J. DIFFERENTIAL GEOMETRY
69 (2005) 43-74

COMPARISON THEOREM FOR KÄHLER MANIFOLDS AND POSITIVITY OF SPECTRUM

Peter Li & Jiaping Wang

Abstract

The first part of this paper is devoted to proving a comparison theorem for Kähler manifolds with holomorphic bisectional curvature bounded from below. The model spaces being compared to are $\mathbb{CP}^m$, $\mathbb{C}^m$, and $\mathbb{CH}^m$. In particular, it follows that the bottom of the spectrum for the Laplacian is bounded from above by m^2 for a complete, m-dimensional, Kähler manifold with holomorphic bisectional curvature bounded from below by -1. The second part of the paper is to show that if this upper bound is achieved and when $m = 2$, then it must have at most four ends.

0. Introduction

In 1975, Cheng [1] proved a comparison theorem for the first Dirichlet eigenvalues of the Laplacian on geodesic balls. One of the consequences is a sharp upper bound for the bottom of the spectrum on a complete manifold with Ricci curvature bounded from below.

Theorem 0.1 (Cheng)**.** *Let M^n be a complete Riemannian manifold of dimension n. Suppose the Ricci curvature of M has a lower bound given by*

$$\mathrm{Ric}_M \geq -(n-1).$$

Then, the bottom of the spectrum of the Laplacian must satisfy the upper bound

$$\lambda_1(M) \leq \frac{(n-1)^2}{4}.$$

Cheng's estimate is sharp and equality is achieved by the hyperbolic space form $\mathbb{H}^n$. A key ingredient of Cheng's theorem is the Laplacian comparison theorem asserting that the Laplacian of the distance function Δr has an upper bound for manifolds whose Ricci curvature is bounded from below.

Received 08/09/2004.

A natural question is to study those manifolds satisfying the equality case in Cheng's theorem, i.e., M satisfying

$$\mathrm{Ric}_M \geq -(n-1) \tag{0.1}$$

and

$$\lambda_1(M) = \frac{(n-1)^2}{4}. \tag{0.2}$$

Other than the fact that $\mathbb{H}^n$ is an example of the equality case, it was not known what can be said about this class of manifolds.

More examples of complete manifolds satisfying (0.1) and (0.2) can be found by considering hyperbolic manifolds $M = \mathbb{H}^n/\Gamma$ obtained by the quotient of $\mathbb{H}^n$ with a Kleinian group Γ. According to a theorem of Sullivan [**14**], the bottom of the spectrum, $\lambda_1(M)$, can be expressed by the Hausdorff dimension, $\delta(\Gamma)$, of the limit set of Γ. In fact, he proved that if Γ is geometrically finite, then

$$\lambda_1(M) = \begin{cases} \frac{(n-1)^2}{4}, & \text{if} \quad \delta(\Gamma) \leq \frac{n-1}{2} \\ \delta(\Gamma)(n-1-\delta(\Gamma)), & \text{if} \quad \delta(\Gamma) \geq \frac{n-1}{2}. \end{cases}$$

Hence, (0.2) is equivalent to $\delta(\Gamma) \leq \frac{n-1}{2}$ for geometrically finite Γ.

In 1995, Lee [**6**] proved that if M is a conformally compact Einstein manifold with

$$\mathrm{Ric}_M = -(n-1),$$

whose conformal infinity has non-negative Yamabe invariant, then (0.2) is valid. This theorem provided more examples of manifolds satisfying (0.1) and (0.2).

In [**12**], the authors proved the following theorems:

Theorem 0.2. *Let M^n be a complete Riemannian manifold of dimension $n \geq 3$. Suppose M satisfies (0.1) and (0.2). Then, M must either be:*

(1) *A warped product manifold $M = \mathbb{R} \times N$ of dimension $n = 3$ with metric given by*

$$ds_M^2 = dt^2 + \cosh^2 t\, ds_N^2,$$

where N^2 is a compact manifold with Gaussian curvature bounded from below by -1;

(2) *A warped product manifold $M = \mathbb{R} \times N$ with metric given by*

$$ds_M^2 = dt^2 + e^{2t}\, ds_N^2,$$

where N^{n-1} is a compact manifold with non-negative Ricci curvature; or

(3) *M has only one end.*

When $n = 2$, they proved that

Theorem 0.3. *Let M^2 be a complete Riemannian surface. Suppose the Gaussian curvature of M satisfies*

$$K_M \geq -1$$

and

$$\lambda_1(M) = \frac{1}{4}.$$

Then, M must either be:

