bar{s}} \exp \left(\frac{Bs^{1/n}}{C_n^{1/n}} \right) \frac{V_+^{\prime 2}}{c_-^*} - \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \exp \left(\frac{Bs^{1/n}}{C_n^{1/n}} \right) V_+^2 ds}{\int_0^{\bar{s}} \frac{s^{-2+2/n}}{n^2 C_n^{2/n}} \exp \left(\frac{Bs^{1/n}}{C_n^{1/n}} \right) V_+^2 ds} \leq 0$$

which gives the absurd.

Using again (2.11) we have

$$\left\{ \begin{aligned}
 & - \left(\exp \left(\frac{Bs^{1/n}}{C_n^{1/n}} \right) \frac{V'}{c_-^*} \right)' \leq 0 \\
 & V(0) = 0 \\
 & V'(|u > 0|) \leq 0.
 \end{aligned} \right.$$

Integrating between s and $|u > 0|$, we obtain $V'(s) \leq 0$, that is $w(s) \leq 0$. The result in the case $c_-^* \geq 0$ can be proved by approximation techniques. \square

Making use of the techniques of the proof of theorem 2.1 it is possible to estimate the L^2 -norm of ∇u . We have the following

Theorem 2.3. *If one of the conditions of proposition 2.1 is verified and u, v are solution respectively of (1.2), (1.3) with the assumptions (1.1), then*

$$\int_{\Omega} |\nabla u|^2 dx \leq \int_{\Omega^*} |\nabla v|^2 dx.$$

Proof. Starting from (2.5) and squaring both sides we get

$$\begin{aligned} & -\frac{d}{dt} \int_{u>t} |\nabla u|^2 dx \leq \\ & \leq -\mu'(t) \left[Bt + \frac{\mu(t)^{-1+1/n}}{nC_n^{1/n}} \int_0^{\mu(t)} (f^*(\sigma) + c_-^*(\sigma)u^*(\sigma)) d\sigma \right]^2. \end{aligned}$$

Integrating between 0 and $\sup u$ and making a change of variables we obtain

$$\begin{aligned} & \int_{\Omega} |\nabla u|^2 dx \leq \\ & \leq \int_0^{|\sup u|} \left[Bu^*(s) + \frac{s^{-1+1/n}}{nC_n^{1/n}} \int_0^s (f^*(\sigma) + c_-^*(\sigma)u^*(\sigma)) d\sigma \right]^2 ds. \end{aligned}$$

By Theorem 2.2 we have $u^*(s) \leq v^*(s)$, which implies also $|u>0| \leq |v>0|$, therefore we conclude observing that

$$\begin{aligned} & \int_{\Omega^*} |\nabla v|^2 dx = \\ & = \int_0^{|\sup v|} \left[Bv^*(s) + \frac{s^{-1+1/n}}{nC_n^{1/n}} \int_0^s (f^*(\sigma) + c_-^*(\sigma)v^*(\sigma)) d\sigma \right]^2 ds. \quad \square \end{aligned}$$

Using the previous results we can get also an estimate for $a(u, u)$.

Proposition 2.3. *Under the same assumptions of theorem 2.2, the following estimates hold*

- (α) $a(u, u) + \int_{\Omega} c_{-}(x)u^2 dx \leq a^{*}(v, v) + \int_{\Omega^{*}} c_{-}^{*}(x)v^2 dx;$
- (β) $a(u, u) \leq a^{*}(v, v) \quad \text{if } b_i(x) = 0.$

Proof. If in (1.2) we choose as test function $U=0$ and $U=2u$ we obtain

$$a(u, u) = \int_{\Omega} f(x)u dx.$$

By Hardy inequality we have

$$a(u, u) + \int_{\Omega} c_{-}(x)u^2 dx \leq \int_0^{|\Omega|} (f^{*}(s)u^{*} + c_{-}^{*}(s)u^{*2}) ds,$$

and integration by parts gives

$$\int_0^{|\Omega|} (f^{*}(s) + c_{-}^{*}(s)u^{*})u^{*} ds = \int_0^{|\Omega|} \int_0^s (f^{*}(t) + c_{-}^{*}(t)u^{*}) dt (-du^{*}(s)).$$

