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COMPARISON TECHNIQUES FOR CERTAIN OVERDAMPED HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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1. **Introduction.** We use comparison techniques to obtain sharp error bounds for approximate solutions of the linear hyperbolic partial differential equation

$$\epsilon(U_{xx} - U_{tt}) + A(x, t, \epsilon)U_x - B(x, t, \epsilon)U_t - C(x, t, \epsilon)U = -F(x, t, \epsilon) \tag{1.1}$$

for small values of the positive parameter ϵ, subject to various initial and/or boundary conditions of interest. The functions A, B, C and F are assumed to be given continuous functions of the three real variables x, t and ϵ for all small nonnegative values of ϵ and for all points (x, t) in a suitable domain D contained in the (x, t)-plane. In addition, the functions A and B are assumed to be differentiable with respect to x and t, and the resulting partial derivatives A_x, A_t, B_x, and B_t are assumed to be bounded on (compact subsets of) D, uniformly for all small nonnegative values of ϵ. The domain D varies from problem to problem depending on the type of boundary and/or initial conditions employed.

The equation (1.1) represents a mathematical model for certain overdamped vibration problems such as the motion of a vibrating string imbedded in a highly viscous medium, the propagation of radiation or gas through a highly absorbing medium, and the propagation of electrical signals along a conducting wire of large resistance or loss. For example, the Heaviside telegraph equation can be written in the form (1.1) with

$$\epsilon = \sqrt{\ell c}/(gr),\ A = 0,\ B = (c/g) + (\ell/r), \text{ and } C = \sqrt{\ell c}, \tag{1.2}$$

where the constants ℓ, c, g and r are the usual coefficients of inductance, capacitance, loss, and resistance, and where in this case the original (physical) time variable τ has been replaced with $t = \tau/\sqrt{\ell c}$.

In all of these applications the parameter ϵ can be taken to be inversely proportional to some suitable coefficient of damping, resistance, absorption, or loss, and the following inequalities always hold,

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$$
\begin{aligned}
& B(x,t,\epsilon)^2 - A(x,t,\epsilon)^2 \\
& \qquad \geqq 2\epsilon[2C(x,t,\epsilon) + A_x(x,t,\epsilon) \\
& \qquad\qquad + B_x(x,t,\epsilon) - A_t(x,t,\epsilon) - B_t(x,t,\epsilon)]
\end{aligned} \tag{1.3}
$$

and

$$
\begin{aligned}
& B(x,t,\epsilon)^2 - A(x,t,\epsilon)^2 \\
& \qquad \geqq 2\epsilon[2C(x,t,\epsilon) + A_x(x,t,\epsilon) \\
& \qquad\qquad - B_x(x,t,\epsilon) + A_t(x,t,\epsilon) - B_t(x,t,\epsilon)]
\end{aligned} \tag{1.4}
$$

for all points (x, t) in D and for all sufficiently small nonnegative values of ϵ. From a physical standpoint the conditions (1.3) and (1.4) guarantee that the equation (1.1) is *overdamped*, whereas from a mathematical standpoint these conditions imply that the two Laplace invariants of (1.1) are nonnegative (see Smith and Weinstein [4]). We shall assume that the conditions (1.3) and (1.4) hold (for small ϵ). For the telegraph equation it follows with (1.2) that these conditions hold for *all* nonnegative values of ϵ, not just for small values.

We also assume that the coefficient B of the differential equation is positive and dominates the coefficient A in the sense that there hold

$$B(x,t,\epsilon) \geqq |A(x,t,\epsilon)| \tag{1.5}$$

and

$$B(x,t,\epsilon) \geqq \kappa > 0 \tag{1.6}$$

for some suitable positive constant κ (which may depend on ϵ). The condition (1.5) is a stability condition (for $t \to \infty$ or $\epsilon \to 0$), whereas the condition (1.6) serves to rule out certain "turning point" behavior. The conditions (1.5) and (1.6) actually hold in the usual applications mentioned above.

In § 2 we study the Cauchy problem for (1.1), and in § 3 we study a one-point boundary-initial value problem which has application in various signaling and radiation problems. In both sections we show how to use elementary comparison techniques to study the validity of asymptotic approximations to the solutions for small ϵ.

