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COMPARISON RESULTS FOR SOLUTIONS OF PARABOLIC EQUATIONS WITH A SINGULAR POTENTIAL

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We consider the solution u of the Cauchy-Dirichlet problem for a class of linear parabolic equations in which the coefficient of the zero order term could have a singularity at the origin of the type $1/|x|^2$. We prove that u can be compared "in the sense of rearrangements" with the solution v of a problem whose data are radially symmetric with respect to the space variable.

1. Introduction.

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Let us consider the Cauchy-Dirichlet problem

$$
\begin{cases} u_t - \Delta u + cu = f & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x) & x \in \Omega, \end{cases} \tag{1.1}
$$

where Ω is an open bounded set of $\mathbb{R}^N$ $(N \geq 1)$, $f \in L^2(\Omega \times (0, T))$, $u_0 \in L^2(\Omega)$. In [4], under the assumption $c \in L^p(\Omega)$ with $p > N/2$ in the case $N \geq 2$, $p \geq 1$ if $N = 1$, it is proved that the solution u of

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problem (1.1) can be compared "in the sense of rearrangements" with the solution v of the symmetrized problem

$$\begin{cases} v_t - \Delta v + \left[(c^+)_\# - (c^-)^\#\right] v = f^\# & \text{in } \Omega^\# \times (0,T) \\ v = 0 & \text{on } \partial\Omega^\# \times (0,T) \\ v(x,0) = u_0^\#(x) & x \in \Omega^\#, \end{cases} \tag{1.2}$$

where $\Omega^\#$ is the ball of $\mathbb{R}^N$ centered at the origin having the same measure as Ω, c^+ and c^- are the positive and negative part of c, $(c^+)_\#$ is the *increasing spherical rearrangement* of c^+, and $(c^-)^\#$, $u_0^\#$ are the *decreasing spherical rearrangements* of c^-, u_0 respectively (see section 2 for definitions). Besides, the decreasing spherical rearrangement $f^\#$ of f is meant to be calculated with respect to x, for t fixed.

More precisely, we have that for all $t \in [0,T]$ the following inequality holds

$$\int_0^s u^*(\sigma,t)d\sigma \le \int_0^s v^*(\sigma,t)d\sigma, \ \ \forall s \in [0,|\Omega|], \tag{1.3}$$

where u^* and v^* are the *decreasing rearrangements* of u and v, for t fixed. Similar results are contained in [1], [7], [8], [18], [23], [26].

In the case $c \in L^p(\Omega)$ with $p > N/2$, $N \ge 2$, existence, uniqueness and behaviour of solutions to problem (1.1) do not differ too much from the corresponding properties of solutions to the same problem when the potential c is bounded. Indeed, in both cases the operator $-\Delta u + cu$ can be made coercive, provided to multiply both sides of the equation by $e^{-\lambda t}$ for a suitable real number λ.

However, the situation is remarkably different if the potenzial c is very singular. A really interesting case is when Ω is an open bounded set containing the origin and

$$c(x) = -\frac{\lambda}{|x|^2}.$$

Equations with similar potentials appear in several contests: in the elliptic case, when we consider the Schrödinger equation in quantum mechanics (see [19]) or, in the parabolic case, when we linearize the equation

$$u_t - \Delta u = 2(N-2)e^u$$

with respect to the stationary solution $S(x) := \log\left(1/|x|^2\right)$ (see [12],

[14], [24]). The equation

$$u_t - \Delta u = \frac{\lambda}{|x|^2} u + f \tag{1.4}$$

is a borderline case in the classical theory of parabolic equations, since the potential $\lambda/|x|^2$ belongs to L^p if and only if $1 \leq p < N/2$, therefore it is not possible to use traditional uniqueness and regularity results.

This kind of problems were firstly studied by Baras and Goldstein in [9], with the assumptions $f, u_0 \geq 0$, $f, u_0 \not\equiv 0$ (see also [12] for the case $N = 2$). In [9] it is proved that the behaviour of solutions depends on the value of the parameter λ. More precisely, there exists a critical value $\lambda_N := (N-2)^2/4$, corresponding to the best constant in the classical Hardy inequality, such that for $\lambda \leq \lambda_N$, the Cauchy-Dirichlet problem associated to (1.4) has a solution, while in the case $\lambda > \lambda_N$ the same problem has no local solution for any f, $u_0 \not\equiv 0$.

Afterwards, this problem was studied in [24] removing the sign assumptions on the data and pursuing a deeper analysis of the critical case $\lambda = \lambda_N$, and in [14], where the corresponding nonlinear case is treated.

The subcritical case $\lambda < \lambda_N$ is easier to study than the case $\lambda = \lambda_N$. Indeed, it is possible to use the classical methods of the Calculus of Variations, since by the classical Hardy inequality (see section 2) it follows that the operator

$$Lu = -\Delta u - \frac{\lambda}{|x|^2} u$$

is coercive. Then, for all $f \in L^2\big(\Omega \times (0,T)\big)$ and $u_0 \in L^2(\Omega)$ there exists a unique solution $u \in C\big([0,T]; L^2(\Omega)\big) \cap L^2\big(0,T; H_0^1(\Omega)\big)$ (see [9], [14], [24]).

The situation is very different in the critical case $\lambda = \lambda_N$, in which the classical Hardy inequality fails to provide the coercivity of the differential operator L in $H_0^1(\Omega)$. The existence of a solution in the sense of distribution whose gradient is in a suitable Marcinkiewicz space $\mathcal{M}^p$ is proved in [14]; however, the uniqueness of this solution could fail. In this case, a deeper analysis has been proposed by Vazquez and Zuazua in [24]. More precisely, the authors show an improvement of the Hardy inequality (see section 2) which allows to define a suitable Hilbert space $H(\Omega)$, larger than $H_0^1(\Omega)$, in which existence and uniqueness of solution is guaranteed. The space $H(\Omega)$ is the completion of $H_0^1(\Omega)$ with respect to the norm

$$\|u\|_{H(\Omega)} := \int_\Omega \Big[|\nabla u|^2 - \lambda_N \frac{u^2}{|x|^2}\Big] dx. \tag{1.5}$$

It can be proved that the compact embedding

$$H(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H'(\Omega)$$

holds, that $H(\Omega)$ is larger than $H_0^1(\Omega)$ and smaller than $\bigcap_{q<2} W_0^{1,q}$: then, by classical existence and regularity theorems for evolution equations (see, for instance, [10]) it follows that *there exists a unique*

$$u \in L^2\big(0,T;H(\Omega)\big) \cap C\big([0,T];L^2(\Omega)\big),\, u_t \in L^2\big(0,T;L^2(\Omega)\big)$$

which is a weak solution of the Cauchy-Dirichlet problem associated to equation (1.4), with initial condition $u(\cdot,0) = u_0$ and zero Dirichlet data.

