ANNALES DE L'I. H. P., SECTION C

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Annales de l'I. H. P., section C, tome 7, nº 2 (1990), p. 37-65
<http://www.numdam.org/item?id=AIHPC_1990__7_2_37_0>

Ann. Inst. Henri Poincaré,

Vol. 7, n° 2, 1990, p. 37-65. *Analyse non linéaire*

Comparison results for elliptic and parabolic equations via Schwarz symmetrization

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ABSTRACT. — We study various extensions to general linear or nonlinear, elliptic or parabolic operators of a celebrated result due to G. Talenti. We give several comparison results for solutions of such problems involving the solutions of conveniently symmetrized problems, using Schwarz spherical symmetrization.

Key words : Schwarz symmetrization, comparison results, elliptic equations, parabolic equations, first-order terms, quasilinear equations.

RÉSUMÉ. — Nous étudions diverses extensions à des opérateurs elliptiques ou paraboliques généraux, linéaires ou non linéaires, d'un résultat célèbre dû à G. Talenti. Nous donnons aussi divers résultats de majoration des solutions de tels problèmes par les solutions de problèmes convenablement symétrisés, à l'aide de la symétrisation de Schwarz.

Classification A.M.S. : 35 J 25-35 J 65-35 K 20.

1. INTRODUCTION

It is well known that by means of Schwarz symmetrization it is possible to establish sharp estimates for solutions of second order elliptic and parabolic equations. To be more specific let us consider (*see* [7], [24], [28]) the following problem

$$-\sum_{ij}(a_{ij}(x)\,u_{x_i})_{x_j}=f \quad \text{in } \Omega, \qquad u\in H_0^1(\Omega) \tag{1.1}$$

where the coefficients $a_{ij}(x)$ $(i,j=1,\ldots,n)$ are measurable functions such that

$$\sum_{ij} a_{ij}(x)\,\xi_i\,\xi_j \geqq \nu\,|\,\xi\,|^2, \qquad \forall\,\xi\in\mathbb{R}^n \quad \text{with } \nu\geqq 0. \tag{1.2}$$

Moreover if $\Omega^\#$ is the ball of $\mathbb{R}^n$ centered in 0 such that $|\Omega^\#|=|\Omega|$ and $f^\#$ is the symmetrized function of f (*see* [6]), let us consider the following problem

$$-\nu\,\Delta v=f^\# \quad \text{in } \Omega^\#, \qquad v\in H_0^1(\Omega^\#). \tag{1.3}$$

If $u(x)$, $v(x)$ are the solutions of (1.1), (1.3) respectively, then $u^\#(x)\leqq v(x)$. Obviously such a result allows us to estimate any Orlicz norm of $u(x)$ simply evaluating the same norm of the solution $v(x)$ of (1.3).

The arguments leading to the above result have been extended to general elliptic equations by either weakening ellipticity condition (1.2) (*see* [3], [4]) or taking into account lower order terms (*see* [5], [6], [11], [19], [25], [26]).

In this paper we first study linear elliptic equations of a general form that is with first and zero order terms. And we give two comparison results (Theorems 1 and 2) with different constraints on the coefficients of the lower order terms. In all cases we obtain spherically symmetric problems whose structures depend on the hypotheses on the coefficients. From Theorem 1, following an idea of [27], we derive a comparison result for solutions of parabolic equations. Finally we consider quasilinear equations (*see* also [23] for a similar result). Most of these results have been announced in [1].

2. ELLIPTIC EQUATIONS: MAIN RESULTS

If Ω is an open, bounded set of $\mathbb{R}^n$, let $\Omega^\#$ be the ball of $\mathbb{R}^n$, centered at 0, whose measure is $|\Omega|$; we set $|\Omega^\#|=C_n R_\Omega^n$ where R_Ω is the radius of $\Omega^\#$ and C_n is the measure of the unit ball of $\mathbb{R}^n$. If $\varphi\in L^1(\Omega)$ the function

$$\mu(t)=|\{x\in\Omega : |\varphi(x)|>t\}|, \qquad t\geqq 0$$

is the *distribution function* of φ and

$$\varphi^*(s)=\sup\{t\geqq 0:\mu(t)\geqq s\},\qquad s\in[0,|\Omega|]$$

is the *decreasing rearrangement of* φ. The *spherically symmetric decreasing rearrangement* (or *symmetrized function*) of $\varphi(x)$ is defined by

$$\varphi^\#(x)=\varphi^*(C_n|x|^n),\qquad x\in\Omega^\#.$$

In addition to the above rearrangements it is useful to consider the *increasing rearrangement* of φ, that is the function

$$\varphi_*(s)=\varphi^*(|\Omega|-s),\qquad s\in[0,|\Omega|];$$

likewise we define by

$$\varphi_\#(x)=\varphi_*(C_n|x|^n),\qquad x\in\Omega^\#$$

the *spherically symmetric increasing rearrangement* of φ.

For an exhaustive statement of the properties of rearrangements we refer to [2], [6], [12], [16], [17] and to the appendix of [25]; we just want to point out the Hardy inequality

$$\int_{[0,|\Omega|]} f^*(s)g_*(s)\,ds\leqq\int_\Omega |f(x)g(x)|\,dx\leqq\int_{[0,|\Omega|]} f^*(s)g^*(s)\,ds \qquad (2.1)$$

where $f(x)$, $g(x)$ are measurable functions.

Furthermore we recall the following known result.

LEMMA 1. — *Let $f(s)$, $g(s)$ measurable, positive functions such that*

$$\int_{[0,r]} f(s)\,ds\leqq\int_{[0,r]} g(s)\,ds,\qquad r\in[0,\alpha];$$

if $h(s)\geqq 0$ is a decreasing function then

$$\int_{[0,r]} f(s)h(s)\,ds\leqq\int_{[0,r]} g(s)h(s)\,ds,\qquad r\in[0,\alpha].$$

Now let us consider the following general elliptic operator

$$Lu=-\sum_{ij}(a_{ij}(x)u_{x_i})_{x_j}+\sum_i(b_i(x)u)_{x_i}+\sum_i d_i(x)u_{x_i}+c(x)u$$

and the Dirichlet problem

$$Lu=f \text{ in } \Omega,\qquad u\in H_0^1(\Omega). \qquad (2.2)$$

Besides (1.2) we require the additional conditions

$$\sum_i |b_i(x)+d_i(x)|^2 \leqq R^2 \qquad (R \geqq 0); \tag{2.3}$$

$$\sum_i (b_i(x))_{x_i} + c(x) \geqq c_0(x) \quad \text{in } \mathscr{D}'(\Omega) \tag{2.4}$$

with $c_0(x) \in L^\infty(\Omega)$.

Finally let us consider the symmetrized problem

$$-\nu\Delta v + R|x|^{-1}\sum_1 x_i v_{x_i} + (c_0^+)_\#(x)v - (c_0^-)^\#(x)v = f^\# \quad \text{in } \Omega^\#, \qquad v \in H_0^1(\Omega^\#) \tag{2.5}$$

where $c_0^+(x) = \max(c_0(x), 0)$, $c_0^-(x) = \max(-c_0(x), 0)$; we have the following comparison result.

THEOREM 1. — *We assume that the coefficients of* L *satisfy* (1.2), (2.3) *and* (2.4)*; if the problem* (2.5) *has a spherically symmetric decreasing solution* $v(x) = v^\#(x)$ (*this condition is certainly satisfied if* $c_0(x) \geqq 0$) *then the Dirichlet problem* (2.2) *has a solution* $u(x)$. *Moreover*

(i) *if* $c_0(x) \geqq 0$ *and* $c_0(x) \not\equiv 0$, *then*

$$u^*(s) \leqq v^*(s) \tag{2.6}$$

holds for all $s \in [0, s_1]$ *where* $s_1 = \sup\{s : (c_0)_*(s) = 0\}$ *and*

$$\int_0^s \exp(-R\sigma^{1/n}/(\nu C_n^{1/n}))\, u^*(\sigma)\, d\sigma \leqq \int_0^s \exp(-R\sigma^{1/n}/(\nu C_n^{1/n}))\, v^*(\sigma)\, d\sigma \tag{2.7}$$

holds for $s \in [s_1, |\Omega|]$;

(ii) *if* $c_0(x) \leqq 0$ *then* (2.6) *holds for* $s \in [0, \Omega]$;

(iii) *if* $c_0^+, c_0^- \not\equiv 0$ *then* (2.6) *holds in* $[0, s_2]$ *and* (2.7) *in* $]s_2, |\Omega|]$ *where*

$$s_2 = \inf\{s : (c_0^+)_*(s) > 0\}.$$

Compared to other known results Theorem 1 appears to be the most general in that we are able to handle (in a non trivial way) all the lower order terms. Obviously if for example $b_1 = d_i = 0$, we recover known results (*see* [6], [11], [25] for the cases (i), (ii) and [19] for the case (iii)).

We want here to give an example showing that part (i) of Theorem 1 is in general optimal. Indeed one possible way to test the optimality of part (i) is to ask what is the smallest nonnegative constant δ such that

$$\int_{[0,s]} \exp(-\delta\sigma^{1/n})\, u^*(\sigma)\, d\sigma \leqq \int_{[0,s]} \exp(-\delta\sigma^{1/n})\, v^*(\sigma)\, d\sigma \quad \text{for all } s \in [0, |\Omega|].$$

Our example shows that one has to take $\delta>0$ in general and then by a simple scaling argument one sees that «the optimal δ" is of the form $\delta_n R/\nu$ where δ_n is a constant depending only on n. Part (i) gives $\delta_n \leqq 1/C_n^{1/n}$ and the determination of δ_n is an open question.