2. **A Singularly Perturbed Cauchy Problem.** In this section we study equation (1.1) for small ϵ subject to specified Cauchy data given along a specified segment of the initial line $t = 0$. We take this initial segment to be given as

$$\Gamma = \{(x,t) : t = 0, x_0 < x < x_1\} \tag{2.1}$$

for arbitrary given numbers x_0 and x_1 $(x_0 < x_1)$, and we let D be the region of exclusive influence of Γ in the upper half-plane for (1.1) with

$$D = \{(x, t) : x_0 < x_0 + t < x < x_1 - t < x_1\}. \tag{2.2}$$

Either or both of x_0 and x_1 may be finite or infinite, provided that $x_0 < x_1$, and provided also that D is interpreted as being the appropriate unbounded region in the upper half-plane if x_0 or x_1 is infinite.

Along with the equation (1.1) we impose on the solution function $U(x, t, \epsilon)$ the initial conditions

$$\begin{aligned} &U(x, 0, \epsilon) = G(x, \epsilon), \\ &U_t(x, 0, \epsilon) = H(x, \epsilon) \text{ for } x_0 \leqq x \leqq x_1, \end{aligned} \tag{2.3}$$

where G and H are given continuous functions of x and ϵ for all $x_0 < x < x_1$ and for all small nonnegative values of ϵ, with G of class C^2 and H of class C^1 with respect to x. [These regularity conditions can be weakened in a study of weak solutions.]

The problem (1.1), (2.3) can be solved explicitly in terms of an appropriate Volterra resolvent kernel (see, for example, Smith and Weinstein [4]), and the resulting solution function can be studied directly as $\epsilon \to 0$. This approach is particularly feasible in the special case in which the given coefficients A, B and C are *constants* since in that case the Volterra resolvent kernel can be given in terms of elementary functions such as the exponential function and certain Bessel functions. Integral transform techniques can also be conveniently used to study the solution function as $\epsilon \to 0$ in the constant coefficient case (cf. Whitham [6]).

An alternative approach is to use direct perturbation techniques such as matching techniques or multivariable techniques to obtain a suitable approximation to the solution for small values of ϵ. The use of matching techniques is illustrated for a special, constant coefficient problem of the type (1.1), (2.3) by Cole [1, pp. 133–136] without proof, while the multivariable approach is illustrated by Smith [3], where a Gronwall type argument is used in the proof.

In all of these approaches one seeks a suitable, easily interpretable function $V = V(x, t, \epsilon)$ which is expected to provide a suitable approximation to the exact solution U for small values of ϵ. The function V is always chosen so as to satisfy (1.1) and (2.3) *approximately*, with

$$\begin{aligned} &\epsilon(V_{xx} - V_{tt}) + AV_x - BV_t - CV = -F(x, t, \epsilon) + \delta F(x, t, \epsilon), \\ &\qquad\qquad \text{for } (x, t) \text{ in } D, \end{aligned} \tag{2.4}$$

and

$$
(2.5) \qquad \begin{aligned} V(x, 0, \epsilon) &= G(x, \epsilon) - \delta G(x, \epsilon), \\ V_t(x, 0, \epsilon) &= H(x, \epsilon) - \delta H(x, \epsilon) \text{ for } x_0 < x < x_1, \end{aligned}
$$

for suitable functions δF, δG, and δH which are known to be small for small ϵ. In order to verify the validity of any such expected approximation, one must obtain a suitable estimate for the *error term* $E = E(x, t, \epsilon)$, where E is defined as the difference of the exact solution and the proposed approximate solution,

$$
(2.6) \qquad E(x, t, \epsilon) = U(x, t, \epsilon) - V(x, t, \epsilon),
$$

and where it then follows from (1.1), (2.3), (2.4), (2.5), and (2.6) that E is a solution of the related Cauchy problem

$$
(2.7) \qquad \begin{aligned} \epsilon(E_{xx} - E_{tt}) + A(x, t, \epsilon)E_x - B(x, t, \epsilon)E_t - C(x, t, \epsilon)E \\ = -\delta F(x, t, \epsilon) \text{ for } (x, t) \text{ in } D, \end{aligned}
$$

and

$$
(2.8) \qquad \begin{aligned} E(x, 0, \epsilon) &= \delta G(x, \epsilon), \\ E_t(x, 0, \epsilon) &= \delta H(x, \epsilon) \text{ for } x_0 < x < x_1. \end{aligned}
$$