Motivated by the interest of this kind of problems, in this paper we consider the more general problem

$$\begin{cases} u_t - \sum_{i,j=1}^{N} (a_{ij}(x,t)u_{x_i})_{x_j} + cu = f & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & x \in \Omega, \end{cases} \tag{1.6}$$

where we assume that the operator is uniformly parabolic, i.e.

$$\sum_{i,j=1}^{N} a_{ij}(x,t)\xi_i\xi_j \geq |\xi|^2 \quad \textit{for a.e. } (x,t) \in \Omega \times (0,T),\ \forall \xi \in \mathbb{R}^N, \tag{1.7}$$

the data and the coefficients are such that

$$\begin{gathered} \frac{\partial a_{ij}}{\partial t} \in C^0(\overline{\Omega} \times [0,T]),\ i,j, = 1,\dots,N, \\ u_0 \in H_0^1(\Omega),\ \ f \in L^2(\Omega \times (0,T)), \end{gathered} \tag{1.8}$$

$c \in L(N/2,\infty)$ (where $N > 2$), $c \leq 0$ and:

$$c^\#(x) \leq \frac{\lambda}{|x|^2}\ ,\ \ \forall x \in \Omega^\# \backslash \{0\},\ \ 0 < \lambda \leq \lambda_N. \tag{1.9}$$

We compare the solution u of (1.6) with the solution v of the symmetrized problem

$$(1.10) \qquad \begin{cases} v_t - \Delta v = \dfrac{\lambda}{|x|^2} v + f^{\#} & \text{in } \Omega^{\#} \times (0,T) \\ v = 0 & \text{on } \partial\Omega^{\#} \times (0,T) \\ v(x,0) = u_0^{\#}(x) & x \in \Omega^{\#}. \end{cases}$$

More precisely in the subcritical case we prove the following:

Theorem 1. *Let $\lambda < \lambda_N$ and assume that (1.7), (1.8), (1.9) hold. Let u and v be the weak solutions to problems (1.6) and (1.10) respectively. Then, for a.e. $t \in [0,T]$ inequality (1.3) holds.*

As regards the critical case, in order to guarantee existence and uniqueness of a solution to problem (1.6), we also suppose that the coefficients a_{ij} do not depend on time, i.e.

$$(1.11) \qquad a_{ij} = a_{ij}(x),$$

and add the following condition on the zero order term c :

$$(1.12) \qquad |c(x)| \leq \frac{\lambda_N}{|x|^2}, \forall x \in \Omega \backslash \{0\}.$$

Then as in [24] we have existence and uniqueness of a weak solution

$$u \in L^2(0,T;\widetilde{H}(\Omega)) \cap C\big([0,T];L^2(\Omega)\big), u_t \in L^2\big(0,T;L^2(\Omega)\big)$$

of problem (1.6), where $\widetilde{H}(\Omega)$ is the completion of $H_0^1(\Omega)$, with respect to the norm

$$(1.13) \qquad \|u\|_{\widetilde{H}(\Omega)} := \left(\int_\Omega [a_{ij}(x) u_{x_i} u_{x_j} + c(x) u^2] dx \right)^{1/2}$$

and the comparison result can be stated as follows:

Theorem 2. *Let $\lambda = \lambda_N$ and assume that (1.7), (1.8), (1.11), (1.12) hold. Let u and v be the weak solutions to problems (1.6) and (1.10) respectively. Then, for a.e. $t \in [0,T]$ inequality (1.3) still holds.*

In section 3 we prove theorem 1 and theorem 2 following an approach that is quite standard when one deals with this kind of problems. Namely we approximate the solutions u and v to problems (1.6), (1.10) with solutions of Cauchy-Dirichlet problems whose potentials are bounded, to which we can apply the comparison result explained at the beginning.

In section 4 we give an alternative proof which uses the implicit time discretization scheme. This method consists in replacing the time derivative with a difference quotient, so that we are reduced to find a comparison result for a sequence of elliptic problems with a zero order term of the form

$$\begin{cases} -\big(a_{ij}^{(r)}(x)u_{x_i}^{(r)}\big)_{x_j} + \Big(c + \dfrac{1}{t_r - t_{r-1}}\Big)u^{(r)} = f^{(r)} + \dfrac{u^{(r-1)}}{t_r - t_{r-1}} \\ u^{(r)} \in H_0^1(\Omega), \end{cases} \tag{1.14}$$

where $0 = t_0 < \ldots < t_n = T$ is a partition of the interval $[0, T]$, $u^{(r)} = u(x, t_r)$ and $f^{(r)}$, $a_{ij}^{(r)}$, for $r = 1, \ldots, n$, are suitable discretization of the functions $f = f(x, t)$, $a_{ij} = a_{ij}(x, t)$. Then we reach the aim by passing to limit (see [1], [23], [25]). For this reason, at the beginning of section 4 we prove a comparison result for elliptic problems of the type (1.14) where the coefficient $c \in L(N/2, \infty)$ satisfy (1.9) if $\lambda < \lambda_N$ and (1.12) if $\lambda = \lambda_N$.

Comparison theorems for elliptic equations with lower order terms are known in literature only when the zero order term c is in L^p with $p \geq N/2$ (see [2], [4]). Also the result has its own interest since it provides existence of solutions to problems of type (1.14) reducing the study to the spherically symmetric situation.