In order to show that the above inequality cannot hold in general, we now sketch how to build a counterexample. We consider the example when $n=1$, $f\in\mathscr{D}_+(\mathbb{R})$, $\Omega=(-R, R)$, $\varepsilon>0$ and we introduce the solutions u_ε^R, v_ε^R of

$$-\varepsilon(u_\varepsilon^R)''-|(u_\varepsilon^R)'|+u_\varepsilon^R=f \text{ in } (-R, R), \qquad u_\varepsilon^R(\pm R)=0$$
$$-\varepsilon(v_\varepsilon^R)''-|(v_\varepsilon^R)'|+v_\varepsilon^R=f^\# \text{ in } (-R, R), \qquad v_\varepsilon^R(\pm R)=0.$$

Then, if the agove inequality were valid with $\delta>0$, a simple argument yields that we would deduce

$$\int_{[0,s]}(u_\varepsilon^R)^*(\sigma)\,d\sigma \leqq \int_{[0,s]}(v_\varepsilon^R)^*(\sigma)\,d\sigma, \qquad s\in[0, 2R]. \tag{2.8}$$

Then we would let R go to $+\infty$ and then ε go to 0, thus obtaining

$$\int_{[0,s]}u^*(\sigma)\,d\sigma \leqq \int_{[0,s]}v^*(\sigma)\,d\sigma, \qquad s\in[0, +\infty] \tag{2.9}$$

where u and v are respectively the unique viscosity solutions in BUC($\mathbb{R}$) of

$$-|u'|+u=f \text{ on } \mathbb{R}, \qquad -|v'|+v=f^\# \text{ in } \mathbb{R}.$$

Indeed the convergence for fixed $\varepsilon>0$ as R goes to $+\infty$ is easily proved by ODE considerations (for example) while we may apply the general results on viscosity solutions of M. G. Crandall and P.-L. Lions [13] in order to deduce the convergence as ε goes to 0: observe that in both cases the convergence is uniform in $\mathbb{R}$ (extending by 0 to $\mathbb{R}$ the functions u_ε^R, v_ε^R), thus allowing to pass to the limit in (2.8).

Therefore, we will have obtained the desired counter example if we show that (2.9) is not true in general. To this end, we observe that since v is even we have for all $x\geqq 0$

$$v(x)=v(-x)=\int_{[0,x]}f^\#(x-s)\,e^{-s}\,ds+f^\#(0)\,e^{-x}$$

thus

$$\|v\|_{L^1}=\int_{[0,\infty]}v^*(s)\,ds=2f^\#(0)+\|f^\#\|_{L^1}=2\|f\|_{L^\infty}+\|f\|_{L^1};$$

while u is even if f is even and we have, assuming in addition that f is constant on [1, 2],

$$u(x)\geqq f(1)\,e^{-(1-x)} \quad \text{if } x\in[0, 1],$$

$$u(x) \geqq f(1) \quad \text{if } x \in [1,2], \qquad u(x) \geqq f(1)\, e^{2-x} \quad \text{if } x \geqq 2.$$

Therefore

$$\| u \|_{L^1} = \int_{[0,\infty]} u^*(s)\, ds \geqq f(1)\, 2\, (3 - e^{-1}).$$

and we conclude choosing f even in $\mathscr{D}_+(\mathbb{R})$ such that $\|f\|_{L^\infty} = f(1) = 1$, f is constant on $[1,2]$ and $\|f\|_{L^1} < 2(2-e^{-1})$. Indeed in such a case, (2.9) cannot hold for arbitrary large s.

Now we assume that the coefficients of L satisfy (1.2) and, instead of (2.3), (2.4), the following conditions

$$\sum_1 b_1^2 \leqq B^2, \qquad \sum_i d_i^2 \leqq D^2, \tag{2.10}$$

$$c(x) \geqq 0. \tag{2.11}$$

In agreement with these constraints let us consider the symmetrized problem

$$-\nu \Delta v - B \sum_i (v\, x_i / | x |)_{x_i} + D \sum_i v_{x_i} x_i / | x | = f^\# \quad \text{in } \Omega^\#, \qquad v \in H_0^1(\Omega^\#). \tag{2.12}$$

Then we obtain the following comparison result

THEOREM 2. — *If conditions* (1.2), (2.10), (2.11) *hold and the problem* (2.12) *has a solution* $v(x) = v^\#(x)$, *then there exists a solution* $u(x)$ *of* (2.2)*; moreover* (2.6) *holds for all* $s \in [0, |\Omega|]$.

The above result is known provided that only one of the two terms

$$\sum_i (b_i(x)\, u)_{x_i}, \qquad \sum_i d_i(x)\, u_{x_i} \tag{2.13}$$

is present (*see* [5], [25]). Therefore Theorem 2 solves completely the problem when both terms (2.13) are in the structure of the operator L.

3. PROOF OF THEOREM 1

As well as in the proofs of other similar results, the basic idea is, first, to derive a differential inequality for the rearrangement u^* of the solution $u(x)$ of (1.2) and then to gain the desired result making use of maximum principles. The first aim is achieved by integrating on the level sets of $u(x)$ and using, as main tools, the isoperimetric inequality, the coarea-formula, Schwarz and Hardy inequalities.

If $h > 0$ and $t \in [0, \sup |u|[$, let us write

$$\varphi_h(x) = \begin{cases} h \operatorname{sign} u & \text{if } |u(x)| > t+h \\ (|u(x)| - t) \operatorname{sign} u & \text{if } t < |u(x)| \leqq t+h \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

In view of the definition of weak solution of (2.2), using (3.1) as test functions, we have:

$$\sum_{ij}(1/h)\int_{t<|u|\leqq t+h} a_{ij}(x)\,u_{x_i}u_{x_j}\,dx-\sum_j(1/h)\int_{t<|u|\leqq t+h} b_j(x)\,u_{x_j}u\,dx$$
$$=(1/h)\int_{t<|u|\leqq t+h}\Big(f(x)-c(x)\,u-\sum_j d_j(x)\,u_{x_j}\Big)(|u|-t)\,\mathrm{sign}\,u\,dx$$
$$+\int_{|u|>t+h}\Big(f(x)-c(x)\,u-\sum_j d_j(x)\,u_{x_j}\Big)\,\mathrm{sign}\,u\,dx.$$

By the ellipticity condition (1.2), letting h tend to zero we obtain

$$-\nu\, d/dt\int_{|u|>t}|\nabla u|^2\,dx+\sum_j d/dt\int_{|u|>t} b_j(x)\,u_{x_j}u\,dx$$
$$\leqq\int_{|u|\geqq t}\Big(f(x)-c(x)\,u-\sum_j d_j(x)\,u_{x_j}\Big)\,\mathrm{sign}\,u\,dx; \quad (3.2)$$

here we have used the fact that

$$(1/h)\int_{t<|u|\leqq t+h}\Big(f(x)-c(x)\,u-\sum_j d_j(x)\,u_{x_j}\Big)(|u|-t)\,\mathrm{sign}\,u\,dx$$

goes to zero as $h\to 0$; we rewrite (3.2) in the form

$$-\nu\, d/dt\int_{|u|>t}|\nabla u|^2\,dx\leqq-\sum_j d/dt\int_{|u|>t} b_j(x)\,u_{x_j}u\,dx$$
$$+\sum_j\int_{|u|>t} b_j(x)\,u_{x_j}\,\mathrm{sign}\,u\,dx-\int_{|u|>t} c(x)\,u\,\mathrm{sign}\,u\,dx$$
$$=\sum_j\int_{|u|>t}(b_j(x)+d_j(x))\,u_{x_j}\,\mathrm{sign}\,u\,dx+\int_{|u|>t} f(x)\,\mathrm{sign}\,u\,dx. \quad (3.3)$$

and proceed to evaluate all the terms by the following inequalities

$$n\,C_n^{1/n}\,\mu(t)^{1-1/n}\leqq-d/dt\int_{|u|>t}|\nabla u|\,dx$$
$$\leqq(-\mu'(t))^{1/2}\left(-d/dt\int_{|u|>t}|Du|^2\,dx\right)^{1/2}. \quad (3.4)$$

where $\mu(t)$ denotes the distribution function of $u(x)$.

$$-\sum_j d/dt \int_{|u|>t} b_j(x) u_{x_j} u dx + \sum_j \int_{|u|>t} b_j(x) u_{x_j} \operatorname{sign} u \, dx$$
$$- \int_{|u|>t} c(x) u \operatorname{sign} u \, dx \leqq - \int_{|u|>t} c_0(x) |u(x)| \, dx$$
$$\leqq \int_{[0, \mu(t)]} [(c_0^-)^*(s) - (c_0^+)_*(s)] u^*(s) \, ds. \quad (3.5)$$

$$\left| \sum_j \int_{|u|>t} (b_j(x) + d_j(x)) u_{x_j} \operatorname{sign} u \, dx \right|$$
$$\leqq \mathrm{R}/(n(\mathrm{C}_n)^{1/n}) \int_{[t, +\infty]} \mu(s)^{-1+1/n} (-\mu'(s)) \left(-d/ds \int_{|u|>s} |\nabla u|^2 dx \right) ds. \quad (3.6)$$

$$\left| \int_{|u|>t} f(x) \operatorname{sign} u \, dx \right| \leqq \int_{[0, \mu(t)]} f^*(s) \, ds. \quad (3.7)$$

The inequalities (3.4) are consequence of the isoperimetric inequality [14], Fleming-Rishel coarea formula [15] and Schwarz inequality (we refer to [24] for a complete proof), (3.7) can be easily deduced from Hardy inequality (2.1). With regard to (3.6), from (2.3) we obtain

$$\left| \sum_j \int_{|u|>t} (b_j(x) + d_j(x)) u_{x_j} \operatorname{sign} u \, dx \right| \leqq \mathrm{R} \int_{|u|>t} |\nabla u| \, dx;$$

on the other hand

$$\int_{|u|>t} |\nabla u| \, dx = \int_{[t, +\infty]} (-d/ds \int_{|u| \geqq s} |\nabla u| \, dx) \, ds \quad \text{[by (3.4)]}$$
$$\leqq \int_{[t, +\infty]} (-\mu'(s))^{1/2} (-d/ds \int_{|u|>s} |\nabla u|^2 dx)^{1/2} dx;$$

since from (3.4)

$$1 \leqq (n \mathrm{C}_n^{1/n})^{-1} \mu(t)^{-1+1/n} (-\mu'(t))^{1/2} (-d/dt \int_{|u|>t} |\nabla u|^2 dx)^{1/2} \quad (3.8)$$

we easily obtain (3.6). It remains to show (3.5) and for this purpose we observe that

$$-d/dt \int_{|u|>t} b_j(x) u_{x_j} u \, dx = \lim (1/h) \int_{t<|u| \leqq t+h} b_j(x) u_{x_j} u \, dx$$
$$= \lim (1/h) \int_{t<|u| \leqq t+h} b_j(x) u_{x_j} (u - t \operatorname{sign} u) \, dx$$
$$+ t \lim (1/h) \int_{t<|u| \leqq t+h} b_j(x) u_{x_j} \operatorname{sign} u \, dx$$

$$= t \lim (1/h) \int_{t<|u|\leqq t+h} b_j(x)\, u_{x_j} \operatorname{sign} u\, dx$$

$$= t \lim (1/h) \int_{\Omega} b_j(x)\, (|\varphi_h|)_{x_j}\, dx$$

and then by (2.4)

$$-\sum_j d/dt \int_{|u|>t} b_j(x)\, u_{x_j} u\, dx$$

$$= t \lim (1/h) \int_{\Omega} c(x)\, |\varphi_h|\, dx$$

$$+ t \lim (1/h) \int_{\Omega} \Big(\sum_j b_j(x)\, (|\varphi_h|)_{x_j} - c(x)\, |\varphi_h|\Big)\, dx$$

$$\leqq t \lim (1/h) \int_{\Omega} (c(x) - c_0(x))\, |\varphi_h|\, dx = t \int_{|u|>t} (c(x) - c_0(x))\, dx.$$