Hence one is led to seek a suitable estimate for E directly from (2.7) and (2.8) in terms of the known quantities $\delta F, \delta G$, and δH. For example, one often has estimates of the following type which are known to hold for δF, δG, and δH as a consequence of the method of construction of V,

$$
(2.9) \qquad \delta F(x, t, \epsilon) = 0(\epsilon^{\nu}) \text{ uniformly for } (x, t) \text{ in } D,
$$

and

$$
(2.10) \qquad \begin{aligned} &\delta G(x, \epsilon) = 0(\epsilon^{\nu}),\ \partial[\delta G(x, \epsilon)]/\partial x = 0(\epsilon^{\nu - 1}), \text{ and} \\ &\delta H(x, \epsilon) = 0(\epsilon^{\nu - 1}) \text{ uniformly for } x_0 < x < x_1, \end{aligned}
$$

for some suitable, given positive number ν, and one then wishes to prove that E satisfies some suitable corresponding estimate such as

$$
(2.11) \qquad \begin{aligned} &E(x, t, \epsilon) = 0(\epsilon^{\nu}),\ E_x(x, t, \epsilon) = 0(\epsilon^{\nu - 1}), \text{ and} \\ &E_t(x, t, \epsilon) = 0(\epsilon^{\nu - 1}) \text{ uniformly on } D \end{aligned}
$$

as $\epsilon \to 0$. We shall use the following differential inequality to obtain quantitative, sharp estimates of this type which are best possible.

THEOREM 2.1. *Let $U(x, t, \epsilon)$ satisfy the equation* (1.1) *in the region D of* (2.2), *and assume that the conditions* (1.3), (1.4), *and* (1.5) *hold in D. Furthermore assume that the parameter ϵ is positive and that the* *forcing term F is nonnegative in D, and assume that the initial data satisfy the conditions $U \geqq 0$ and $|U_x| \leqq U_t$ for $t = 0$, $x_0 \leqq x \leqq x_1$. Then everywhere in D there hold $U \geqq 0$ and $|U_x + (AU)/(2\epsilon)| \leqq U_t + (BU)/(2\epsilon)$.* [*These same conclusions hold in D without the condition* (1.5) *if the initial data satisfy the modified conditions $U \geqq 0$ and $|U_x + (AU)/(2\epsilon)| \leqq U_t + (BU)/(2\epsilon)$ for $t = 0$, $x_0 \leqq x \leqq x_1$.*]

PROOF. We transform the independent variables from x, t to $r = t + x$, $s = t - x$, and then the function $u(r, s) = u(r, s, \epsilon)$ defined as $u(r, s) = U((r - s)/2, (r + s)/2, \epsilon)$ is seen to satisfy the conditions of Theorem 3.1 of Smith and Weinstein [4]. The stated results then follow directly from this referenced Theorem 3.1. We omit the details.

Protter and Weinberger [2; pp. 195–207] have used different methods to obtain certain related, though different, results for the Cauchy problem which only yield an inequality for U. The fact that our Theorem 2.1 yields inequalities for both U and the first derivatives of U enables us to obtain the following estimates for the solution E of the Cauchy problem (2.7), (2.8).

THEOREM 2.2. *Let $E(x, t, \epsilon)$ satisfy* (2.7) *and* (2.8), *and assume that* (1.3), (1.4), (1.5), *and* (1.6) *hold in D. Then E satisfies the inequalities*

$$\begin{gathered} |E(x, t, \epsilon)| \leqq W(t, \epsilon) \text{ and} \\ |E_x(x, t, \epsilon)|, |E_t(x, t, \epsilon)| \leqq W'(t, \epsilon) + \frac{1}{\epsilon} B(x, t, \epsilon) W(t, \epsilon) \end{gathered} \tag{2.12}$$

everywhere in D, where $W'(t, \epsilon) = \partial W(t, \epsilon)/\partial t$ and

$$\begin{aligned} W(t, \epsilon) = & \\ = & \|\delta G\| \frac{\rho_1 e^{\rho_2 t/\epsilon} - \rho_2 e^{\rho_1 t/\epsilon}}{\rho_1 - \rho_2} \\ & + \epsilon(\|(\delta G)_x\| + \|\delta H\|) \frac{e^{\rho_1 t/\epsilon} - e^{\rho_2 t/\epsilon}}{\rho_1 - \rho_2} \\ & + \int_0^t \left(\frac{e^{\rho_1 (t-\tau)/\epsilon} - e^{\rho_2 (t-\tau)/\epsilon}}{\rho_1 - \rho_2} \right) \|\delta F\|_\tau \, d\tau \end{aligned} \tag{2.13}$$

with

$$\rho_1 = (-\kappa + \sqrt{\kappa^2 + 4\epsilon \|C\|})/2 \tag{2.14}$$

and

$$\rho_2 = (-\kappa - \sqrt{\kappa^2 + 4\epsilon\|C\|}\,)/2,$$

where $\|C\|$ *denotes a positive upper bound on the absolute value of* C *in* D, *and*