As usual the proof can be split into two steps. First a differential inequality for the rearrangement of the solution of (1.14) is deduced, an inequality that becomes an equality in the case of symmetrized problem. We do not go into details on how getting this inequality, since it is basically well known. Then, by means of maximum proncipale arguments, one obtains the desired comparison result. At this stage, we essentially use the coercivity of the operator in the case $\lambda < \lambda_N$, while the analysis of the critical case $\lambda = \lambda_N$ requires a one-dimensional form of the improved Hardy inequality.

2. Definitions and preliminary results.

Let Ω be a bounded open subset of $\mathbb{R}^N$ and u be a real measurable function on Ω, we define the distribution function μ_u of u as

$$\mu_u(\theta) = |\{x \in \Omega : |u(x)| > \theta\}|, \theta \geq 0,$$

the decreasing and the increasing rearrangement of u as

$$u^*(s) = \sup\{\theta \geq 0 : \mu_u(\theta) > s\}, s \in (0, |\Omega|)$$

$$u_*(s) = u^*(|\Omega| - s), s \in (0, |\Omega|)$$

Furthermore, if ω_N is the measure of the unit ball in $\mathbb{R}^N$ and $\Omega^\#$ is the ball of $\mathbb{R}^N$ centered at the origin having the same measure as Ω, the functions

$$u^\#(x) = u^*(\omega_N |x|^N), x \in \Omega^\#$$

$$u_\#(x) = u_*(\omega_N |x|^N), x \in \Omega^\#$$

are the decreasing spherical rearrangement and the increasing spherical rearrangement of u respectively. For an exhaustive treatment of rearrangements we refer to [8], [16] and to the appendix of [21]. Here we just recall the well known Hardy-Littlewood inequality (see [15])

$$\int_0^{|\Omega|} u^*(s)v_*(s)ds \leq \int_\Omega |u(x)v(x)|dx \leq \int_0^{|\Omega|} u^*(s)v^*(s)ds. \tag{2.1}$$

where u, v are measurable functions on Ω, and the *Pólya-Szegö* principle:

Theorem 3. *If* $u \in W_0^{1,p}(\Omega)$ *and* $p \geq 1$, *then* $u^\# \in W_0^{1,p}(\Omega^\#)$ *and*

$$\| |\nabla u^\#| \|_{L^p(\Omega^\#)} \leq \| |\nabla u| \|_{L^p(\Omega)}. \tag{2.2}$$

Since we deal with solutions of parabolic equations, we will consider real functions u defined on the set $\Omega \times (0, T)$, where T is a real positive number, that are measurable with respect to the space variable x and denote by $\mu_u(\theta, t)$, $u^*(s, t), u_*(s, t)$, $u^\#(x, t)$, $u_\#(x, t)$ the distribution function and the rearrangements of $u(x, t)$, with respect to x for t fixed. In other words, $u^\#(x, t)$ is the Steiner symmetrization of $u(x, t)$ with respect to the line $x = 0$.

Finally we quote the following result, essentially due to Hardy (see [15]):

Theorem 4. *Assume $N \geq 3$. Then for every $u \in H^1(\mathbb{R}^N)$ we have*

$$\lambda_N \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx, \tag{2.3}$$

where the constant $\lambda_N := (N-2)^2/4$ is optimal.

The same result applies for $u \in H_0^1(\Omega)$, if Ω is an open subset of $\mathbb{R}^N$ containing the origin, with integrals in Ω.

We also recall the following improvement of the Hardy inequality contained in [24]:

Theorem 5. *Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 3$. Then for any $1 \leq q < 2$ there exists a constant $C = C(q, \Omega) > 0$ such that*

$$\int_{\Omega} \left[|\nabla u|^2 - \lambda_N \frac{u^2}{|x|^2} \right] dx \geq C \|u\|^2_{W_0^{1,q}(\Omega)} \tag{2.4}$$

holds for all $u \in H_0^1(\Omega)$.

3. A first method to prove the comparison result: the truncature approach.

In this section we simultaneously prove theorem 1 and theorem 2. In both cases we approximate the solutions u and v to problems (1.6), (1.10) with sequences of solutions of problems having bounded zero order coefficient. More precisely, let $\{u_n\}$ and $\{v_n\}$ be the solutions of the following truncated problems (we use the convention on repeated indices):

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \dfrac{\partial}{\partial x_j}\left(a_{ij} \dfrac{\partial u_n}{\partial x_i}\right) + c_n u_n = f & \text{in} \quad \Omega \times (0,T) \\ u_n = 0 & \text{on} \quad \partial\Omega \times (0,T) \\ u_n(x,0) = u_0(x) & x \in \Omega, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \dfrac{\partial v_n}{\partial t} - \Delta v_n = W_n v_n + f^\# & \text{in} \quad \Omega^\# \times (0,T) \\ v_n = 0 & \text{on} \quad \partial\Omega^\# \times (0,T) \\ v_n(x,0) = u_0^\#(x) & x \in \Omega^\#, \end{cases} \tag{3.2}$$

where $c_n(x)$ is the truncation at level $n \in \mathbb{N}$ of the potential $c(x)$, i.e. the function

$$c_n(x) = \begin{cases} c(x) & \text{if } c(x) \geq -n \\ -n & \text{if } c(x) \leq -n \end{cases}$$

and $W_n(x)$ is the truncation at level $n \in \mathbb{N}$ of the potential $\lambda/|x|^2$. As $c_n \in L^\infty(\Omega)$, we have (see [4], [26]) that for all $t \in [0, T]$,

$$\int_0^s u_n^*(\sigma, t)d\sigma \leq \int_0^s z_n^*(\sigma, t)d\sigma, \;\; \forall s \in [0, |\Omega|], \tag{3.3}$$

where z_n is the solution of the symmetrized problem

$$\begin{cases} \dfrac{\partial z_n}{\partial t} - \Delta z_n - (c_n)^\# z_n = f^\# & \text{in } \;\; \Omega^\# \times (0, T) \\ z_n = 0 & \text{on } \;\; \partial\Omega^\# \times (0, T) \\ z_n(x, 0) = u_0^\#(x) & x \in \;\; \Omega^\#. \end{cases}$$