Writting $\varphi(x) = \max(|u(x)| - t, 0)$ we get

$$-\sum_j d/dt \int_{|u|>t} b_j(x)\, u_{x_j} u\, dx + \sum_j \int_{|u|>t} b_j(x)\, u_{x_j} \operatorname{sign} u\, dx$$

$$- \int_{|u|>t} c(x)\, u \operatorname{sign} u\, dx$$

$$\leqq -\sum_j \int_{|u|>t} b_j(x)\, (|u| - t)_{x_j}\, dx$$

$$- \int_{|u|>t} c(x)\, (|u| - t)\, dx + \int_{|u|>t} c_0(x)\, (|u| - t)\, dx$$

$$- \int_{|u|>t} c_0(x)\, |u|\, dx = \int_{\Omega} \Big(\sum_j b_j(x)\, \varphi_{x_j} - c(x)\, \varphi(x) + c_0(x)\, \varphi(x)\Big)\, dx$$

$$- \int_{|u|>t} c_0(x)\, |u|\, dx \quad [\text{by } (2.4)]$$

$$\leqq \int_{|u|>t} c_0^-(x)\, |u|\, dx - \int_{|u|>t} c_0^+(x)\, |u|\, dx \quad [\text{by Hardy inequality } (2.1)]$$

$$\leqq \int_{[0,\, \mu(t)]} [(c_0^-)^*(s) - (c_0^+)_*(s)]\, u^*(s)\, ds.$$

Collecting (3.5), (3.6), (3.7) we thus have

$$-\nu\, d/dt \int_{|u|>t} |\nabla u|^2\, dx \leqq$$

$$\leqq R/(n C_n^{1/n}) \int_{[t,\,+\infty)} \mu(s)^{-1+1/n} (-\mu'(s)) (-d/ds \int_{|u|>s} |Du|^2\, dx)$$

$$+ \int_{[0,\,\mu(t)]} [f^*(s) + ((c_0^-)^*(s) - (c_0^+)_*(s))\, u^*(s)]\, ds.$$

We now make use of Gronwall's lemma:

$$-\nu\, d/dt \int_{|u|>t} |\nabla u|^2\, dx \leqq \exp(R(\nu C_n^{1/n})^{-1} \mu(t)^{1/n})$$

$$\times \int_{[t,\,+\infty]} \exp(-R(\nu C_n^{1/n})^{-1} \mu(s)^{1/n}$$

$$\times \{ f^*(\mu(s)) + [(c_0^-)^*(\mu(s)) - (c_0^+)_*(\mu(s))]\, u^*(\mu(s)) \} (-\mu'(s))\, ds$$

so that, by (3.8)

$$1/(-\mu'(t)) \leqq \nu^{-1} n^{-2} C_n^{-2/n} \mu(t)^{-2+2/n} \exp(R(\nu C_n^{1/n})^{-1} \mu(t)^{1/n})$$

$$\times \int_{[0,\,\mu(t)]} \exp(-R(\nu C_n^{1/n})^{-1} \sigma^{1/n})$$

$$\times \{ f^*(\sigma) + [(c_0^-)^*(\sigma) - (c_0^+)_*(\sigma)]\, u^*(\sigma) \}\, d\sigma.$$

Hence, by standard arguments (*see* [25])

$$-(u^*)'(s) \leqq \nu^{-1} n^{-2} C_n^{-2/n} s^{-2+2/n} \exp(R(\nu C_n^{1/n})^{-1} s^{1/n})$$

$$\times \int_{[0,\,s]} \exp(-R(\nu C_n^{1/n})^{-1} \sigma^{1/n})$$

$$\times \{ f^*(\sigma) + [(c_0^-)^*(\sigma) - (c_0^+)_*(\sigma)]\, u^*(\sigma) \}\, d\sigma. \quad (3.9)$$

Let us consider problem (2.5) and its solution $v(x) = v^\#(x)$; obviously the arguments leading to (3.9) proceed in the same way except that equalities now replace inequalities in the details. Thus in place of (3.9) we obtain the differential equality

$$-(v^*)'(s) = \nu^{-1} n^{-2} C_n^{-2/n} s^{-2+2/n} \exp(R(\nu C_n^{1/n})^{-1} s^{1/n})$$

$$\times \int_{[0,\,s]} \exp(-R(\nu C_n^{1/n})^{-1} \sigma^{1/n})$$

$$\times \{ f^*(\sigma) + [(c_0^-)^*(\sigma) - (c_0^+)_*(\sigma)]\, v^*(\sigma) \}\, d\sigma \quad (3.10)$$

where $v^*(s)$ is the decreasing rearrangement of $v(x)$.

Remark 1. – If $g(x)=g^\#(x)$ is such that

$$\int_{[0,s]} \exp(-R(\nu C_n^{1/n})^{-1}\sigma^{1/n}) f^*(\sigma)\,d\sigma \leq \int_{[0,s]} \exp(-R(\nu C_n^{1/n})^{-1}\sigma^{1/n}) g^*(\sigma)\,d\sigma \quad (3.11)$$

we can insert $g^*(s)$ instead of $f^*(s)$ in (3.9), (3.10). Obviously now the function $v^*(s)$ in (3.10) is the rearrangement of the solution of (2.5) with $f^\#(x)$ replaced by $g^\#(x)$.

For the discussion of (3.9), (3.10), we distinguish different cases depending upon the sign of $c_0(x)$.

We begin by considering the simple case $c_0(x)=0$. We then have

$$-(u^*)'(s) \leq -(v^*)'(s) \quad \text{in } [0,|\Omega|], \qquad u^*(|\Omega|)=v^*(|\Omega|)=0;$$

integrating on $[s,|\Omega|]$ we obtain (2.6). This result is already known if $b_i=0$ or b_i are "sufficiently smooth" (*see* [25]).

Case (i). – $c_0(x)\geq 0$ and $c_0(x)\not\equiv 0$. We note that this case could fall within the previous one simply disregarding the zero order term; in such a way, however, we can just compare $u^*(s)$ with the rearrangement $v_0^*(s)$ of the solution of the problem

$$-\nu\Delta v_0 + (R/|x|)\sum_i x_i (v_0)_{x_i} = f^\#(x) \quad \text{in } \Omega^\#, \qquad v_0 \in H_0^1(\Omega^\#).$$

On the other hand one can yield more precise estimates for $u^*(s)$ by handling carefully the zero order term in order to compare $u(x)$ with the smaller function $v(x)\,(\leq v_0(x))$. For example, if $b_i=d_i=0$, (2.6) fails but it is replaced by the weaker inequality

$$\int_{[0,s]} u^*(r)\,dr \leq \int_{[0,s]} v^*(r)\,dr, \qquad s\in[0,|\Omega|]. \quad (3.12)$$

The previous inequality is fully satisfactory for our ends because it allows us to estimate Orlicz norm of u by the same Orlicz norm of v (*see* [6], [11], [19]).

Let us write $w(s)=u^*(s)-v^*(s)$ and $s_1=\sup\{s:(c_0)_*(s)=0\}$; from (3.9), (3.10) we have

$$-w'(s) \leq -\nu^{-1} n^{-2} C_n^{-2/n} s^{-2+2/n} \exp(R(\nu C_n^{1/n})^{-1} s^{1/n}) \times \int_{[s_1,s]} \exp(-R(\nu C_n^{1/n})^{-1}\sigma^{1/n})(c_0)_* w(\sigma)\,d\sigma, s\in[s_1,|\Omega|]. \quad (3.13)$$

Writing

$$W(s)=\int_{[s_1,s]} \exp(-R(\nu C_n^{1/n})^{-1}\sigma^{1/n})(c_0)_*(\sigma)\,w(\sigma)\,d\sigma$$

(3.13) can be interpreted in terms of the following problem

$$-(\exp(R(\nu C_n^{1/n})^{-1} s^{1/n})(c_0)_*(s)^{-1} W')' + \nu^{-1} n^{-2} C_n^{-2/n} s^{-2+2/n} \exp(R(\nu C_n^{1/n})^{-1} s^{1/n}) W \leqq 0 \quad \text{in }]s_1, |\Omega|[$$
$$W(s_1) = W'(|\Omega|) = 0.$$

By the maximum principle we have $W(s) \leqq 0$ that is

$$\int_{[s_1, s]} \exp(-R(\nu C_n^{1/n})^{-1} \sigma^{1/n})(c_0)_*(\sigma) u^*(\sigma)\, d\sigma \leqq \int_{[s_1, s]} \exp(-R(\nu C_n^{1/n})^{-1} \sigma^{1/n})(c_0)_*(\sigma) v^*(\sigma)\, d\sigma;$$

moreover by virtue of Lemma 1 [with $h(\sigma) = (c_0)_*(s)^{-1}$] we obtain

$$\int_{[s_1, s]} \exp(-R(\nu C_n^{1/n})^{-1} \sigma^{1/n}) u^*(\sigma)\, d\sigma \leqq \int_{[s_1, s]} \exp(-R(\nu C_n^{1/n})^{-1} \sigma^{1/n}) v^*(\sigma)\, d\sigma. \quad (3.14)$$

From (3.14) it follows that $u^*(s_1) \leqq v^*(s_1)$; on the other hand in $[0, s_1]$ (3.13) is replaced by $(-u^*)' \leqq (-v^*)$; therefore we get $u^*(s) \leqq v^*(s)$ in $[0, s_1]$. This completes the proof in case (i).