$$(2.15)\qquad \begin{aligned} \|\delta G\| &= \max_{x_0 \leqq x \leqq x_1} |\delta G(x,\epsilon)|, \\ \|(\delta G)_x\| &= \max_{x_0 \leqq x \leqq x_1} |\partial(\delta G(x,\epsilon))/\partial x|, \\ \|\delta H\| &= \max_{x_0 \leqq x \leqq x_1} |\delta H(x,\epsilon)|, \text{ and} \\ \|\delta F\|_\tau &= \max_{x_0+\tau \leqq x \leqq x_1-\tau} |\delta F(x,\tau,\epsilon)|. \end{aligned}$$

PROOF. The function $W(t)$ given by (2.13) is the unique solution of the initial value problem

$$(2.16)\qquad \begin{aligned} \epsilon W'' + \kappa W' - \|C\|W &= \|\delta F\|_t \text{ for } 0 < t < (x_1 - x_0)/2, \\ W(0) = \|\delta G\|,\ W'(0) &= \|(\delta G)_x\| + \|\delta H\|, \end{aligned}$$

and W is easily seen to satisfy the inequalities

$$(2.17)\qquad W(t) \geqq 0 \text{ and } W'(t) \geqq 0 \text{ for } 0 \leqq t < (x_1 - x_0)/2.$$

If we put $U = -E + W$ and use (2.7), (2.8), (2.16), (2.17), (1.3), (1.4), (1.5) and (1.6), we find that U satisfies the conditions of Theorem 2.1 with $F = -\delta F + \|\delta F\|_t + (B - \kappa)W' + (\|C\| + C)W \geqq 0$, from which we find the results

$$(2.18)\qquad \begin{aligned} &E(x,t,\epsilon) \leqq W(t,\epsilon) \text{ and} \\ &|-E_x + A(-E + W)/(2\epsilon)| \leqq -E_t + W' + B(-E + W)/(2\epsilon) \end{aligned}$$

everywhere in D. Similarly, if we put $U = +E + W$, we find that this function $E + W$ also satisfies the conditions of Theorem 2.1 (with $F = +\delta F + \|\delta F\|_t + (B - \kappa)W' + (\|C\| + C)W \geqq 0$), from which we conclude the results

$$(2.19)\qquad \begin{aligned} &-E(x,t,\epsilon) \leqq W(t,\epsilon) \text{ and} \\ &|E_x + A(E + W)/(2\epsilon)| \leqq E_t + W' + B(E + W)/(2\epsilon) \end{aligned}$$

everywhere in D. The stated results of Theorem 2.2 now follow from (2.18), (2.19), (1.5), and the triangle inequality. We omit the details.

Since $\rho_1/\epsilon = (\|C\|/\kappa) + 0(\epsilon)$ while ρ_2 is negative, one sees that Theorem 2.2 yields directly the desired uniform estimates (2.11) whenever (2.9) and (2.10) hold, provided that κ can be taken to be *constant* as $\epsilon \to 0$.

Theorem 2.2 makes no distinction between the cases $C(x, t, \epsilon) \geqq 0$ and $C(x, t, \epsilon) \leqq 0$. The case $C \geqq 0$ is important in vibration problems, and in this case we can obtain the following improved result.

THEOREM 2.3. *Let the function C which appears in* (2.7) *be nonnegative in D, let E satisfy* (2.7) *and* (2.8), *and assume that* (1.3), (1.4), (1.5), *and* (1.6) *hold in D. Then E satisfies the inequalities* (2.12) *everywhere in D where now W is given in this case as*

$$W(t, \epsilon) = \|\delta G\| + \epsilon(\|(\delta G)_x\| + \|\delta H\|)\frac{1 - e^{-\kappa t/\epsilon}}{\kappa} + \int_0^t \frac{1 - e^{-\kappa(t-\tau)/\epsilon}}{\kappa}\|\delta F\|_\tau \, d\tau \tag{2.20}$$

where $\|\delta G\|$, $\|(\delta G)_x\|$, $\|\delta H\|$, *and* $\|\delta F\|_\tau$ *are again given by* (2.15).

PROOF. The function $W(t)$ given by (2.20) is the unique solution of the initial value problem

$$\epsilon W'' + \kappa W' = \|\delta F\|_t \text{ for } 0 < t < (x_1 - x_0)/2,$$
$$W(0) = \|\delta G\|, \; W'(0) = \|(\delta G)_x\| + \|\delta H\|,$$

and again one sees that W satisfies (2.17). The remaining proof follows along the lines of the previous proof of Theorem 2.2, and we omit the details.