On the other hand, the assumption (1.9) in the case $\lambda < \lambda_N$ and (1.12) in the case $\lambda = \lambda_N$, implies that $c_n^\# \leq W_n$, therefore by the maximum principle we deduce that $z_n \leq v_n$. Hence by (3.3) we get that, for all $t \in [0, T]$,

$$\int_0^s u_n^*(\sigma, t)d\sigma \leq \int_0^s v_n^*(\sigma, t)d\sigma \;\; , \;\; \forall s \in [0, |\Omega|] \;\; , \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Thus it remains to prove that $\{u_n\}$ and $\{v_n\}$ converge to the solutions u and v of problems (1.6) and (1.10), and that we can pass to the limit in (3.4). At this point we have to treat separately the two cases $\lambda < \lambda_N$ and $\lambda = \lambda_N$. The arguments we use are quite standard (see [11], [13], [14]). If $\lambda < \lambda_N$ we first prove that $\{u_n\}$ is bounded both in $L^\infty\big(0, T; L^2(\Omega)\big)$ and in $L^2\big(0, T; H_0^1(\Omega)\big)$. Indeed, if we choose u_n as test function in the weak formulation of problem (3.1) we have that u_n verifies

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega |u_n(x, t)|^2 dx + \int_\Omega a_{ij}(x, t)\frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_j}dx + \int_\Omega c_n(x)u_n^2(x, t)dx \\ &\qquad = \int_\Omega f(x, t)u_n(x, t)dx. \end{aligned} \tag{3.5}$$

On the other hand by ellipticity condition (1.7), by (2.1), (1.9) and the classical Hardy and Pólya-Szegö inequalities (2.3), (2.2), we find

$$
\begin{aligned}
&\int_\Omega \Big[a_{ij}(x,t)\frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_j} + c_n(x)u_n^2(x,t)\Big]dx \\
&\qquad \geq \int_\Omega |\nabla u_n(t)|^2 dx + \int_\Omega c_n u_n^2(t)dx \\
&\qquad \geq \int_\Omega |\nabla u_n(t)|^2 dx - \int_{\Omega^\#} c^\# u_n^\#(t)^2 dx \\
&\qquad \geq \int_\Omega |\nabla u_n(t)|^2 dx - \lambda \int_{\Omega^\#} \frac{u_n^\#(t)^2}{|x|^2} dx \\
&\qquad \geq \int_\Omega |\nabla u_n(t)|^2 dx - \lambda\lambda_N^{-1} \int_{\Omega^\#} \left|\nabla u_n^\#(t)\right|^2 dx \\
&\qquad \geq \alpha \int_\Omega |\nabla u_n(t)|^2 dx,
\end{aligned} \tag{3.6}
$$

for all $n \in \mathbb{N}$ and $t \in (0,T)$, where $\alpha := (1-\lambda\lambda_N^{-1})$. Hence using (3.6) in (3.5) and estimating the right hand side of (3.5) by Hölder-Young inequalities we deduce that

$$
\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + 2\alpha\|u_n(t)\|^2_{H_0^1(\Omega)} \leq \|u_n(t)\|^2_{L^2(\Omega)} + \|f(t)\|^2_{L^2(\Omega)}. \tag{3.7}
$$

Therefore, Gronwall inequality implies

$$
\max_{t\in[0,T]} \|u_n(t)\|^2_{L^2(\Omega)} \leq C\Big(\|u_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega\times(0,T))}\Big), \tag{3.8}
$$

that is the boundness of $\{u_n\}$ in $L^\infty\big(0,T;L^2(\Omega)\big)$. Finally, integrating (3.7) in $(0,T)$ and using (3.8) we obtain the following energy estimate:

$$
\begin{aligned}
&\max_{t\in[0,T]} \|u_n(t)\|_{L^2(\Omega)} + \|u_n\|_{L^2(0,T;H_0^1(\Omega))} \\
&\qquad\qquad \leq C\Big[\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega\times(0,T))}\Big].
\end{aligned} \tag{3.9}
$$

Since $\{u_n\}$ is bounded in $L^2\big(0,T;H_0^1(\Omega)\big)$, we can extract a subsequence, still denoted by $\{u_n\}$, such that

$$
u_n \rightharpoonup u \quad \textit{weakly } \text{in} L^2\big(0,T;H_0^1(\Omega)\big). \tag{3.10}
$$

Moreover, by the equation of problem (3.1) we easily infer that the sequence $\{\partial u_n/\partial t\}$ is bounded in $L^2\big(0,T;H^{-1}(\Omega)\big)$: then, using a compactness lemma of Aubin's type (see [6], [20]) we get that $\{u_n\}$ is relatively compact in $L^2\big(\Omega \times (0,T)\big)$. Therefore up to subsequences

$$u_n \to u \ strong \text{ in } L^2\big(\Omega \times (0,T)\big) \text{ and a.e..} \tag{3.11}$$

The convergences (3.10) and (3.11) allow us to conclude that $\{u_n\}$ converges to the weak solution of problem (1.6).

On the other hand, by (3.9) we find

$$u_n(t) \to u(t) \ strong \text{ in } L^1(\Omega) \text{ for a.e. } t \in (0,T),$$

hence

$$\int_0^s u_n^*(\sigma,t)d\sigma \to \int_0^s u^*(\sigma,t)d\sigma.$$

In a similar way we prove that $\{v_n\}$ converges to the weak solution of problem (1.10) and that

$$\int_0^s v_n^*(\sigma,t)d\sigma \to \int_0^s v^*(\sigma,t)d\sigma.$$

Hence theorem 1 is proved. As regards the critical case $\lambda = \lambda_N$, the proof is quite similar, we only have to change the functional setting, due to the definition of weak solution we gave in the introduction. We first prove that $\{u_n\}$ (and we can use analogous arguments for $\{v_n\}$) is bounded in $L^\infty\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;\widetilde{H}(\Omega)\big)$. We do that simply by replacing (3.6) with the estimate

$$\int_\Omega \Big[a_{ij}\frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_j} + c_n u_n^2\Big]dx \ge \int_\Omega \Big[a_{ij}(x)\frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_j} + c u_n^2\Big]dx = \|u_n(t)\|^2_{\tilde{H}(\Omega)},$$

hence the same arguments we used for (3.7) allow us to find

$$\frac{d}{dt}\|u_n(t)\|^2_{L^2(\Omega)} + 2\|u_n(t)\|^2_{\tilde{H}(\Omega)} \le \|u_n(t)\|^2_{L^2(\Omega)} + \|f(t)\|^2_{L^2(\Omega)}.$$