Case (ii). – $c_0(x) \leqq 0$ and $c_0(x) \not\equiv 0$. Let us assume initially $c_0(x) < 0$ a.e. in Ω so that $(c_0^-)^*(s) > 0$ in $[0, |\Omega|[$. From (3.9), (3.10) we obtain

$$-w'(s) \leqq \nu^{-1} n^{-2} C_n^{-2/n} s^{-2+2/n} \exp(R(\nu C_n^{1/n})^{-1} s^{1/n}) \times \int_{[0, s]} \exp(-R(\nu C_n^{1/n})^{-1} \sigma^{1/n})(c_0^-)^*(\sigma) w(\sigma)\, d\sigma$$

where $w(s) = u^*(s) - v^*(s)$. Writing

$$W(s) = \int_{[0, s]} \exp(-R(\nu C_n^{1/n})^{-1} \sigma^{1/n})(c_0^-)^*(\sigma) w(\sigma)\, d\sigma$$

we have

$$-(\exp(R(\nu C_n^{1/n})^{-1} s^{1/n})(c_0^-)^*(s)^{-1} W')' \leqq \nu^{-1} n^{-2} C_n^{-2/n} s^{-2+2/n} \exp(R(\nu C_n^{1/n})^{-1} s^{1/n}) W \quad (3.15)$$
$$W(0) = W'(|\Omega|) = 0.$$

We note here the importance of the hypothesis on the existence of a symmetrically decreasing solution $v(x) = v^\#(x)$ of problem (2.5); indeed it provides a maximum principle for (3.15) by arguments involving the first eigenvalue λ_1 of the following problem

$$-(\exp(R(\nu C_n^{1/n})^{-1} s^{1/n})(c_0^-)^*(s)^{-1} \varphi')' = \lambda \nu^{-1} n^{-2} C_n^{-2/n} s^{-2+2/n} \exp(R(\nu C_n^{1/n})^{-1} s^{1/n}) \varphi$$
$$\varphi(0) = \varphi'(|\Omega|) = 0.$$

In fact the problem

$$-(\exp(\mathrm{R}(\nu \mathrm{C}_n^{1/n})^{-1} s^{1/n})(c_0^-)^*(s)^{-1} \mathrm{Z}')' = \nu^{-1} n^{-2} \mathrm{C}_n^{-2/n} s^{-2+2/n} \exp(\mathrm{R}(\nu \mathrm{C}_n^{1/n})^{-1} s^{1/n}) \mathrm{Z}$$
$$+\nu^{-1} n^{-2} \mathrm{C}_n^{-2/n} s^{-2+2/n} \int_{[0,s]} \exp(-\mathrm{R}(\nu \mathrm{C}_n^{1/n})^{-1} s^{1/n}) f^*(\sigma)\, d\sigma$$
$$\mathrm{Z}(0)=\mathrm{Z}'(|\Omega|)=0$$

has [*see* (3.10)] the following positive solution

$$\mathrm{Z}(s)=\int_{[0,s]} \exp(-\mathrm{R}(\nu \mathrm{C}_n^{1/n})^{-1} \sigma^{1/n})(c_0^-)^*(\sigma)\, v^*(\sigma)\, d\sigma;$$

hence by using, with slight modifications, the same arguments than in [6] (*see* also [18]) we obtain that λ_1 is greater then one: thus we can conclude (*see* [6] again) that

$$\mathrm{W}(s) \leqq 0, \qquad \mathrm{W}'(s) \leqq 0$$

i. e. (2.6). Finally we remark that (2.6) also provides an existence result for problem (2.2).

In order to dispense with the initial assumptions concerning $c_0(x)$ we proceed by approximation. For example we consider the following problem

$$-\nu \Delta v_\varepsilon + (\mathrm{R}/|x|) \Sigma x_i (v_\varepsilon)_{x_i} - (c_0^-)^\# v_\varepsilon - \varepsilon v_\varepsilon = f^\#(x) \quad \text{in } \Omega^\#, \qquad v_\varepsilon \in \mathrm{H}_0^1(\Omega^\#)$$

If ε is small enough this problem has a symmetrically decreasing solution $v_\varepsilon(x)=v_\varepsilon^\#(x)$. By the above result (we replace c_0 by $c_0-\varepsilon$) we obtain $u^*(s) \leqq v_\varepsilon^*(s)$ for all $s \in [0, |\Omega|]$. Since we can estimate (uniformly with respect to ε) L^2 and H_0^1 norms of v_ε, by continuity arguments, v_ε converges in L^2 to the solution of (2.5) and then $v^*(s)=\lim v_\varepsilon^*(s)$; so we obtain (2.6) again.

Case (iii). — $c_0(x)=c_0^+(x)-c_0^-(x)$ and $c_0^+(x)$, $c_0^-(x) \not\equiv 0$. Let us denote by

$$s_1'=\inf\{s:(c_0^+)_*(s)>0\}, \qquad s_0'=\sup\{s:(c_0^-)^*(s)>0\};$$

we assume initially

$$(c_0^-)^*(s) \text{ is continuous at } s_0'. \tag{3.16}$$

If $w(s)=u^*(s)-v^*(s)$ we have

$$-w'(s) \leqq \nu^{-1} n^{-2} \mathrm{C}_n^{-2/n} s^{-2+2/n} \exp(\mathrm{R}(\nu \mathrm{C}_n^{1/n})^{-1} s^{1/n}) \times \int_{[0,s]} \exp(-\mathrm{R}(\nu \mathrm{C}_n^{1/n})^{-1} \sigma^{1/n})(c_0^-)^*(\sigma)\, w(\sigma)\, d\sigma$$
$$-\nu^{-1} n^{-2} \mathrm{C}_n^{-2/n} s^{-2+2/n} \exp(\mathrm{R}(\nu \mathrm{C}_n^{1/n})^{-1} s^{1/n}) \times \int_{[0,s]} \exp(-\mathrm{R}(\nu \mathrm{C}_n^{1/n})^{-1} \sigma^{1/n})(c_0^+)_*(\sigma)\, w(\sigma)\, d\sigma. \tag{3.17}$$

Writing

$$W_1(s)=\int_{[0,s]}\exp(-R(\nu C_n^{1/n})^{-1}\sigma^{1/n})(c_0^-)^*(\sigma)\,w(\sigma)\,d\sigma,\qquad s\in[0,s_0']$$

from (3.17), (3.16) we obtain

$$\begin{aligned}&-(\exp(R(\nu C_n^{1/n})^{-1}s^{1/n})(c_0^-)^*(s)^{-1}W_1')'\\&\qquad\leqq \nu^{-1}n^{-2}C_n^{-2/n}s^{-2+2/n}\exp(R(\nu C_n^{1/n})^{-1}s^{1/n})W_1\\&\qquad W_1(0)=W_1'(s_0')=0.\end{aligned}$$

Proceeding as in case (ii) we have $W_1'(s)\leqq 0$ in $[0,s_0']$ *i.e.*

$$u^*(s)\leqq v^*(s)\quad\text{in }[0,s_0'].\tag{3.18}$$

Writing now

$$W_2(s)=\int_{[s_1',s]}\exp(-R(\nu C_n^{1/n})^{-1}\sigma^{1/n})(c_0^+)_*(\sigma)\,w(\sigma)\,d\sigma,\qquad s\in]s_1',|\Omega|]$$

from (3.17) and (3.18) we obtain

$$\begin{aligned}&-(\exp(R(\nu C_n^{1/n})^{-1}s^{1/n})(c_0^+)_*(s)^{-1}W_2')'\\&\qquad\leqq \nu^{-1}n^{-2}C_n^{-2/n}s^{-2+2/n}\exp(R(\nu C_n^{1/n})^{-1}s^{1/n})\\&\qquad\times\int_{[0,s_0']}\exp(-R(\nu C_n^{1/n})^{-1}\sigma^{1/n})(c_0^-)^*(\sigma)\,w(\sigma)\,d\sigma\\&\qquad-\nu^{-1}n^{-2}C_n^{-2/n}s^{-2+2/n}\exp(R(\nu C_n^{1/n})^{-1}s^{1/n})W_2(s)\\&\qquad\leqq-\nu^{-1}n^{-2}C_n^{-2/n}s^{-2+2/n}\exp(R(\nu C_n^{1/n})^{-1}s^{1/n})W_2(s)\\&\qquad\qquad W_2(s_1')=W_2'(|\Omega|)=0.\end{aligned}$$

Proceeding as in case (i) we have

$$\int_{[s_1',s]}\exp(-R(\nu C_n^{1/n})^{-1}\sigma^{1/n})u^*(\sigma)\,d\sigma\leqq\int_{[s_1',s]}\exp(-R(\nu C_n^{1/n})^{-1}\sigma^{1/n})v^*(\sigma)\,d\sigma\tag{3.19}$$

and also

$$u^*(s_1')\leqq v^*(s_1').\tag{3.20}$$

Finally from (3.17), (3.18) we deduce

$$w'(s)\leqq 0\quad\text{in }[s_0',s_1'];$$

integrating between $s\in[s_0',s_1']$ and s_1', using (3.20), we get

$$u^*(s)\leqq v^*(s)\quad\text{in }[s_0',s_1'].$$

This completes the proof of case (iii). At last we can remove the hypothesis (3.16) proceeding by approximations.