If we consider the Cauchy problem (2.7), (2.8) in the entire upper half-plane with $x_0 = -\infty$, $x_1 = +\infty$, and $D = \{(x, t) : t \geqq 0\}$, and if the data functions δG, δH, and δF are all *globally* bounded along with $(\delta G)_x$, then Theorem 2.3 leads directly to the estimate

$$|E(x, t, \epsilon)| \leqq \|\delta G\| + \epsilon(\|(\delta G)_x\| + \|\delta H\|)/\kappa + \int_0^t \|\delta F\|_\tau \, d\tau/\kappa \tag{2.21}$$

uniformly in the upper half-plane, with related estimates for E_x and E_t. (In this case the weaker estimates of Theorem 2.2 would permit E to grow exponentially with increasing t.)

Finally, if C is uniformly positive with $C(x, t, \epsilon) \geqq \lambda > 0$ everywhere in the upper half-plane for some positive constant λ, then the bound (2.21) can be further improved as

$$|E(x, t, \epsilon)| \leqq \|\delta G\| + \epsilon(\|(\delta G)_x\| + \|\delta H\|)/\kappa + |\|\delta F\||/\lambda \tag{2.22}$$

uniformly in the upper half-plane, with related bounds for E_x and E_t, where $|||F||| = \sup_{\tau \geqq 0} \|\delta F\|_\tau = \sup_{t \geqq 0, -\infty < x < \infty} |\delta F(x, t, \epsilon)|$. The inequality (2.22) provides a global bound on E, whereas (2.21) still permits E to grow linearly with increasing t. The proof of (2.22) makes use of the functions $U_1 = -E + W + |||\delta F|||/\lambda$ and $U_2 = +E + W + |||\delta F|||/\lambda$ where $W(t)$ is the solution of the problem $\epsilon W'' + \kappa W' = 0$ for $t > 0$, $W(0) = \|\delta G\|$, $W'(0) = \|(\delta G)_x\| + \|\delta H\|$. One can show easily that both U_1 and U_2 satisfy the conditions of Theorem 2.1, and the estimate (2.22) follows directly. We omit details.

From (2.22) we obtain in this case the global estimate $|E(x, t, \epsilon)| = 0(\epsilon^\nu)$, uniformly in the upper half-plane whenever there hold the uniform estimates $\delta F = 0(\epsilon^\nu)$, $\delta G = 0(\epsilon^\nu)$, $(\delta G)_x = 0(\epsilon^{\nu-1})$, and $\delta H = 0(\epsilon^{\nu-1})$. These results, as well as those of Theorem 2.3, represent improvements over the results obtained in Smith [3] with the use of the Gronwall argument.

The estimates (2.12), (2.13), (2.20), (2.21), and (2.22) are all sharp in their various cases, as is seen by simple (constant coefficient) examples.

The present approach based on the comparison result of Theorem 2.1 also handles easily certain overdamped linear hyperbolic equations which involve *several* small parameters, as occurs in the Heaviside equation if, for example, both the loss and the resistance are large (see (1.2)). We omit these details here. [For an indication of these results see Weinstein and Smith [5] where such results are discussed for ordinary differential equations.]

3. **A Singularly Perturbed Signaling Problem.** In this section we extend the results of § 2 to the signaling problem for equation (1.1) in the region

$$D = \{(x, t) : 0 \leqq x < x_1, 0 \leqq t < x_1 - x\}, \tag{3.1}$$

where x_1 is an arbitrary given nonnegative number which may be finite or infinite. [If $x_1 = +\infty$, then D is the entire first quadrant in the (x, t)-plane.] Along with the differential equation (1.1) we impose on $U(x, t, \epsilon)$ the initial conditions

$$U(x, 0, \epsilon) = G(x, \epsilon),\ U_t(x, 0, \epsilon) = H(x, \epsilon) \text{ for } 0 \leqq x \leqq x_1,$$

and the boundary condition

$$U(0, t, \epsilon) = K(t, \epsilon) \text{ for } 0 \leqq t \leqq x_1,$$

where G, H, and K are given smooth functions of their arguments.