An application of the Gronwall lemma to this last estimate leads to the energy estimate

$$\max_{t\in[0,T]} \|u_n(t)\|_{L^2(\Omega)} + \|u_n\|_{L^2(0,T;\tilde{H}(\Omega))} \le C\Big(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega\times(0,T))}\Big):$$

therefore, by following the same steps contained in the proof of theorem 1, we have

$$u_n \rightharpoonup u \quad weak \quad in \ L^2\big(0,T;\tilde{H}(\Omega)\big)$$
$$u_n \to u \ strong \quad in \ L^2\big(\Omega \times (0,T)\big).$$

Analogously, for the solution v it follows

$$v_n \rightharpoonup v \quad weak \quad in \ L^2\big(0,T;H(\Omega^\#)\big)$$
$$v_n \to v \quad strong \quad in \ L^2\big(\Omega^\# \times (0,T)\big)$$

hence by passing to the limit in (3.4) we obtain (1.3), and theorem 2 is also proved. □

4. A second method to prove the comparison results: the discretization scheme.

The main goal of this section is to determine inequality (1.3) by using the alternative tool of the implicit time discretization scheme. As we explained in the introduction, this method allows us to reduce problem (1.6) (and the symmetrized problem (1.10)) to a sequence of elliptic problems with a zero order term that could present, in this framework, a singularity of the type $1/|x|^2$. For this reason, we use the entire following subsection to show a comparison result for solutions of Dirichlet problems, related to linear elliptic operators having zero order terms that eventually can fall in this singular case.

4.1. A comparison result for a class of linear elliptic operators.

We consider the solution u of the Dirichlet problem

$$\text{(4.1)} \qquad \begin{cases} -(\tilde{a}_{ij}(x)u_{x_i})_{x_j} + [\lambda k + c(x)]u = g & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $k \geq 0$, $0 < \lambda \leq \lambda_N$, $g \in L^2(\Omega)$, $c \in L(N/2, \infty)$ verify the assumptions (1.9) or (1.12) and the coefficients $\tilde{a}_{ij}$ verify

$$\text{(4.2)} \qquad \tilde{a}_{ij}(x)\xi_i\xi_j \geq |\xi|^2, \quad \text{for a.e. } x \in \Omega \ , \forall \xi \in \mathbb{R}^N.$$

Let v be the solution of the symmetrized problem

$$\text{(4.3)} \qquad \begin{cases} -\Delta v + \lambda\Big[k - \dfrac{1}{|x|^2}\Big]v = h & \text{in } \Omega^\# \\ v = 0 & \text{on } \partial\Omega^\#, \end{cases}$$

where $h \in L^2(\Omega^\#)$ is such that

$$\int_0^s g^*(\sigma)d\sigma \leq \int_0^s h^*(\sigma)d\sigma,$$

i.e. $g \prec h$ *in the sense of rearrangements*. Then, we prove the following two lemmas:

Lemma 1. *Let $\lambda < \lambda_N$ and assume that (4.2), (1.9) hold. Let $u \in H_0^1(\Omega)$ and $v \in H_0^1(\Omega^\#)$ be the weak solutions of problem (4.1),(4.3) respectively. Then*

$$\int_0^s u^*(\sigma)d\sigma \le \int_0^s v^*(\sigma)d\sigma, \ \ \forall s \in [0, |\Omega|]. \tag{4.4}$$

Lemma 2. *Let $\lambda = \lambda_N$ and assume that (4.2), (1.12) hold. Let $u \in \tilde{H}(\Omega)$ and $v \in H(\Omega^\#)$ be the weak solutions of problem (4.1), (4.3) respectively. Then inequality (4.4) still holds.*

Proof. of lemma 1. As in [1], [3], [21], [22], we derive from (4.1) the following integro-differential inequality:

$$-p(s)u^{*\prime}(s) + \lambda \int_0^s \left[k - \left(\frac{\omega_N}{\sigma}\right)^{2/N}\right] u^*(\sigma)d\sigma \le \int_0^s g^*(\sigma)d\sigma, \tag{4.5}$$

a.e. in $[0, |\Omega|]$, where $p(s) := N^2\omega_N^{2/N}s^{2-(2/N)}$. On the other side, as for the solution v we obtain

$$-p(s)v^{*\prime}(s) + \lambda \int_0^s \left[k - \left(\frac{\omega_N}{\sigma}\right)^{2/N}\right] v^*(\sigma)d\sigma = \int_0^s h^*(\sigma)d\sigma,$$

so by setting

$$w := u^* - v^*$$

we find

$$-p(s)w'(s) + \lambda \int_0^s \left[k - \left(\frac{\omega_N}{\sigma}\right)^{2/N}\right] w(\sigma)d\sigma \le 0. \tag{4.6}$$

Now we follow an approach that is quite similar to [3].
We first consider the case $k = 0$. Then we have

$$-p(s)w'(s) - \lambda \int_0^s \left(\frac{\omega_N}{\sigma}\right)^{2/N} w(\sigma)d\sigma \le 0. \tag{4.7}$$

Set

$$W(s) := \int_0^s \left(\frac{\omega_N}{\sigma}\right)^{2/N} w(\sigma)d\sigma,$$

and suppose that

$$W^+ \not\equiv 0.$$

Multiplying inequality (4.7) by W^+ and integrating on $(0,|\Omega|)$ we have

$$-\int_0^{|\Omega|} w'(s)W^+(s)ds \le \lambda \int_0^{|\Omega|} \frac{1}{p(s)} W\ W^+ds.$$