Remark 2. — The above proof shows in fact that if, for instance, c_0 is a nonnegative constant and we set

$$U(x)=\int_{|y|<|x|} u^{\#}(y)\exp(-R\nu^{-1}|y|)\,dy,$$
$$V(x)=\int_{|y|<|x|} v(y)\exp(-R\nu^{-1}|y|)\,dy,$$

then we have for $x\in\Omega^{\#}$

$$-\nu\Delta U+(2\nu(n-1)|x|^{-1}-R)|x|^{-1}\sum_i x_i U_{x_i}+c_0 U \leqq \int_{|y|<|x|} f^{\#}(y)\exp(-R\nu^{-1}|y|)\,dy$$

$$-\nu\Delta V+(2\nu(n-1)|x|^{-1}-R)|x|^{-1}\sum_i x_i V_{x_i}+c_0 V = \int_{|y|<|x|} f^{\#}(y)\exp(-R\nu^{-1}|y|)\,dy$$

and U, V satisfy homogeneous Neumann conditions on $\partial\Omega^{\#}$. In particular this yields on $(0, R_\Omega)$

$$w'+\varphi w\leqq 0 \qquad \text{where} \quad w(r)=(U-V)(|x|) \quad \text{with} \quad x\in\Omega^{\#},\ |x|=r,$$

and φ solves

$$\nu\varphi'+(R-\nu(n-1)t^{-1}-\nu\varphi)\varphi=-c_0 \quad \text{on} \quad (0, R_\Omega) \qquad \text{with} \quad \varphi(R_\Omega)=0.$$

4. PROOF OF THEOREM 2

In this section we assume that the coefficients of L satisfy hypotheses (1.2), (2.10), (2.11). If $u(x)$ is a solution of (2.2), proceeding as in the previous section, from (3.2) and (2.11) we have

$$-\nu\, d/dt\int_{|u|>t}|\nabla u|^2\,dx \leqq \sum_j d/dt\int_{|u|>t} b_j(x)\,u_{x_j}u\,dx-\sum_i\int_{|u|>t} d_i(x)\,u_{x_i}\operatorname{sign} u\,dx + \int_{|u|>t} f(x)\operatorname{sign} u\,dx. \quad (4.1)$$

The bounds of the terms in the right hand side can be achieved as follows:

$$-\sum_j d/dt \int_{|u|>t} b_j(x)\, u_{x_j} u\,dx = \lim (1/h) \sum_j \int_{t<|u|\leqq t+h} b_j(x)\, u_{x_j} u\,dx$$

$$= \lim (1/h) \sum_j \int_{t<|u|\leqq t+h} b_j(x)\, u_{x_j} (u - t \operatorname{sign} u)\, dx$$

$$+ t \lim (1/h) \int_{t<|u|\leqq t+h} \sum_j b_j(x)\, u_{x_j} \operatorname{sign} u\, dx$$

$$= t \lim (1/h) \int_{t<|u|\leqq t+h} \sum_j b_j(x)\, u_{x_j} \operatorname{sign} u\,dx \leqq \mathrm{B}\, t\left(-d/dt \int_{|u|>t} |\nabla u|\, dx\right)$$

and

$$\sum_i \int_{|u|>t} d_i(x)\, u_{x_i} \operatorname{sign} u\,dx \leqq \mathrm{D} \int_{|u|>t} |\nabla u|\, dx$$

Recalling (3.7) thus we have

$$-\nu\, d/dt \int_{|u|>t} |\nabla u|^2\, dx \leqq \mathrm{B}\, t\left(-d/dt \int_{|u|>t} |\nabla u|\, dx\right)$$

$$+ \mathrm{D} \int_{|u|>t} |\nabla u|\, dx + \int_{[0,\,\mu(t)]} f^*(s)\, ds \quad [\text{by } (3.4)]$$

$$\leqq \mathrm{B}\, t[-\mu'(t)]^{1/2}\left(-d/dt \int_{|u|>t} |\nabla u|^2\, dx\right)^{1/2}$$

$$+ \mathrm{D} \int_{|u|>t} |\nabla u|\, dx + \int_{[0,\,\mu(t)]} f^*(s)\, ds \quad [\text{by } (3.8)]$$

$$\leqq \mathrm{B}\, t[-\mu'(t)]^{1/2}\left(-d/dt \int_{|u|>t} |\nabla u|^2\, dx\right)^{1/2}$$

$$+ (n\,\mathrm{C}_n^{1/n})^{-1}\, \mu(t)^{-1+1/n} [-\mu'(t)]^{1/2}\left(-d/dt \int_{|u|>t} |\nabla u|^2\, dx\right)^{1/2} \mathrm{J}$$

where

$$\mathrm{J} = \mathrm{D} \int_{|u|>t} |\nabla u|\, dx + \int_{[0,\,\mu(t)]} f^*(s)\, dx;$$

hence

$$\mu(t)^{1-1/n} [-\mu'(t)]^{-1/2} \left(-d/dt \int_{|u|>t} |\nabla u|^2\, dx\right)^{1/2} \leqq \mathrm{B}\,\nu^{-1}\, t\,\mu(t)^{1-1/n} + (\nu\, n\, \mathrm{C}_n^{1/n})^{-1}\, \mathrm{J} \quad (4.2)$$

for a.e. $t \in [0, \sup|u|[$. Denoting by $\psi(t)$ the function on the left side of (4.2), since $t\,\mu(t)^{1-1/n}$ converges as t goes to $+\infty$, we deduce that $\psi(t)$ is a bounded function; moreover

$$|\mathrm{J}| \leqq \mathrm{D} \int_{[t,\,+\infty]} \left(-d/ds \int_{|u|>s} |\nabla u|\, dx\right) ds + \int_{[0,\,\mu(t)]} f^*(s)\, ds \quad [\text{by } (3.4)]$$

$$\leqq \mathrm{D} \int_{[t,\,+\infty]} [-\mu'(s)]^{-1/2} \left(-d/ds \int_{|u|>s} |\nabla u|^2\, dx\right)^{1/2} ds + \int_{[0,\,\mu(t)]} f^*(s)\, ds$$

$$\leqq \mathrm{D} \int_{[t,\,+\infty]} \psi(s)\, \mu(s)^{-1+1/n} [-\mu'(s)]\, ds + \int_{[0,\,\mu(t)]} f^*(s)\, ds. \quad (4.3)$$

Thus by (4.2) and (4.3) we can write

$$\psi(t) \leqq \mathrm{D}(\nu n \mathrm{C}_n^{1/n})^{-1} \int_{[t,\,+\infty]} \psi(s)\, \mu(s)^{-1+1/n} [-\mu'(s)]\, ds + \mathrm{B}\nu^{-1} t\, \mu(t)^{1-1/n} + (\nu n \mathrm{C}_n^{1/n})^{-1} \int_{[0,\,\mu(t)]} f^*(s)\, ds.$$

By Gronwall's lemma we have

$$\psi(t) \leqq \exp(\mathrm{D}(\nu \mathrm{C}_n^{1/n})^{-1} \mu(t)^{1/n}) \times \int_{[t,\,+\infty]} \exp(-\mathrm{D}(\nu \mathrm{C}_n^{1/n})^{-1} \mu(s)^{1/n}) \times \{(\nu n \mathrm{C}_n^{1/n})^{-1} f^*(\mu(s))[-\mu'(s)] - \mathrm{B}\nu^{-1}(s\,\mu(s)^{1-1/n})'\}\, ds;$$

hence from (3.4) we obtain

$$n \mathrm{C}_n^{1/n} \mu(t)^{2-2/n} [-\mu'(t)]^{-1} \leqq \exp(\mathrm{D}(\nu \mathrm{C}_n^{1/n})^{-1} \mu(t)^{1/n}) \times \int_{[0,\,\mu(t)]} \exp(-\mathrm{D}(\nu \mathrm{C}_n^{1/n})^{-1} \sigma^{1/n}) \times \{(\nu n \mathrm{C}_n^{1/n})^{-1} f^*(\mu(\sigma)) - \mathrm{B}\nu^{-1}(u^*(\sigma)\,\sigma^{1-1/n})'\}\, d\sigma.$$

Consequently, setting $\mu(t)=s$, since $u^*(\sigma)\,\sigma^{1-1/n}$ goes to 0 as $\sigma \to 0$, we get

$$-(u^*(s))' \leqq (\nu n^2 \mathrm{C}_n^{2/n})^{-1} s^{-2+2/n} \exp(\mathrm{D}(\nu \mathrm{C}_n^{1/n})^{-1} s^{1/n}) \times \int_{[0,\,s]} \exp(-\mathrm{D}(\nu \mathrm{C}_n^{1/n})^{-1} \sigma^{1/n}) f^*(\sigma)\, d\sigma$$
$$+ \mathrm{BD}(\nu n \mathrm{C}_n^{1/n})^{-2} s^{-2+2/n} \exp(\mathrm{D}(\nu \mathrm{C}_n^{1/n})^{-1} s^{1/n}) \times \int_{[0,\,s]} \exp(-\mathrm{D}(\nu \mathrm{C}_n^{1/n})^{-1} \sigma^{1/n})\, u^*(\sigma)\, d\sigma$$
$$+ \mathrm{B}(\nu n \mathrm{C}_n^{1/n})^{-1} u^*(s)\, s^{-1+1/n}. \quad (4.4)$$

As in the previous section our objective is to compare $u^*(s)$ with the solution of the following equation

$$\begin{aligned}-(v^*(s))' = (\nu n^2 C_n^{2/n})^{-1} s^{-2+2/n} \exp(D(\nu C_n^{1/n})^{-1} s^{1/n}) \\ \times \int_{[0,s]} \exp(-D(\nu C_n^{1/n})^{-1} \sigma^{1/n}) f^*(\sigma)\, d\sigma \\ + BD(\nu n C_n^{1/n})^{-2} s^{-2+2/n} \exp(D(\nu C_n^{1/n})^{-1} s^{1/n}) \\ \times \int_{[0,s]} \exp(-D(\nu C_n^{1/n})^{-1} \sigma^{1/n})\, v^*(\sigma)\, d\sigma \\ + B(\nu n C_n^{1/n})^{-1} v^*(s) s^{-1+1/n}. \quad (4.5)\end{aligned}$$

Obviously $v^*(s)$, the rearrangement of the spherically symmetric decreasing solution $v^\#(x)$ of (2.12), is solution of (4.5): indeed (4.5) can be deduced in the same way as (4.4), starting from the problem (2.12), by using only equalities.

From (4.4), (4.5) writing

$$W(s) = \int_{[0,s]} \exp(-D(\nu C_n^{1/n})^{-1} \sigma^{1/n}) (u^*(\sigma) - v^*(\sigma))\, d\sigma$$

we have

$$\begin{aligned}-(\exp((B+D)(\nu C_n^{1/n})^{-1} s^{1/n}) W')' \\ \leqq BD(\nu n C_n^{1/n})^{-2} s^{-2+2/n} \exp(D(\nu C_n^{1/n})^{-1} s^{1/n}) W \\ W(0) = W'(|\Omega|) = 0.\end{aligned}$$

As well as case (iii) of Theorem 1 (*see* section 3) we are now in position to assert that $W'(s) \leqq 0$ and then $u^*(s) \leqq v^*(s)$ for all $s \in [0, |\Omega|]$. Thus the Theorem is proved.