(1) *A warped product manifold $M = \mathbb{R} \times \mathbb{S}^1$ with metric given by*
$$ds_M^2 = dt^2 + e^{2t}\, ds_{\mathbb{S}^1},$$
where $\mathbb{S}^1$ is the circle; or

(2) *M has no finite volume ends.*

At this point, we should point out that in a previous work [**11**] of the authors where they generalized the theorems of Witten–Yau [**17**], Cai–Galloway [**3**], and Wang [**16**], they proved that:

Theorem 0.4. *Let M^n be a complete Riemannian manifold of dimension $n \geq 3$. Suppose M satisfies (0.1) and*

$$\lambda_1(M) \geq (n-2).$$

Then, M must be either:

(1) *A warped product manifold $M = \mathbb{R} \times N$ with metric given by*
$$ds_M^2 = dt^2 + \cosh(2t)\, ds_N^2,$$
where N^{n-1} is a compact manifold with Ricci curvature bounded from below by $-(n-2)$; or

(2) *M has only one end with infinite volume.*

The purpose of this article is to investigate the corresponding setting for complete Kähler manifolds. The authors observed in [**11**] that on a Kähler manifold, one can rule out the existence of two infinite volume ends much easier than the Riemannian case, hence prompted this study. A major new ingredient in this paper is a comparison theorem (Theorems 1.5 and 1.6) for Kähler manifolds whose holomorphic bisectional curvature is bounded from below. It is a general principle that holomorphic bisectional curvature is more suitable for the Kähler category. Though assumptions on the holomorphic bisectional curvature are more restrictive compared to assumptions on the Ricci curvature, the results obtained, however, should be sharper.

Now, let us assume that M^m is a Kähler manifold of complex dimension m. Let $\{e_1, \cdots, e_m\}$ be a unitary frame for the $(1,0)$-part of the complexified tangent space, $T_x^{1,0}M$. The holomorphic bisectional curvature is denoted by

$$R_{\alpha\bar{\alpha}\beta\bar{\beta}} = \langle R_{e_\alpha e_{\bar{\alpha}}} e_\beta, e_{\bar{\beta}} \rangle$$

for $\alpha, \beta = 1, \cdots, m$.

Definition 0.5. Let M^m be a Kähler manifold of complex dimension m. We say that the holomorphic bisectional curvature of M is bounded from below by a constant K, denoted by

$$BK_M \geq K$$

if

$$R_{\alpha\bar{\alpha}\beta\bar{\beta}} \geq K(1 + \delta_{\alpha\beta})$$

for any unitary frame $\{e_1, e_2, \ldots, e_m\}$.

Note that for the simply connected complex space forms $\mathbb{CP}^n$, $\mathbb{C}^n$ and $\mathbb{CH}^n$, their holomorphic bisectional curvatures satisfy

$$R_{\alpha\bar{\alpha}\beta\bar{\beta}} = K(1 + \delta_{\alpha\beta}),$$

where $K = 1$, 0 and -1, respectively.

We would like to point out that a complex Hessian comparison theorem for the Busemann function was proved by Greene–Wu [**5**] in 1978 for Kähler manifolds with non-negative holomorphic bisectional curvature, i.e., $BK_M \geq 0$. In their recent paper [**4**], Cao–Ni proved the complex Hessian comparison theorem for the distance function on a Kähler manifold with $BK_M \geq 0$. Since the assumption $BK_M \geq -1$ is not the same for the cases $\alpha = \beta$ and $\alpha \neq \beta$, it is difficult to come up with a comparison theorem. In Section 1, we gave a new proof of the Hessian comparison theorem for the Riemannian case which allows us to generalize to the Kähler case.

A consequence of the comparison theorem (Theorem 1.6) is a version of Cheng's upper bound for $\lambda_1(M)$ for Kähler manifolds with $BK_M \geq -1$. In fact, we proved (Corollary 1.7) that

$$\lambda_1(M) \leq m^2.$$

Similar to Cheng's estimate, this estimate is also sharp as equality is achieved by the complex hyperbolic space form $\mathbb{CH}^m$. Of course, one now faces the question of what can be said about those Kähler manifolds satisfying

$$BK_M \geq -1 \tag{0.3}$$

and

$$\lambda_1(M) = m^2. \tag{0.4}$$

In Section 3, we proved that (Theorem 3.1) if M^m satisfies (0.3) and

$$\lambda_1(M) \geq m,$$

then M must have only one end with infinite volume. In particular, for those manifolds satisfying (0.3) and (0.4), we will only have to content with finite volume ends.