An integration by parts gives

$$\int_{W>0} \left(\frac{\omega_N}{s}\right)^{2/N} w^2(s)ds \le \lambda \int_{W>0} \frac{1}{p(s)} W^2(s)ds,$$

i.e.

$$\int_{W>0} \left(\frac{s}{\omega_N}\right)^{2/N} (W')^2(s)ds \le \lambda \int_{W>0} \frac{1}{p(s)} W^2(s)ds. \tag{4.8}$$

Since

$$\frac{1}{\lambda_N} = \sup\left\{ \frac{\int_0^{|\Omega|} \frac{1}{p(s)} V^2 ds}{\int_0^{|\Omega|} \left(\frac{s}{\omega_N}\right)^{2/N} (V')^2 ds} : V \in H^1(0,|\Omega|),\, V(0) = V'(|\Omega|) = 0 \right\}$$

the last inequality gives $\lambda \ge \lambda_N$, that is a contradiction. Hence

$$W^+ \equiv 0.$$

By (4.7) we find $w' \ge 0$, and since $w\big(|\Omega|\big) = 0$ we have

$$w(s) \le 0$$

or, that is the same,

$$u^*(s) \le v^*(s),\ \ \forall s \in [0,|\Omega|].$$

Note that this pointwise estimate is *stronger* than inequality (4.4).
If $k > 0$ we distinguish two different cases.

a) $\omega_N/k^{N/2} \ge |\Omega|$. In this case, set

$$W(s) := \int_0^s \left[\left(\frac{\omega_N}{\sigma}\right)^{2/N} - k\right] w(\sigma)d\sigma.$$

and assume

$$W^+ \not\equiv 0.$$

Proceeding as in the previous case, we derive

$$\int_{W>0} \left[\left(\frac{\omega_N}{s}\right)^{2/N} - k\right] w^2(s)ds \le \lambda \int_{W>0} \frac{1}{p(s)} W^2(s)ds.$$

Since $0 < [(\omega_N/s)^{2/N} - k] < (\omega_N/s)^{2/N}$ for all $s \in (0, |\Omega|)$, we find

$$\int_{W>0} \left(\frac{s}{\omega_N}\right)^{2/N} \left[\left(\frac{\omega_N}{s}\right)^{2/N} - k\right]^2 w^2(s)ds \leq \lambda \int_{W>0} \frac{1}{p(s)} W^2(s)ds,$$

that can be written as

$$\int_{W>0} \left(\frac{s}{\omega_N}\right)^{2/N} \left(W'\right)^2(s)ds \leq \lambda \int_{W>0} \frac{1}{p(s)} W^2(s)ds,$$

that is inequality (4.8), so we reach again a contradiction. We conclude as in the case $k = 0$ and get

$$u^*(s) \leq v^*(s), \ \ \forall s \in [0, |\Omega|].$$

b) $\omega_N/k^{N/2} < |\Omega|$.

Set

$$W(s) := \int_0^s \left| k - \left(\frac{\omega_N}{\sigma}\right)^{2/N} \right| w(\sigma)d\sigma \ , \ \ \forall s \in [0, |\Omega|]. \tag{4.9}$$

Notice that

$$k - \left(\frac{\omega_N}{s}\right)^{2/N} \geq 0 \Longleftrightarrow s \geq \frac{\omega_N}{k^{N/2}},$$

hence by (4.6) it follows that for $s \in [0, \omega_n/k^{N/2}]$ we have

$$-w'(s) - \frac{\lambda}{p(s)} W(s) \leq 0 \tag{4.10}$$

and for $s \in [\omega_N/k^{N/2}, |\Omega|]$,

$$-w'(s) - \frac{\lambda}{p(s)} W\Big(\omega_N/k^{N/2}\Big) + \tag{4.11}$$

$$+\frac{\lambda}{p(s)} \int_{\omega_N/k^{N/2}}^{s} \left[k - \left(\frac{\omega_N}{\sigma}\right)^{2/N}\right] w(\sigma)d\sigma \leq 0.$$

Now we prove that

$$W(s) \leq 0.$$

Suppose that

$$W(\omega_N/k^{N/2}) \leq 0 \tag{4.12}$$

and

$$W^+ \not\equiv 0.$$

Since $W(0) = W'(|\Omega|) = 0$, we can find an interval $[a, b] \subseteq [0, |\Omega|]$ such that $W(a) = W'(b) = 0$ and $W > 0$ in $(a, b]$. By (4.12), two different cases are possible: $[a, b] \subseteq \left[0, \omega_N / k^{N/2}\right]$; $[a, b] \subseteq \left[\omega_N / k^{N/2}, |\Omega|\right]$.

If $[a, b] \subseteq \left[0, \omega_N / k^{N/2}\right]$, we get a a contradiction proceeding exactly as in the case (a), replacing the interval $[0, |\Omega|]$ by the interval $[0, b]$.

Otherwise, if $[a, b] \subseteq \left[\omega_N / k^{N/2}, \left|\Omega\right|\right]$, from (4.12) and (4.11), we have that for a.e. $s \in \left[\omega_N / k^{N/2}, |\Omega|\right]$

$$-w'(s) \leq -\frac{\lambda}{p(s)} V(s), \tag{4.13}$$

where

$$V(s) = \int_{\omega_N / k^{N/2}}^{s} \left[k - \left(\frac{\omega_N}{\sigma}\right)^{2/N}\right] w(\sigma) d\sigma. \tag{4.14}$$