5. PARABOLIC EQUATIONS

Let Q denote the cylindrical domain of $\mathbb{R}^{n+1}$ given by $\Omega \times [0,T]$ $(T>0)$; we consider the initial boundary-value problem

$$\begin{aligned}u_t - \sum_{ij} (a_{ij}(x,t) u_{x_i})_{x_j} + \sum_i (b_i(x,t) u)_{x_i} \\ + \sum_i d_i(x,t) u_{x_i} + c(x,t) u = f(x,t) \quad \text{in } Q \quad (5.1) \\ u \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; L^2(\Omega)), \qquad u(x,0) = u_0(x)\end{aligned}$$

where the coefficients $a_{ij}(x,t)$, $b_i(x,t)$, $d_i(x,t)$, $c(x,t) \in L^\infty(Q)$, $f(x,t) \in L^2(Q)$, $u_0(x) \in L^2(\Omega)$; furthermore we assume

$$\sum_{ij} a_{ij}(x,t) \xi_i \xi_j \geqq \nu(t) |\xi|^2, \quad \forall \xi \in \mathbb{R}^n \quad \text{for a.e. } t \in [0,T], \quad (5.2)$$

with $\nu(t)\in L^\infty(0,T)$ and $\nu(t)\geqq \nu_0>0$;

$$\sum_i |b_i+d_i|^2 \leqq R(t)^2 \tag{5.3}$$

with $R(t)\in L^\infty(0,T)$ and $0\leqq R(t)\leqq R_0$;

$$\sum_i (b_i(x,t))_{x_i}+c(x,t)\geqq c_0(t) \quad \text{in } \mathscr{D}'(\Omega) \qquad \text{for a. e. } t\in[0,T] \tag{5.4}$$

with $c_0(t)\in L^\infty(0,T)$, Finally, we assume

$$R(t)/\nu(t) \text{ is increasing} \tag{5.5}$$

and we set $A_0=\inf(R(t)/\nu(t))$.

Besides we consider the following "symmetrized" problem in the cylindrical domain $Q^\#=\Omega^\#\times[0,T]$

$$\begin{gathered} v_t-\nu(t)\Delta v+R(t)|x|^{-1}\sum_i x_i v_{x_i}+c_0 v=g(x,t) \quad \text{in } Q^\# \\ v\in L^2(0,T;H_0^1(\Omega^\#))\cap C([0,T];L^2(\Omega^\#)), \qquad v(x,0)=v_0(x) \end{gathered} \tag{5.1$'$}$$

where $g(x,t)\in L^2(Q^\#)$ and $v_0(x)\in L^2(\Omega^\#)$.

In all this section we adopt the following convention: if $h(x,t)$ is defined in Q we denote by $h^\#(x,t)$ the symmetrized function, with respect to x, of $h(x,t)$ for t fixed.

Then we assume

$$g(x,t)=g^\#(x,t), \quad \forall x\in\Omega^\# \qquad \text{for a. e. } t\in[0,T]; \tag{5.6}$$

$$\int_{|x|<r} f^\#(x,t)\exp(-R(t)/\nu(t)|x|)\,dx \leqq \int_{|x|<r} g(x,t)\exp(-R(t)/\nu(t)|x|)\,dx, \qquad r\in[0,R_\Omega]; \tag{5.7}$$

$$v_0(x)=v_0^\#(x), \quad \forall x\in\Omega^\#; \tag{5.8}$$

$$\int_{|x|<r} u_0^\#(x)\exp(-A_0|x|)\,dx\leqq \int_{|x|<r} v_0(x)\exp(-A_0|x|)\,dx, \quad r\in[0,R_\Omega]. \tag{5.9}$$

THEOREM 3. — *Let $u(x,t)$, $v(x,t)$ denote the solutions of* (5.1), (5.1)′ *respectively; if conditions from* (5.2) *to* (5.9) *are fulfilled then*

$$U(x,t)\leqq V(x,t), \qquad x\in\Omega^\# \quad \textit{for a.e. } t\in[0,T]$$

where

$$U(x,t)=\int_{|y|<|x|} u^\#(y,t)\exp(-R(t)/\nu(t)|y|)\,dy,$$

$$V(x,t)=\int_{|y|<|x|} v(y,t)\exp(-R(t)/\nu(t)|y|)\,dy.$$

It suffices to prove the theorem for the case $c_0(t) \geqq c_0 > 0$ otherwise we replace $u(x, t)$ by $e^{-\lambda t} u(x, t)$ where λ is a sufficiently large constant.

Initially we assume $R(t)/\nu(t)$ piecewise constant, *i.e.* there exists a subdivision

$$0 = \tau_0 < \tau_1 < \ldots < \tau_k = T$$

of $[0, T]$ such that

$$R(t) = R_i, \qquad \nu(t) = \nu_1, \qquad t \in [\tau_i, \tau_{i+1}[;$$

we put $A_i = R_i/\nu_i$: obviously $A_i \leqq A_{i+1}$. Moreover we divide $[0, T]$ into $m \geqq k$ subintervals by introducing the points

$$0 = t_0 < t_1 < \ldots < t_m = T;$$

we assume that there exist $k-1$ indices $j_1 < j_2 < \ldots < j_{k-1}$ such that

$$t_{j_1} = \tau_1, \qquad t_{j_2} = \tau_2, \ldots, t_{j_{k-1}} = \tau_{k-1};$$

moreover

$$t_{i+1} - t_i \leqq h(m) \qquad \text{and} \qquad h(m) \to 0 \quad \text{as } m \to \infty.$$

Now, following an idea of [27], we replace the term u_t in (5.1) by a difference quotient; we begin by writing

$$a_{ij}^{(1)}(x) = (t_2 - t_1)^{-1} \int_{[t_1, t_2]} a_{ij}(x, t)\, dt,$$

$$b_i^{(1)}(x) = (t_2 - t_1)^{-1} \int_{[t_1, t_2]} b_i(x, t)\, dt,$$

$$d_i^{(1)}(x) = (t_2 - t_1)^{-1} \int_{[t_1, t_2]} d_i(x, t)\, dt,$$

$$c^{(1)}(x) = (t_2 - t_1)^{-1} \int_{[t_1, t_2]} c(x, t)\, dt,$$

$$f^{(1)}(x) = (t_2 - t_1)^{-1} \int_{[t_1, t_2]} f(x, t)\, dt;$$

we thus consider the problem

$$\begin{gathered} -\sum_{ij} (a_{ij}^{(1)}(x) u_{x_i}^{(1)})_{x_j} + \sum_i (b_i^{(1)}(x) u^{(1)})_{x_i} + \sum_i d_i^{(1)}(x) u_{x_i}^{(1)} \\ + c^{(1)}(x) u^{(1)} + (t_2 - t_1)^{-1} u^{(1)} = f^{(1)}(x) + (t_2 - t_1)^{-1} u^{(0)} \\ u^{(1)} \in H_0^1(\Omega). \end{gathered} \tag{5.10}$$

where $u^{(0)} = u_0$.

If for example $t_2 \leqq \tau_1$ we obtain from (5.2), (5.3).

$$\sum_{ij} a_{ij}^{(1)}(x) \xi_1 \xi_j \geqq \nu_0 |\xi|^2, \qquad \xi \in \mathbb{R}^n; \tag{5.11}$$

$$\sum_i |b_i^{(1)}+d_i^{(1)}|^2 \leqq R_0^2; \tag{5.12}$$

furthermore setting

$$c_0^{(1)}=(t_2-t_1)^{-1}\int_{[t_1,\,t_2]} c_0(t)\,dt$$

we have, if $\varphi \in \mathscr{D}_+(\Omega)$

$$\int_\Omega [(c^{(1)}(x)+(t_2-t_1)^{-1})\,\varphi(x)-\sum_i b_i^{(1)}(x)\,\varphi_{x_i}]\,dx$$

$$=(t_2-t_1)^{-1}\int_{[t_1,\,t_2]} dt\int_\Omega [(c(x)+(t_2-t_1)^{-1})\,\varphi(x)-\sum_i b_i(x)\,\varphi_{x_i}]\,dx \quad (\text{by } (5.4))$$

$$\geqq (c_0^{(1)}+(t_2-t_1)^{-1})\int_\Omega \varphi\,dx;$$

hence

$$\sum_i (b_i^{(1)}(x))_{x_i}+c^{(1)}(x)+(t_2-t_1)^{-1} \geqq c_0^{(1)}+(t_2-t_1)^{-1} \quad \text{in } \mathscr{D}'(\Omega). \tag{5.13}$$

At last writing

$$g^{(1)}(x)=(t_2-t_1)^{-1}\int_{[t_1,\,t_2]} g(x,t)\,dt,$$

we have

$$\int_{|x|<r} (f^{(1)}(x)+(t_2-t_1)^{-1}\,u^{(0)})^{\#}\exp(-A_0|x|)\,dx$$

$$\leqq \int_{|x|<r} (g^{(1)}(x)+(t_2-t_1)^{-1}\,v_0)^{\#}\exp(-A_0|x|)\,dx, \qquad r\in[0,R_\Omega]. \tag{5.14}$$

In fact let $e(x)$ be a function (*see* [12]) such that

$$e^{\#}(x)=\begin{cases}\exp(-A_0|x|) & \text{if } |x|<r\\ 0 & \text{if } |x|\geqq r\end{cases}$$

and

$$\int_{|x|<r} (f^{(1)}(x)+(t_2-t_1)^{-1}\,u^{(0)})^{\#}\exp(-A_0|x|)\,dx$$

$$=\int_\Omega (f^{(1)}(x)+(t_2-t_1)^{-1}\,u^{(0)})\,e(x)\,dx;$$

then

$$\int_{|x|<r} (f^{(1)}(x)+(t_2-t_1)^{-1}u^{(0)})^{\#}\exp(-A_0|x|)\,dx$$

[by Hardy inequality (2.1)]

$$\leqq \int_{|x|<r} f^{(1)\#}(x)\exp(-A_0|x|)\,dx+(t_2-t_1)^{-1}\int_{|x|<r} u^{(0)\#}\exp(-A_0|x|)\,dx;$$

recalling (5.7) and (5.9) we obtain (5.14).

Therefore (5.10) is an elliptic problem: let $u^{(1)}$ denote its solution. From (5.11), (5.12), (5.13) and (5.14), by virtue of Theorem 1 and Remark 1 we infer

$$\int_{|x|<r} (u^{(1)})^{\#}\exp(-A_0|x|)\,dx \leqq \int_{|x|<r} v^{(1)}\exp(-A_0|x|)\,dx \quad (5.15)$$

where $v^{(1)}$ is the solution of

$$-\nu_0\Delta v^{(1)}+R_0|x|^{-1}\sum_i x_i v^{(1)}_{x_i}+(c_0^{(1)}+(t_2-t_1)^{-1})v^{(1)} = g^{(1)}(x)+(t_2-t_1)^{-1}v^{(0)}, \qquad v^{(1)}\in H_0^1(\Omega^{\#}) \quad (5.12)'$$

with $v^{(0)}=v_0$.