By (2.4) we have

$$\int_0^s (f^{*}(t) + c_{-}^{*}(t)u^{*}) dt \geq 0,$$

moreover (2.1), (2.8) and theorem 2.1 imply

$$-\frac{du^{*}}{ds} \leq -\frac{dv^{*}}{dv}.$$

Then

$$\begin{aligned} \int_0^{|\Omega|} (f^{*}(s)u^{*} + c_{-}^{*}(s)u^{*2}) ds &\leq \int_0^{|\Omega|} \int_0^s (f^{*}(t) + c_{-}^{*}(t)v^{*}) dt (-dv^{*}(s)) = \\ &= \int_0^{|\Omega|} (f^{*}(s) + c_{-}^{*}(s)v^{*})v^{*} ds = a^{*}(v, v) + \int_{\Omega^{*}} c_{-}^{*}(x)v^2 dx, \end{aligned}$$

that is (α) or, which is the same, (β) when $c_{-}(x) = 0$.

If $b_i(x) = 0$, then the bilinear form $a^{*}(u, v)$ is symmetric and hence we get

$$(2.14) \quad \begin{aligned} a(u, u) &= \int_{\Omega} f(x)u \, dx \leq \int_{\Omega^*} f^*(x)u^* \, dx = a^*(v, u^*) \leq \\ &\leq \sqrt{a^*(v, v)} \sqrt{a^*(u^*, u^*)}. \end{aligned}$$

Ellipticity condition, Pólya Szegő and Hardy inequalities give

$$(2.15) \quad \begin{aligned} a^*(u^*, u^*) &= \int_{\Omega^*} |\nabla u^*|^2 \, dx - \int_{\Omega^*} c^*(x)u^{*2} \, dx \leq \\ &\leq \int_{\Omega} |\nabla u|^2 \, dx - \int_{\Omega} c_-(x)u^2 \, dx \leq a(u, u). \end{aligned}$$

By (2.14) and (2.15) (β) follows. \square

Remark. We can apply the previous results to the obstacle problem

$$(2.16) \quad a(u, \varphi - u) \geq \int_{\Omega} f(\varphi - u) \quad \forall \varphi \in H_0^1(\Omega) \quad \varphi, u \geq \psi$$

where the obstacle $\psi \in H_0^1(\Omega)$ and

$$A\psi = -(a_{ij}(x)\psi_{x_i})_{x_j} + (b_i(x)\psi)_{x_i} + c(x)\psi \quad \text{in } L^2(\Omega).$$

Setting $\bar{u} = u - \psi$ and $g = f - A\psi$ we have that \bar{u} satisfies

$$a(\bar{u}, \varphi - \bar{u}) \geq \int_{\Omega} g(\varphi - \bar{u}) \quad \varphi \in H_0^1(\Omega) \quad \varphi, \bar{u} \geq 0.$$

Hence we can compare \bar{u} with the solution of the symmetrized problem

$$a^*(v, V - v) \geq \int_{\Omega^*} g^*(x)(V - v) \, dx \quad \forall V \in H_0^1(\Omega^*) \quad V, v \geq 0.$$

Acknowledgement. We would like to thank prof. J. I. Díaz for having suggested to estimate the energy in complete form.

APPENDIX

Proof of Proposition 2.1. Following [AMT] at first we prove that conditions (i), (ii), (iii) are equivalent. In order to do that let us observe that

$$(A.1) \quad A^*Z=H \Leftrightarrow - (e^{B|x}Z_x)_x - \frac{e^{B|x}B}{|x|} Z - e^{B|x} c^*(x)Z = e^{B|x}H$$

then the first eigenvalue of (2.6) can be characterized by the Rayleigh principle

$$(A.2) \quad \lambda_1 = \min_{\Phi \in H_0^1(\Omega^*)} \frac{\int_{\Omega^*} e^{B|x} (|\nabla \Phi|^2 - B \frac{(n-1)}{|x|} \Phi^2 - c^*(x) \Phi^2) dx}{\int_{\Omega^*} e^{B|x} p(x) \Phi^2 dx}.$$

(i) \Rightarrow (ii)

By (A.1)

$$\lambda_1 \int_{\Omega^*} e^{B|x} p(x) \Phi_1 Z dx = \int_{\Omega^*} e^{B|x} H \Phi_1 dx,$$

where Φ_1 is the first eigenfunction of (2.6). Then, since Φ_1 has constant sign in Ω^* , we have $\lambda_1 > 0$.