We assume that we have a proposed approximate solution V which satisfies the equation (2.4) in D, the initial conditions (2.5) for $0 \leqq x \leqq x_1$, and the boundary condition

$$V(0, t, \epsilon) = K(t, \epsilon) - \delta K(t, \epsilon) \text{ for } 0 \leqq t \leqq x_1$$

for suitable functions δF, δG, δH, and δK, and we then seek a suitable estimate on the error term E defined by (2.6). [The actual construction of a suitable approximate solution V is given in the constant coefficient case in unpublished work of M. B. Weinstein. If the coefficient A is nonzero, then only ordinary boundary layers appear in the first approximation, while if A is zero, then both ordinary and parabolic boundary layers appear.]

The error term E satisfies equation (2.7) in D and it satisfies the initial conditions

$$E(x, 0, \epsilon) = \delta G(x, \epsilon),\ E_t(x, 0, \epsilon) = \delta H(x, \epsilon) \text{ for } 0 \leqq x \leqq x_1 \tag{3.2}$$

and the boundary condition

$$E(0, t, \epsilon) = \delta K(t, \epsilon) \text{ for } 0 \leqq t \leqq x_1. \tag{3.3}$$

In the following estimates it will be convenient to write $D = D_1 \cup D_2$ with

$$\begin{aligned} D_1 &= \{(x, t) : 0 \leqq t \leqq x < x_1, t + x < x_1\}, \\ D_2 &= \{(x, t) : 0 \leqq x \leqq t < x_1, t + x < x_1\}. \end{aligned} \tag{3.4}$$

The following Theorem 3.1 will be used along with Theorem 2.1 to obtain estimates for E.

THEOREM 3.1. *Let* $U(x, t, \epsilon)$ *satisfy the equation* (1.1) *in the region* D_2 *of* (3.4), *and assume that the conditions* (1.3), (1.4), *and* (1.5) *hold in* D_2. *Furthermore assume that the parameter* ϵ *is positive and that the forcing term* F *is nonnegative in* D_2, *and assume that* U *satisfies the conditions* $U \geqq 0$ *for* $x = 0,\ 0 \leqq t < x_1$, *and* $U_x + U_t + (A + B)U/(2\epsilon) \geqq 0$ *for* $0 \leqq t = x < x_1/2$. *Then* $U \geqq 0$ *and* $U_x + U_t + (A + B)U/(2\epsilon) \geqq 0$ *everywhere in* D_2. *If in addition* U *satisfies the condition* $-U_x + U_t + (-A + B)U/(2\epsilon) \geqq 0$ *for* $x = 0, 0 \leqq t < x_1$, *then also* $|U_x + (AU)/(2\epsilon)| \leqq U_t + (BU)/(2\epsilon)$ *everywhere in* D_2.

PROOF. The stated results follow from Theorem 4.3 of Smith and Weinstein [4] upon transforming the independent variables to $r = t + x$, $s = t - x$. We omit the details.

We now have the following estimates for E.

THEOREM 3.2. *Let* E *satisfy* (2.7) *in the region* $D = D_1 \cup D_2$ *of* (3.1) *and* (3.4), *let* E *satisfy the initial and boundary conditions of* (3.2) *and* (3.3), *and assume that* (1.3), (1.4), (1.5) *and* (1.6) *hold in* D. *Then* E *satisfies the inequalities* (2.12) *everywhere in* D_1 *with* W *given by* (2.13), (2.14), *and* (2.15) (*with* $x_0 = 0$), *while in* D_2 *there hold*

$$|E(x, t, \epsilon)| \leqq W_1(t, \epsilon) \text{ and}$$
(3.5)
$$|E_x(x, t, \epsilon) + E_t(x, t, \epsilon)| \leqq W_1'(t, \epsilon) + \frac{2}{\epsilon} B(x, t, \epsilon) W_1(t, \epsilon),$$

where $W_1 = W + \delta_1 W$ *with* W *given by* (2.13) *and*

$$\delta_1 W(t, \epsilon) = \|\delta K\| \frac{\rho_1 e^{\rho_2 t/\epsilon} - \rho_2 e^{\rho_1 t/\epsilon}}{\rho_1 - \rho_2} \tag{3.6}$$

with ρ_1 *and* ρ_2 *given by* (2.14), *and* $\|\delta K\| = \max_{0 \leqq t \leqq x_1} |\delta K(t, \epsilon)|$. *Finally, in* D_2 *there also hold*

$$|E_x(x, t, \epsilon)|, |E_t(x, t, \epsilon)|$$
$$\leqq W_2'(t, \epsilon) + \frac{1}{\epsilon} B(x, t, \epsilon) W_2(t, \epsilon) \tag{3.7}$$