Now we want to prove inductively that

$$\int_{|x|<r} (u^{(s)})^{\#}\exp(-R(t_{s-1})/\nu(t_{s-1})|x|)\,dx \leqq \int_{|x|<r} v^{(s)}\exp(-R(t_{s-1})/\nu(t_{s-1})|x|)\,dx \quad (5.16)$$

where $u^{(s)}$ is the solution of

$$-\sum_{ij}(a_{ij}^{(s)}(x)u^{(s)}_{x_i})_{x_j}+\sum_i(b_i^{(s)}(x)u^{(s)})_{x_i}+\sum_i d_i^{(s)}(x)u^{(s)}_{x_i} + c^{(s)}(x)u^{(s)}+(t_{s+1}-t_s)^{-1}u^{(s)}=f^{(s)}(x)+(t_{s+1}-t_s)^{-1}u^{(s-1)}, \qquad u^{(s)}\in H_0^1(\Omega)$$

and $v^{(s)}$ is the solution of

$$-\nu(t_s)\Delta v^{(s)}+R(t_s)|x|^{-1}\sum_i x_i v^{(s)}_{x_i}+(c_0^{(s)}+(t_{s+1}-t_s)^{-1})v^{(s)} = g^{(s)}(x)+(t_{s+1}-t_s)^{-1}v^{(s-1)}, \qquad v^{(s)}\in H_0^1(\Omega^{\#});$$

obviously the functions $a_{ij}^{(s)}(x)$, $b_i^{(s)}(x)$, etc. are defined like $a_{ij}^{(1)}(x)$, $b_i^{(1)}(x)$, etc.

In order to prove (5.16) we proceed in the same way as in the previous case if, for example, $t_s<\tau_1$; if $t_s=\tau_1$ then the condition (5.14) becomes

$$\int_{|x|<r} (f^{(s)}(x)+(t_{s+1}-t_s)^{-1} u^{(s-1)})^{\#} \exp(-A_1|x|)\,dx \leqq \int_{|s|<r} (g^{(s)}(x)+(t_{s+1}-t_s)^{-1} v^{(s-1)}) \exp(-A_1|x|)\,dx, \qquad r\in[0, R_\Omega]. \quad (5.17)$$

Since

$$\int_{|x|<r} u^{(s-1)\#} \exp(-A_0|x|)\,dx \leqq \int_{|x|<r} v^{(s-1)} \exp(-A_0|x|)\,dx,$$

and $A_0\leqq A_1$ ($R(t)/\nu(t)$ is increasing!), by virtue of lemma 1 we have

$$\int_{|x|<r} (u^{(s-1)})^{\#} \exp(-A_1|x|)\,dx \leqq \int_{|x|<r} v^{(s-1)} \exp(-A_1|x|)\,dx.$$

Then proceeding as in the proof of (5.4) we obtain (5.17). In the same way we proceed beyond τ_1.

Finally we set

$$u_m(x,t)=u^{(s)}(x,t), \qquad x\in\Omega \text{ and } t_s\leqq t\leqq t_{s+1} \quad (s=0,\ldots,m-1)$$
$$v_m(x,t)=v^{(s)}(x,t), \qquad x\in\Omega^{\#} \text{ and } t_s\leqq t\leqq t_{s+1} \quad (s=0,\ldots,m-1).$$

From (5.16) we have

$$\int_{|x|<r} u_m^{\#}(x,t) \exp(-R(t)/\nu(t)|x|)\,dx \leqq \int_{|x|<r} v_m(x,t) \exp(-R(t)/\nu(t)|x|)\,dx, \qquad r\in[0, R_\Omega]$$

for a.e. $t\in[0,T]$. Then letting $m\to\infty$, since $\{u_m\}$, $\{v_m\}$ converge to u, v respectively (*see* [20]), we obtain the result.

At last we assume that $R(t)/\nu(t)$ is not piecewise constant. Let ν^n, F^n be piecewise constant functions on $(0,T)$ such that $\nu_0\leqq\nu^n\leqq\|\nu\|_{L^\infty}$, F^n is nondecreasing, $F^n\leqq\|R/\nu\|_{L^\infty}$ and

$$\nu^n, F^n \to \nu, R/\nu \quad \text{as } n\to\infty \text{ a.e. on } (0, T).$$

We then set $R^n=F^n\nu^n$. Clearly, $0\leqq R^n\leqq$(const. ind. of n) and $R^n\to R$ a.e. on $(0,T)$. Let us observe that replacing if necessary R by $R+\delta$ (for all $\delta>0$) we may assume $\operatorname{inf\,ess} R>0$.

Next we consider

$$a_{ij}^n=a_{ij}+(\nu^n-\nu)^+\delta_{ij}, \qquad b_i^n=b_i R^n/R, \qquad d_i^n=d_i R^n/R, \qquad c^n=c R^n/R$$

and we observe that we may now apply the preceding proof to the case when a_{ij}, b_i, d_i, c, ν, R are replaced by a_{ij}^n, b_i^n, d_i^n, c^n, ν^n, R^n. And we conclude easily by letting n go to ∞.

6. QUASILINEAR EQUATIONS

For the sake of simplicity we consider the following Dirichlet problem

$$-\sum_{ij}(a_{ij}(x)\,u_{x_j})_{x_i}=\mathrm{H}(x,\nabla u) \quad \text{in } \Omega, u\in \mathrm{H}_0^1(\Omega)\cap \mathrm{L}^\infty(\Omega) \qquad (6.1)$$

where $a_{ij}(x)$, $\mathrm{H}(x,\xi)$ are measurable functions verifying ellipticity condition (1.2) (with $\nu=1$) and the following growth condition

$$|\mathrm{H}(x,\xi)|\leqq f(x)+\mathrm{C}_0(\sum_i \xi_i^2)^{p/2} \qquad (6.2)$$

with $f\in \mathrm{L}_+^\infty(\Omega)$, $\mathrm{C}_0>0$, $p\in[1,2]$.

THEOREM 4. — *Under the conditions* (1.2), (6.2), *if there exists a solution* $v(x)$ $(=v^\#(x))$ *of the problem*

$$-\Delta v=f^\#(x)+\mathrm{C}_0|\nabla v|^p \quad \text{in } \Omega^\#, \qquad v\in \mathrm{H}_0^1(\Omega^\#)\cap \mathrm{L}^\infty(\Omega^\#) \quad (6.1)'$$

then (6.1) *has a solution* $u(x)$; *moreover*

$$u^\#(x)\leqq v(x) \quad \text{in } \Omega^\#. \qquad (6.3)$$

$$\int_\Omega \beta(|\nabla u|^2)\,dx\leqq\int_{\Omega^\#}\beta(|\nabla v|^2)\,dx \qquad (6.4)$$

for all functions β *concave, nondecreasing on* $[0, \infty)$.

If (6.1) has a weak solution $u(x)$, by using the functions (3.1) as test functions, we have

$$(1/h)\sum_{ij}\int_{t<|u|\leqq t+h} a_{ij}(x)\,u_{x_i}u_{x_j}\,dx$$
$$=1/h\int_{t<|u|\leqq t+h}\mathrm{H}(x,\nabla u)(u-t\,\mathrm{sign}\,u)\,dx+\int_{|u|>t+h}\mathrm{H}(x,\nabla u)\,\mathrm{sign}\,u\,dx.$$

From (1.2) letting $h\to 0$ we obtain

$$-d/dt\int_{|u|>t}|\nabla u|^2\,dx\leqq\int_{|u|>t}\mathrm{H}(x,\nabla u)\,\mathrm{sign}\,u\,dx;$$

hence, by (6.2) and Hardy inequality.

$$-d/dt\int_{|u|>t}|\nabla u|^2\,dx\leqq\int_{[0,\,\mu(t)]}f^*(s)\,ds+\mathrm{C}_0\int_{|u|>t}|\nabla u|^p\,dx.$$

We proceed to estimate the last integral in the above inequality

$$\int_{|u|>t} |\nabla u|^p\, dx = \int_{[t,\,+\infty]} \left(-d/ds \int_{|u|>s} |\nabla u|^p\, dx\right) ds \quad \text{(by Holder inequality)}$$

$$\int_{[t,\,+\infty]} \left(-d/ds \int_{|u|>s} |\nabla u|^2\, dx\right)^{p/2} [-\mu'(s)]^{1-p/2}\, ds \quad [\text{by } (3.8)]$$

$$\leqq (n\,C_n^{1/n})^{-2+p} \int_{[t,\,+\infty]} \mu(s)^{(1-1/n)(p-2)} [-\mu'(s)]^{2-p} \left(-d/ds \int_{|u|>s} |\nabla u|^2\, dx\right) ds.$$

Then we obtain

$$-d/dt \int_{|u|>t} |\nabla u|^2\, dx \leqq \int_{[0,\,\mu(t)]} f^*(s)\, ds + C_0 (n\,C_n^{1/n})^{-2+p}$$
$$\times \int_{[t,\,+\infty]} \mu(s)^{(1-1/n)(p-2)} [-\mu'(s)]^{2-p} \left(-d/ds \int_{|u|>s} |\nabla u|^2\, dx\right) ds$$

and, by Gronwall's lemma,

$$-d/dt \int_{|u|>t} |\nabla u|^2\, dx$$
$$\leqq \int_{[t,\,+\infty]} \exp\left\{ C_0 (n\,C_n^{1/n})^{-2+p} \int_{[t,\,s]} \mu(r)^{(1-1/n)(p-2)} [-\mu'(r)]^{2-p}\, dr \right\}$$
$$\times f^{\#}(\mu(s))\, [-\mu'(s)]\, ds; \quad (6.5)$$

finally by (3.8)

$$n^2\, C_n^{2/n}\, \mu(t)^{2-2/n} [-\mu'(t)]^{-1}$$
$$\leqq \int_{[t,\,+\infty]} \exp\left\{ C_0 (n\,C_n^{1/n})^{-2+p} \int_{[t,\,s]} \mu(r)^{(1-1/n)(p-2)} [-\mu'(r)]^{2-p}\, dr \right\}$$
$$\times f^{\#}(\mu(s))\, [-\mu'(s)]\, ds.$$

Hence, by a standard way (*see* also section 3), we have

$$-(u^*(s))' \leqq (n^2\, C_n^{2/n})^{-1}\, s^{-2+2/n} \int_{[0,\,s]} \exp\left\{ C_0 (n\,C_n^{1/n})^{-2+p} \right.$$
$$\left. \times \int_{[\sigma,\,s]} r^{(1-1/n)(p-2)} [-(u^*(r))']^{p-1}\, dr \right\} f^{\#}(\sigma)\, d\sigma$$

and then, in euclidean coordinates with $u^*(s)$ replaced by the spherically symmetric rearrangement $u^{\#}(x)$,

$$|\nabla u^{\#}|(x) \leqq |x|^{1-n}$$
$$\times \int_{(0\,|x|)} \exp\left\{ C_0 \int_{(|y|,\,|x|)} |\nabla u^{\#}|^{p-1}(z)\, d|z| \right\} f^{\#}(y)\, |y|^{n-1}\, d|y|. \quad (6.6)$$

Obviously we also have

$$|\nabla v^{\#}|(x)=|x|^{1-n} \times \int_{(0,|x|)} \exp\left\{C_0 \int_{(|y|,|x|)} |\nabla v^{\#}|^{p-1}(z)\,d|z|\right\} f^{\#}(y)\,|y|^{n-1}\,d|y| \quad (6.7)$$

where $v(x)$ $(=v^{\#}(x))$ is the solution of (6.1)′.