By multiplying both sides of (4.13) by V^+ and integrating by parts on $\left[\omega_N / k^{N/2}, b\right]$ we obtain

$$-\left[w(s)V^+(s)\right]_{\omega_N / k^{N/2}}^{b} + \int_{\omega_N / k^{N/2}}^{b} w(s) V^{+\prime}(s) ds$$

$$\leq -\lambda \int_{\omega_N / k^{N/2}}^{b} \frac{1}{p(s)} V(s) V^+(s) ds.$$

Since $w(b) = 0$ and $V\left(\omega_N / k^{N/2}\right) = 0$, we have

$$\int_{[\omega_N / k^{N/2}, b] \cap \{V > 0\}} \left[k - \left(\frac{\omega_N}{s}\right)^{2/N}\right] w^2(s) ds \leq -\lambda \int_{[\omega_N / k^{N/2}, b] \cap \{V > 0\}} \frac{V^2(s)}{p(s)} ds$$

then

$$\int_{[\omega_N / k^{N/2}, b] \cap \{V > 0\}} \frac{V^2(s)}{p(s)} ds = 0$$

i.e.

$$\int_{\omega_N / k^{N/2}}^{b} \frac{\left(V^+\right)^2(s)}{p(s)} ds = 0.$$

Therefore

$$V^+ \equiv 0 \quad \text{in } \left[\omega_N / k^{N/2}, b\right]. \tag{4.15}$$

This implies that if $s \in (a, b]$ we get

$$W(s) = W\left(\omega_N / k^{N/2}\right) + V(s) \leq 0,$$

that contradicts the fact that $W > 0$ in $(a, b]$.

Now we must prove that (4.4) holds. If we multiply again (4.13) by V^+ and integrate by parts on $[b, |\Omega|]$, we also find

$$V \le 0 \text{ in } \Big[\omega_N/k^{N/2}, |\Omega|\Big] \tag{4.16}$$

hence for any $s \in [\omega_N/k^{N/2}, |\Omega|]$

$$\int_{\omega_N/k^{N/2}}^{s} \Big[k - (\frac{\omega_N}{\sigma})^{2/N}\Big] u^*(\sigma)d\sigma \le \int_{\omega_N/k^{N/2}}^{s} \Big[k - (\frac{\omega_N}{\sigma})^{2/N}\Big] v^*(\sigma)d\sigma.$$

Since $k - (\omega_N/\sigma)^{2/N}$ is decreasing, a known property of rearrangements implies that

$$\int_{\omega_N/k^{N/2}}^{s} u^*(\sigma)d\sigma \le \int_{\omega_N/k^{N/2}}^{s} v^*(\sigma)d\sigma \ , \forall s \in \Big[\omega_N/k^{N/2}, |\Omega|\Big]. \tag{4.17}$$

Observe that from this last inequality we find

$$w\Big(\omega_N/k^{N/2}\Big) \le 0. \tag{4.18}$$

But from (4.10), since $W(s) \le 0$, it follows that for $s \in \big[0, \omega_N/k^{N/2}\big]$

$$-w'(s) \le 0.$$

Therefore, by (4.18) we obtain

$$w(s) \le 0 \quad \text{for} \quad \text{a.e.} \quad s \in \Big[0, \omega_N/k^{N/2}\Big],$$

that is

$$u^*(s) \le v^*(s) \quad \forall s \in \Big[0, \omega_N/k^{N/2}\Big]. \tag{4.19}$$

By (4.17) and (4.19) we deduce that for all $s \in \big[\omega_N/k^{N/2}, |\Omega|\big]$

$$\begin{aligned} \int_0^s u^* d\sigma &= \int_0^{\omega_N/k^{N/2}} u^* d\sigma + \int_{\omega_N/k^{N/2}}^{s} u^* d\sigma \\ &\le \int_0^{\omega_N/k^{N/2}} v^* d\sigma + \int_{\omega_N/k^{N/2}}^{s} v^* d\sigma \\ &= \int_0^s v^* d\sigma. \end{aligned}$$

In conclusion we find that inequality

$$\int_0^s u^* d\sigma \le \int_0^s v^* d\sigma \tag{4.20}$$

holds for *all* $s \in [0, |\Omega|]$.

We finally prove that the condition (4.12) is true: in fact, if $W\big(\omega_N/k^{N/2}\big) > 0$, we notice that $W'\big(\omega_N/k^{N/2}\big) = 0$, hence there would exist an interval $[a', b'] \subseteq \big[0, \omega_N/k^{N/2}\big]$ such that $W(a') = W'(b') = 0$ e $W > 0$ in $(a', b']$, so the same arguments used in the case $[a, b] \subseteq [0, \omega_N/k^{N/2}]$ can be applied. □

Proof. of lemma 2. The main tool we used in the proof of lemma 1 was essentially the Hardy inequality, which allows us to get a contradiction if $\lambda < \lambda_N$. In the critical case $\lambda = \lambda_N$ the appropriate tool is the improved Hardy inequality, which in one dimension reads as follows: for any $1 \le q < 2$ there exists a constant $C = C(q, \Omega) > 0$ such that

$$\int_0^{|\Omega|} \left[\left(\frac{s}{\omega_N}\right)^{2/N} V'^2(s)ds - \lambda_N \frac{V^2(s)}{p(s)} \right] ds \geq C\left(\int_0^{|\Omega|} \left(\frac{s}{\omega_N}\right)^{2/N} |V'(s)|^q ds \right)^{2/q}, \tag{4.21}$$

holds for *all* $V \in H^1(0, |\Omega|)$, $V(0) = 0$. Proceeding as in the proof of lemma 1, we get the following integro-differential inequality

$$-w'(s) + \frac{\lambda_N}{p(s)} \int_0^s \Big[k - \big(\frac{\omega_N}{\sigma}\big)^{2/N} \Big] w(\sigma) d\sigma \le 0. \tag{4.22}$$

Once obtained (4.22), the proof follows the same steps we used in the proof of lemma 1. For this reason, we only discuss, without going into details, the case $k = 0$, in order to explain where (4.21) is used. As in the proof of lemma 1, set

$$W(s) := \int_0^s \big(\frac{\omega_N}{\sigma}\big)^{2/N} w(\sigma) d\sigma,$$

and assume

$$W^+ \not\equiv 0.$$

By (4.22) we get

$$\int_{W>0} \left(\frac{s}{\omega_N}\right)^{2/N} \big(W'\big)^2(s)ds - \lambda_N \int_{W>0} \frac{1}{p(s)} W^2(s)ds \le 0.$$

Since $W(0) = 0$, using inequality (4.21) we find

$$\left(\int_0^{|\Omega|} \left(\frac{s}{\omega_N}\right)^{2/N} \big|\big(W^+\big)'(s)\big|^q ds \right)^{2/q} \le 0$$

hence $\left(W^{+}\right)' = 0$, which gives the contradiction $W^{+} \equiv 0$.