(ii) \Rightarrow (iii)

Choosing in (ii) $\bar{p}(x) \geq B \frac{(n-1)}{|x|} + c^*(x)$, and using (A.2) we get

$$\begin{aligned} & \int_{\Omega^*} e^{B|x} \left(|\nabla \Phi|^2 - B \frac{(n-1)}{|x|} \Phi^2 - c^* \Phi^2 \right) dx - \alpha \int_{\Omega^*} e^{B|x} |\nabla \Phi|^2 dx = \\ & = (1 - \alpha) \int_{\Omega^*} e^{B|x} |\nabla \Phi|^2 dx - \int_{\Omega^*} e^{B|x} \left(B \frac{(n-1)}{|x|} \Phi^2 + c^* \Phi^2 \right) dx \geq \\ & \geq \lambda_1 (1 - \alpha) \int_{\Omega^*} e^{B|x} p(x) \Phi^2 dx - \alpha \int_{\Omega^*} e^{B|x} \left(B \frac{(n-1)}{|x|} \Phi^2 + c^* \Phi^2 \right) dx \geq \\ & \geq (\lambda_1 - \alpha \lambda_1 - \alpha) \int_{\Omega^*} e^{B|x} \left(B \frac{(n-1)}{|x|} \Phi^2 + c^* \Phi^2 \right) dx. \end{aligned}$$

Taking $0 < \alpha < \frac{\lambda_1}{\lambda_1 + 1}$, (iii) follows.

(iii) \Rightarrow (i)

By (iii) the problem

$$(A.3) \quad \begin{cases} -(e^{B|x|}Z_{x_i})_{x_i} - e^{B|x|} B \frac{(n-1)}{|x|} Z - e^{B|x|} c_{-}^{\#}(x) Z = e^{B|x|} H & \text{in } \Omega^{\#} \\ Z \in H_0^1(\Omega^{\#}) \end{cases}$$

has a unique solution. Moreover this solution is positive. As a matter of fact setting $Z_- = \max(-Z, 0)$ by (A.3)

$$\begin{aligned} \alpha \int_{\Omega^{\#}} e^{B|x|} |\nabla Z_-|^2 dx &\leq \int_{\Omega^{\#}} e^{B|x|} \left(|\nabla Z_-|^2 - B \frac{(n-1)}{|x|} Z_-^2 - c_{-}^{\#} Z_-^2 \right) dx = \\ &= - \int_{\Omega^{\#}} e^{B|x|} H Z_- dx \leq 0 \end{aligned}$$

and then $Z_- = 0$.

With similar arguments, by (iii), we obtain (2.7).

Now we observe that (1.3) is equivalent to the variational inequality

$$\begin{aligned} \int_{\Omega^{\#}} e^{B|x|} \left[v_{x_i} (\Phi - v)_{x_i} - B \frac{(n-1)}{|x|} v (\Phi - v) - c_{-}^{\#}(x) v (\Phi - v) \right] dx &\geq \\ \geq \int_{\Omega^{\#}} e^{B|x|} f^{\#}(x) (\Phi - v) dx &\quad \forall \Phi \in H_0^1(\Omega^{\#}), \Phi, v \geq 0 \end{aligned}$$

which, by (iii), has a unique solution v .

To prove that v is decreasing for the sake of simplicity we suppose v sufficiently smooth. Setting $r = |x|, R = \left(\frac{|v > 0|}{C_n} \right)^{1/n}$, $V = v_r$ and deriving with respect to r the equation

$$A^{\#} v = f^{\#} \quad \text{in } \{x : v(x) > 0\}$$

we obtain

$$\begin{aligned} -V_{rr} - \frac{(n-1)}{r} V_r - B V_r + \left(\frac{(n-1)}{r^2} - B \frac{(n-1)}{r} - c_{-}^{\#}(x) \right) V &= \\ = f_r^{\#} - B \frac{(n-1)}{r^2} v + (c_{-}^{\#})_r v &\quad \text{in } [0, R]. \end{aligned}$$

Since $f_r^{\#} - B \frac{(n-1)}{r^2} v \leq 0$, we have

$$A^*V + \frac{(n-1)}{|x|^2}V \leq 0$$

and $V(R) \leq 0$.