With the help of (6.6), (6.7) we can now prove that

$$|\nabla u^{\#}|(x) \leqq |\nabla v^{\#}|(x), \qquad x\in\Omega^{\#}. \quad (6.8)$$

That is trivial if $p=1$. Thus we assume $p\in]1,2]$. Let A be the set of $\delta\,(>0)$ such that

(*a*) $|\nabla u^{\#}|(x)\leqq|\nabla v^{\#}|(x)$ for $x\in B_\delta=\{x:|x|<\delta\}$,

(*b*) $|\nabla u^{\#}|(x)<|\nabla v^{\#}|(x)$ on a subset of B_δ of positive measure.

We distinguish two cases: in the first case we assume $A\neq\varnothing$. We set $\delta_0=\sup A\,(>0)$. If $\delta_0<R_\Omega$ we have from (*a*), (*b*)

$$\delta_0^{1-n}\int_{(0,\delta_0)} \exp\left\{C_0\int_{(|y|,\delta_0)} |\nabla u^{\#}|^{p-1}(z)\,d|z|\right\} f^{\#}(y)\,|y|^{n-1}\,d|y| \leqq \delta_0^{1-n}\int_{(0,\delta_0)} \exp\left\{C_0\int_{(|y|,\delta_0)} |\nabla v^{\#}|^{p-1}(z)\,d|z|\right\} f^{\#}(y)\,|y|^{n-1}\,d|y|,$$

hence by a continuity argument we obtain

$$|\nabla u^{\#}|(x)\leqq|x|^{1-n}\int_{(0,|x|)} \times\exp\left\{C_0\int_{(|y|,|x|)} |\nabla u^{\#}|^{p-1}(z)\,d|z|\right\} f^{\#}(y)\,|y|^{n-1}\,d|y| < |x|^{1-n}\int_{(0,|x|)} \times\exp\left\{C_0\int_{(|y|,|x|)} |\nabla v^{\#}|^{p-1}(z)\,d|z|\right\} f^{\#}(y)\,|y|^{n-1}\,d|y| = |\nabla v^{\#}|(x)$$

when $\delta_0\leqq|x|<\delta_0+\varepsilon<R_\Omega$ for some $\varepsilon>0$: we have thus arrived at a contradiction.

Finally if $A=\varnothing$ let us consider the problem

$$-\Delta v_\varepsilon=f^{\#}(x)+\varepsilon+C_0|\nabla v_\varepsilon|^p \ \text{ in } \Omega^{\#}, \qquad v_\varepsilon\in H_0^1(\Omega^{\#})\cap L^\infty(\Omega^{\#}); \quad (6.9)$$

if $\varepsilon>0$ is sufficiently small, (6.9) has solution $v_\varepsilon(x)=(v_\varepsilon)^{\#}(x)$.

From (6.6) we get

$$|\nabla u^{\#}|(x)/|x| \leqq |x|^{-n}\exp\left\{C_0\int_{(0,|x|)} |\nabla u^{\#}|^{p-1}(z)\,d|z|\right\} \times\int_{(0,|x|)} \exp\left\{C_0\int_{(0,|y|)} |\nabla u^{\#}|^{p-1}(z)\,d|z|\right\} f^{\#}(y)\,|y|^{n-1}\,d|y|. \quad (6.10)$$

Since

$$\int_{(0,\,|x|)} |\nabla u^{\#}|^{p-1}(z)\,d|z| \leqq \left(\int_{(0,\,|x|)} |\nabla u^{\#}|(z)\,d|z|\right)^{p-1} |x|^{2-p}$$
$$= (\sup u - u^{\#}(x))^{p-1}|x|^{2-p} \qquad \text{if} \quad p\in]1,2[$$
$$= \sup u - u^{\#}(x) \qquad \text{if} \quad p=2$$

we have

$$\lim \exp\left\{ \pm C_0 \int_{(0,\,|x|)} |\nabla u^{\#}|^{p-1}(z)\,d|z| \right\} = 1 \ (\text{as } |x| \to 0);$$

hence from (6.10)

$$\overline{\lim}\, |\nabla u^{\#}|(x)/|x| \leqq 1/n \sup f^{\#} \ (\text{as } |x| \to 0).$$

Likewise from (6.7) (with $v^{\#}, f^{\#}$ replaced by $v_{\varepsilon}^{\#}, f^{\#}+\varepsilon$) we have

$$\lim |\nabla v_{\varepsilon}|(x)/|x| = 1/n \sup (f^{\#}+\varepsilon).$$

Therefore we obtain

$$\overline{\lim}\, |\nabla u^{\#}|(x)/|x| < \lim |\nabla v_{\varepsilon}|(x)/|x|;$$

hence, for some $\delta > 0$

$$|\nabla u^{\#}|(x) < |\nabla v_{\varepsilon}|(x), \qquad 0<|x|<\delta;$$

thus we are again in the first case, therefore

$$|\nabla u^{\#}|(x) < |\nabla v_{\varepsilon}|(x), \qquad 0<|x|<R_{\Omega}.$$

Letting $\varepsilon \to 0$ we obtain (6.8). Obviously (6.8) implies the desired result (6.3).

Furthermore by (6.5), (6.8) we get

$$\int_{\Omega} \beta(|\nabla u|^2)\,dx = \int_{[0,\,+\infty]} \left(-d/dt \int_{|u|>t} \beta(|\nabla u|^2)\,dx\right) dt$$
$$\leqq \int_{[0,\,+\infty]} [-\mu'(t)]\,\beta\left([-\mu'(t)]^{-1}\left(-d/dt\int_{|u|>t} |\nabla u|^2\,dx\right)\right) dt$$
$$\leqq \int_{[0,\,\infty]} [-\mu'(t)] \times \beta\left([-\mu'(t)]^{-1} \int_{[t,\,+\infty]} \exp\left\{ C_0 (n C_n^{1/n})^{-2+p} \right.\right.$$
$$\left.\left. \times \int_{[t,\,s]} \mu(r)^{(1-1/n)(p-2)} [-\mu'(r)]^{2-p}\,dr \right\} f^{\#}(\mu(s)[-\mu'(s)]\,ds\right) dt$$
$$= n C_n \int_{[0,\,R_{\Omega}]} \beta(|\nabla u^{\#}|(x)$$
$$\times \left(\int_{(0,\,|x|)} \exp\left\{ C_0 \int_{(|y|,\,|x|)} |\nabla u^{\#}|^{p-1}(z)\,d|z| \right\} f^{\#}(y)\,|y|^{n-1}\,d|y|\right) d|x|$$

$$\leq n\,C_n \int_{[0,\,R_\Omega]} \beta(|\nabla v^{\#}|(x))$$

$$\times\left(\int_{(0,\,|x|)} \exp\left\{C_0 \int_{(|u|,\,|x|)} |\nabla v^{\#}|^{p-1}(z)\,d|z|\right\} f^{\#}(y)\,|y|^{n-1}\,d|y|\right) d|x|$$

$$= \int_{\Omega^{\#}} \beta(|\nabla v^{\#}|^2)\,dx$$

that is (6.4).

Finally from (6.4) and (6.5), by standard tools (*see* [8], [9], [10]) we can establish an existence result for the Dirichlet problem (6.1).

Remark 3. – The preceeding result can be extended to elliptic operators of the following type

$$-\sum_i (a_i(x, u, \nabla u))_{x_i} + b(x, u) = H(x, \nabla u)$$

where

(*a*) $$\sum_i a_i(x, u, \xi)\,\xi_i \geqq |\xi|^{\alpha}, \qquad \alpha > 1;$$

(*b*) $$H(x, \xi) \leqq f(x) + C_0 \left[\sum_i \xi_i^2\right]^{p/2} \quad \text{with } f \in L^{\infty},\quad C_0 > 0,\quad p \in [1, \alpha];$$

(*c*) $$b(x, s)\,s \geqq 0.$$

The arguments in Theorem 4 proceed in essentially the same way except that

$$1 \leqq (n\,C_n^{1/n})^{-\alpha}\,\mu(t)^{-(1-1/n)\,\alpha}\,[-\mu'(t)]^{\alpha-1}\left(-d/dt \int_{|u|>t} |\nabla u|^{\alpha}\,dx\right)$$

replaces (3.8) in the details.

Remark 4. – The condition $f \in L^{\infty}(\Omega)$ can be relaxed; it is enough to consider a sequence of $L^{\infty}(\Omega)$ functions going to f in some $L^q(\Omega)$. Obviously we need to guarantee the boundness of the solution $v(x)$ of (6.7): to this aim it suffices that $q > n/2$.

Remark 5. – We emphasize that, if $p = 2$, (6.1)′ has a solution iff the first eigenvalue of the operator $(-\Delta - C_0 f^{\#})$ with homogeneous. Dirichlet boundary condition is strictly positive.

Remark 6. – Generally if, for example, $\sup f^{\#}(x)\,|x|^{n/q}$ $(q > n/2)$ is sufficiently small (6.1)′ has a super solution. Hence, by standard tools, we can deduce the existence of a solution $v(x)$ of (6.1)′ (*see* also [23]).

Remark 7. – The estimate (6.4) in its full generality seems to be new. Observe that it clearly applies to Talenti's original result [24] (take $c_0 = 0$) and that $\beta(\sigma) = \sigma^{\alpha}$, for all $\sigma \geqq 0$ and $\alpha \in [0, 1]$, is admissible, hence (6.4) yields a comparison for the L^q norm of $|\nabla u|$ when $0 < q \leqq 2$.

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(*Manuscript received October* 13*th*, 1988.)