Once proved that $W(s) \leq 0$ for all $s \in [0, |\Omega|]$, we conclude as in the proof of lemma 1 that

$$u^{*}(s) \leq v^{*}(s), \quad \forall s \in [0, |\Omega|]. \qquad \square$$

4.2. Proofs of theorem 1 and theorem 2 by time discretization scheme.

Now we prove theorem 1 and 2 by making use of the implicit time discretization scheme. In order to do that, we divide the interval $[0, T]$ by a decomposition $0 = t_0 < t_1 < \ldots < t_n = T$ of lenght

$$t_{r+1} - t_r = \frac{T}{n}\ , \quad \forall r = 0, \ldots, n-1.$$

We approximate the solutions u and v of problems (1.6) and (1.10) by the sequences

$$u_n(x,t) := u^{(r)}(x,t) \quad x \in \Omega,\ t \in \left[\frac{(r-1)T}{n}, \frac{rT}{n}\right]$$

$$v_n(x,t) := v^{(r)}(x,t) \quad x \in \Omega^{\#},\ t \in \left[\frac{(r-1)T}{n}, \frac{rT}{n}\right],$$

where $u^{(r)}$ is the solution of

$$\text{(4.23)} \qquad \begin{cases} \left(u^{(r)} - u^{(r-1)}\right)\dfrac{n}{T} - \left(a_{ij}^{(r)}(x)u_{x_i}^{(r)}\right)_{x_j} + cu^{(r)} = f^{(r)} & \text{in } \Omega \\ u^{(r)} = 0 & \text{on } \partial\Omega, \end{cases}$$

$$a_{ij}^{(r)}(x) := \begin{cases} \dfrac{n}{T}\displaystyle\int_{\frac{(r-1)T}{n}}^{\frac{rT}{n}} a_{ij}(x,t)dt, & \text{if } \lambda < \lambda_N, \\ a_{ij}(x), & \text{if } \lambda = \lambda_N\ , \end{cases}$$

$$f^{(r)}(x) := \frac{n}{T}\int_{\frac{(r-1)T}{n}}^{\frac{rT}{n}} f(x,t)dt$$

for any $r = 1, \ldots, n$ and $u^{(0)} := u_0$, while $v^{(r)}$ is the solution of the symmetrized problem with

$$\text{(4.24)} \qquad \begin{cases} \left(v^{(r)} - v^{(r-1)}\right)\dfrac{n}{T} - \Delta v^{(r)} - \dfrac{\lambda}{|x|^2}v^{(r)} = f^{(r)\#} & \text{in } \Omega^{\#} \\ v^{(r)} = 0 & \text{on } \partial\Omega^{\#}, \end{cases}$$

where $v^{(0)} := u_0^\#$. We prove by induction that

$$\int_0^s u^{(r)*} d\sigma \le \int_0^s v^{(r)*} d\sigma \quad \forall s \in [0, |\Omega|]. \tag{4.25}$$

Rewrite problem (4.23) as

$$\begin{cases} -\left(a_{ij}^{(r)}(x) u_{x_i}^{(r)}\right)_{x_j} + [\lambda k_n + c(x)] u^{(r)} = f^{(r)} + \dfrac{n}{T} u^{(r-1)} & \text{in} \quad \Omega \\ u^{(k)} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.26}$$

where $k_n = (n/(\lambda T))$. For $r = 1$ we have the problem

$$\begin{cases} -\left(a_{ij}^{(1)}(x) u_{x_i}^{(1)}\right)_{x_j} + [\lambda k_1 + c(x)] u^{(1)} = f^{(1)} + \dfrac{n}{T} u_0 & \text{in} \quad \Omega \\ u^{(1)} = 0 & \text{on} \quad \partial\Omega; \end{cases} \tag{4.27}$$

observe that

$$\int_0^s \left(f^{(1)} + \frac{n}{T} u_0 \right)^* d\sigma \le \int_0^s \left(f^{(1)*} + \frac{n}{T} u_0^* \right) d\sigma$$

while

$$\int_0^s \left(f^{(1)\#} + \frac{n}{T} u_0^\# \right)^* d\sigma = \int_0^s \left(f^{(1)*} + \frac{n}{T} u_0^* \right) d\sigma;$$

therefore, by lemma 1 and lemma 2 we can compare the solution $u^{(1)}$ of problem (4.27) with the solution $v^{(1)}$ of the radial problem

$$\begin{cases} -\Delta v^{(1)} + \lambda \left[k_1 - \dfrac{1}{|x|^2} \right] v^{(1)} = f^{(1)\#} + \dfrac{n}{T} u_0^\# & \text{in} \quad \Omega^\# \\ v^{(1)} = 0 & \text{on} \quad \partial\Omega^\#, \end{cases}$$

that is problem (4.24) for $r = 1$. Namely, we get the estimate

$$\int_0^s u^{(1)*} d\sigma \le \int_0^s v^{(1)*} d\sigma \ , \ \forall s \in [0, |\Omega|].$$

Then, if we suppose $u^{(r-1)} \prec v^{(r-1)}$, that is

$$\int_0^s u^{(r-1)*} d\sigma \le \int_0^s v^{(r-1)*} d\sigma,$$

we have

$$\int_0^s \left(f^{(r)} + \frac{n}{T} u^{(r-1)} \right)^* d\sigma \le \int_0^s \left(f^{(r)\#} + \frac{n}{T} v^{(r-1)} \right)^* d\sigma,$$

hence applying again lemma 1 and 2 it follows that

$$\int_0^s u^{(r)*} d\sigma \leq \int_0^s v^{(r)*} d\sigma \quad \forall s \in [0, |\Omega|]$$

Finally the conclusion follows letting n goes to infinity. □

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