where $W_2 = W + \delta_1 W + \delta_2 W$ *with*

$$\delta_2 W(t, \epsilon) = 2\epsilon \|\delta K'\| \frac{e^{\rho_1 t/\epsilon} - e^{\rho_2 t/\epsilon}}{\rho_1 - \rho_2} +$$
(3.8)
$$+ [(4/\kappa)(\max_{0 \leqq t \leqq x_1} B(0, t, \epsilon)) W_1(x_1) + 2\kappa \|\delta K'\| / \|C\|]$$
$$\cdot \frac{\rho_1 e^{\rho_2 t/\epsilon} - \rho_2 e^{\rho_1 t/\epsilon}}{\rho_1 - \rho_2}$$

with $\|\delta K'\| = \max_{0 \leqq t \leqq x_1} |\partial(\delta K(t, \epsilon))/\partial t|$.

PROOF. The inequalities (2.12) in D_1 follow directly from Theorem 2.2. To obtain (3.5) in D_2 we first observe that the function W_1 is the solution of the initial value problem (compare with (2.16))

$$\epsilon W_1'' + \kappa W_1' - \|C\| W_1 = \|\delta F\|_t \text{ for } 0 < t < x_1,$$
(3.9)
$$w_1(0) = \|\delta G\| + \|\delta K\|, W_1'(0) = \|(\delta G)_x\| + \|\delta H\|,$$

with $W_1(t) \geqq 0$ and $W_1'(t) \geqq 0$ for $0 \leqq t < x_1$.

We now put $U = -E + W_1$ and find that U satisfies the conditions of Theorem 2.1 in D_1, from which we conclude that $|U_x + (AU)/(2\epsilon)| \leqq U_t + (BU)/(2\epsilon)$ holds in D_1. This implies in particular the result

$$U_x + U_t + (A + B)U/(2\epsilon) \geqq 0, \text{ for } 0 \leqq t = x < x_1/2. \tag{3.10}$$

Moreover, with (3.3) we also find $U(0, t, \epsilon) = -E(0, t, \epsilon) + W(t, \epsilon) + \delta_1 W(t, \epsilon) \geqq -\delta K(t, \epsilon) + \delta_1 W(0, \epsilon)$ since W is nonnegative and $\delta_1 W$ is monotonic increasing, and then (3.6) leads to the result $U(0, t, \epsilon) \geqq -\delta K(t, \epsilon) + \|\delta K\| \geqq 0$ for $0 \leqq t < x_1$. This last result along with (3.10) and (the first part of) Theorem 3.1 now give the inequalities

$U \geqq 0$ and $U_x + U_t + (A + B)U/(2\epsilon) \geqq 0$ everywhere in D_2, or

$$E \leqq W_1 \text{ and}$$
$$E_x + E_t + (A + B)E/(2\epsilon) \leqq W_1' + (A + B)W_1/(2\epsilon) \tag{3.11}$$

in D_2. Similarly, if we put $U = + E + W_1$, we find

$$-E \leqq W_1 \text{ and}$$
$$-[E_x + E_t + (A + B)E/(2\epsilon)] \leqq W_1' + (A + B)W_1/(2\epsilon)$$

in D_2, and these results with (3.11) imply

$$|E(x, t, \epsilon)| \leqq W_1(t, \epsilon) \text{ and}$$
$$|E_x + E_t + (A + B(E/(2\epsilon)| \leqq W_1' + (A + B)W_1/(2\epsilon) \tag{3.12}$$

in D_2. The stated result (3.5) then follows from (3.12), (1.5), and the triangle inequality.

Finally, to prove the remaining result (3.7) in D_2, we put $U = -E + W_2$ and find as above (for $-E + W_1$) that $-E + W_2 = U$ satisfies the conditions of the first part of Theorem 3.1. Moreover, for $x = 0$ and $0 \leqq t \leqq x_1$, this present function U also satisfies the result $-U_x + U_t + (-A + B)U/(2\epsilon) = [E_x + E_t + (A + B)E/(2\epsilon)] - 2\delta K' - (B\delta K/\epsilon) + W_2' + (-A + B)W_2/(2\epsilon)$, which with (1.5), (3.6), (3.8), (3.12) and the results $W_2 = W_1 + \delta_2 W$, $\delta_2 W \geqq 0$, $W_1 \geqq \delta_1 W \geqq \|\delta K\|$ and $W_1(t) \leqq W_1(x_1)$ (for $0 \leqq t \leqq x_1$), imply $[-U_x + U_t + (-A + B)U/(2\epsilon)] \geqq \delta_2 W'(t) - 2\|\delta K'\|$. Since $\delta_2 W'$ is an increasing function of t, we find with (3.8) the desired result $[-U_x + U_t + (-A + B)U/(2\epsilon)] \geqq 0$ for $x = 0$, $0 \leqq t \leqq x_1$. Hence we can apply the final part of Theorem 3.1 to $U = -E + W_2$, and we find $E_t - E_x + (B - A)E/(2\epsilon) \leqq W_2' + (B - A)W_2/(2\epsilon)$ in D_2. Similarly, if we consider the function $U = +E + W_2$, we find $-E_t + E_x + (B - A)E/(2\epsilon) \leqq W_2' + (B - A)W_2/(2\epsilon)$, and these last two inequalities along with (1.5) and the triangle inequality imply the result $|E_t - E_x| \leqq W_2' + (2BW_2)/\epsilon$ in D_2. The stated result (3.7) then follows from this last result and (3.5).