We conclude observing that the operator $A^* + \frac{(n-1)}{|x|^2}$ has the property (2.7). \square

References

- [ALT1] ALVINO, A.-LIONS, P. L.-TROMBETTI, G.: *On optimization problems with prescribed rearrangements*, Nonlinear Analysis, T.M.A., **13** (1989), 185-220.
- [ALT2] ALVINO, A.-LIONS, P. L.-TROMBETTI, G.: *Comparison results for elliptic and parabolic equations via Schwarz symmetrization*, Ann. Inst. Henri Poincaré, (2) **7** (1990), 37-65.
- [AMT] ALVINO, A.-MATARASSO, S.-TROMBETTI, G.: *Variational Inequalities and Rearrangements*, Rend. Acc. Naz. Linc., (1992).
- [AT] ALVINO, A.-TROMBETTI, G.: *Sulle migliori costanti di maggiorazione per una classe di equazioni ellittiche degeneri*, Ricerche Mat., **27** (1978), 413-428.
- [Ba] BANDLE, C.: *Isoperimetric Inequalities and Applications*, Monographs and Studies in Math., No. 7, Pitman, London, 1980.
- [BM] BANDLE, C.-MOSSINO, J.: *Rearrangement in Variational Inequalities*, Ann. Mat. Pura e Appl., **138** (1984), 1-14.
- [BS] BREZIS, H. R.-STAMPACCHIA, G.: *Sur la régularité de la solution d'inéquations elliptiques*, Bull. Soc. Math. France, **96** (1968), 153-180.
- [CR] CHONG, K. M.—RICE, N. M.: *Equimeasurable rearrangements of functions*, Queen's papers in pure and applied mathematics, No. 28, Queen's University, Ontario, 1971.
- [DG] DE GIORGI, E.: *Su una teoria generale della misura (r-1)-dimensionale in uno spazio ad r dimensioni*, Ann. Mat. Pura e Appl., **36** (1954), 191-213.
- [DM] DÍAZ, J. I.-MOSSINO, J.: *Isoperimetric inequalities in the parabolic obstacle problem*, to appear in Journal de Math. Pures et App.
- [FP] FERONE, V.-POSTERARO, M. R.: *Symmetrization results for elliptic equations with lower order terms*, Atti Sem. Mat. Fis. Modena, **39** (1991), 47-61.
- [FR] FLEMING, W.-RISHEL, R.: *An integral formula for total gradient variation*, Arch. Math., **11** (1960), 218-222.
- [GT] GIARRUSSO, E.-TROMBETTI, G.: *Estimates for solutions of elliptic equations in a limit case*, Bull. Austral. Math. Soc., **36** (1987), 425-434.
- [HLP] HARDY, G. H.-LITTLEWOOD, J. E.-POLYA, G.: *Inequalities*, Cambridge University Press, 1964.

- [Ka] KAWHOL, B.: *Rearrangements and convexity of level sets in PDE*, Lecture Notes in Math., No. 1150, Springer, Berlin-New York, 1985.
- [MS] MADERNA, C.-SALSA, S.: *Some Special Properties of Solutions to Obstacle Problems*, Rend. Sem. Mat. Univ. Padova, **71** (1984), 121-129.
- [Mo] MOSSINO, J.: *Inégalités isopérimétriques et applications en physique*, Collection Travaux en Cours, Hermann, Paris, 1984.
- [Ta1] TALENTI, G.: *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa, (4) **3** (1976), 697-718.
- [Ta2] TALENTI, G.: *Linear Elliptic P.D.E.'s: Level Sets, Rearrangements and a priori Estimates of Solutions*, Boll. U.M.I., (6) 4-B (1985), 917-949.
- [TV] TROMBETTI, G.-VÁZQUEZ, J. L.: *A symmetrization result for elliptic equations with lower order terms*, Ann. Fac. Sci. de Toulouse, **7** (1985), 137-150.

Dip. di Matematica ed Appl.
«Renato Caccioppoli»
Università di Napoli
Complesso Monte S. Angelo
80126 Napoli
Italy

Recibido: 1 de diciembre de 1992