In the important case in which C is nonnegative we have the following improved result which we state without proof. (The proof follows the pattern of the proofs of Theorems 2.3 and 3.2.)

THEOREM 3.3. *Let the function C which appears in* (2.7) *be nonnegative in D, let E satisfy* (2.7) *in the region $D = D_1 \cup D_2$ of* (3.1) *and* (3.4), *let E satisfy the initial and boundary conditions of* (3.2) *and* (3.3), *and assume that* (1.3), (1.4), (1.5) *and* (1.6) *hold in D. Then E satisfies the inequalities* (2.12) *in D_1 with W given by* (2.20) *and* (2.15) (*with* $x_0 = 0$), *while in* D_2 *the inequalities* (3.5) *hold with* $W_1 = W + \|\delta K\| = W + \max_{0 \leqq t \leqq x_1} |\delta K(t, \epsilon)|$.

If we consider the above signaling problem in the entire first quadrant with $x_1 = +\infty$, and if the functions δG, δH, δK, $(\delta G)_x$, and δF are globally bounded, then Theorem 3.3 leads directly to the estimate (2.21) for $0 \leqq t \leqq x$, and to the related estimate

$$(3.13)\qquad \begin{aligned} |E(x, t, \epsilon)| \leqq \|\delta G\| + \|\delta K\| + \epsilon(\|(\delta G)_x\| + \|\delta H\|)/\kappa \\ + \int_0^t \|\delta F\|_\tau \, d\tau/\kappa, \text{ for } 0 \leqq x \leqq t, \end{aligned}$$

where $\|\delta F\|_\tau = \max_{x \geqq 0} |\delta F(x, \tau, \epsilon)|$.

Finally, if C is uniformly positive in the first quadrant, with $C \geqq \lambda$ for some positive constant $\lambda > 0$, then the bound (2.21) can be improved to (2.22) for $0 \leqq t \leqq x$, while the bound (3.13) can be improved to

$$(3.14)\qquad \begin{aligned} |E(x, t, \epsilon)| \leqq \|\delta G\| + \|\delta K\| + \epsilon(\|(\delta G)_x\| \\ + \|\delta H\|)/\kappa + |\|\delta F\||/\lambda, \text{ for } 0 \leqq x \leqq t \end{aligned}$$

with $|\|\delta F\|| = \sup_{x \geqq 0, t \geqq 0} |\delta F(x, t, \epsilon)|$. We omit the proof. [See the discussion following (2.22) for the idea of the proof.]

The estimates of this section are again sharp, as is seen by simple (constant coefficient) examples.

These same comparison techniques can also be used to study the two-point initial-boundary value problem, but we omit these results here.

References

1. Julian D. Cole, *Perturbations Methods in Applied Mathematics,* Blaisdell Publishing Company, Waltham, Mass., 1968.
2. Murray H. Protter and Hans F. Weinberger, *Maximum Principles in Differential Equations,* Prentice-Hall, Inc., Englewood Cliffs, N.J., 1967.
3. Donald R. Smith, *An asymptotic analysis of a certain hyperbolic Cauchy problem,* SIAM J. Math. Anal., 2 (1971), 375–392.
4. Donald R. Smith and Mills B. Weinstein, *Sturm transformations and linear hyperbolic differential equations in two variables,* J. Math. Anal. and Applications (to appear).
5. Mills B. Weinstein and Donald R. Smith, *Comparison techniques for overdamped systems,* SIAM Review **17** (1975), 520–540.
6. G. B. Whitham, *Some comments on wave propagation and shock wave structure with application to magnetohydrodynamics,* Comm. Pure and Applied Math. **12** (1959), 113–158.

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