Finally, in Section 4, we considered complete Kähler surfaces satisfying (0.3) and (0.4). We showed that such a surface must have at most 4 ends, one of which has infinite volume and the rest have finite volumes. Unfortunately, we do not know if this is sharp, and we suspect that it is not. We also suspect that this finiteness phenomenon should also be true in high dimensions.

1. Comparison theorems

In this section, we will prove a sharp comparison theorem for Kähler manifolds satisfying curvature bounds. We will start by giving a new proof for the Riemannian case to illustrate the ideas. This argument, which relies on the commutation formula for covariant derivatives, also gives a slight extension of the Riemannian case.

Let (M^n, g) be a complete Riemannian manifold and let $r(x) = d(x,p)$ be the distance function to a fixed point $p \in M$. For any unit vector V in the unit tangent sphere $S_p^{n-1}(M)$, we define

$$\rho(V) = \sup\{T : \gamma_V(t) = \exp_p(tV) \text{ is minimizing on } [0,T]\}$$

to be the maximum distance for the geodesic in the direction of V to be minimizing. We also let

$$C_p = \{\rho(V)V : \rho(V) < \infty,\ V \in S_p^{n-1}(M)\}$$

to be the tangential cut locus of p. The cut locus of $p \in M$ is denoted by $\text{Cut}\,(p) = \exp_p(C_p)$. Moreover,

$$M = \exp_p(\Sigma(p)) \cup \text{Cut}\,(p),$$

where

$$\Sigma(p) = \{tV : 0 \leq t < \rho(V),\ V \in S_p^{n-1}(M)\}$$

and

$$\exp_p : \Sigma(p) \to \exp_p(\Sigma(p))$$

is a diffeomorphism. It is known that the set $\text{Cut}(p)$ has measure zero in M. The polar coordinate system (r,θ) on the tangent space $T_p(M)$ also induces a coordinate chart on $\exp_p(\Sigma(p))$. The definition of exponential map implies that $r(x) = t$ if $x = \exp_p(t\theta)$ for $t < \rho(\theta)$. Moreover, $r(x)$ is smooth on $\exp_p(\Sigma(p)) \setminus \{p\}$ and $|\nabla r| = 1$ on $\exp_p(\Sigma(p)) \setminus \{p\}$.

We begin by defining the following notion of curvature.

Definition 1.1. For any integer $1 \leq \ell \leq n-1$, we defined the ℓ-sectional curvature of a pair $\{w, V\}$, where $w \in T_pM$ and $V \subset T_pM$ is an ℓ-dimensional subspace perpendicular to w, by

$$K_M^\ell(w, V) = \sum_{i=1}^{\ell} \langle R_{we_i} w, e_i \rangle$$

with $\{e_1, e_2, \ldots, e_\ell\}$ being an orthonormal basis for V.

Note that $K_M^\ell(w, V)$ does not depend on the choice of orthonormal basis $\{e_i\}$. We say that a manifold M has ℓ-sectional curvature bounded from below by a constant K if

$$K_M^\ell(w, V) \geq \ell K$$

for all pairs $\{w, V\}$ at any point $p \in M$. When $\ell = 1$, this is equivalent to saying that the sectional curvature $K_M \geq K$. When $\ell = n-1$, this is equivalent to the Ricci curvature bounded by

$$\mathrm{Ric}_M \geq (n-1)K.$$

To set up our model for the comparison theorem, we consider $M_K^{\ell+1}$ to be the $(\ell+1)$-dimensional, simply connected, space form of constant sectional curvature K. For a fixed origin $\bar{p} \in M_K^{\ell+1}$, we denote the distance function from any point $\bar{x}$ to $\bar{p}$ by $\bar{r}(\bar{x})$.

Theorem 1.2. *Let M be a complete Riemannian manifold of dimension n. Assume that the ℓ-sectional curvatures of M satisfy $K_M^\ell \geq \ell K$. Then, within the cut locus of a fixed point $p \in M$ and for any $V \subset T_xM$ perpendicular to $\nabla r(x)$,*

$$\sum_{i=1}^{\ell} D^2(r)(e_i, e_i) \leq \sum_{i=1}^{\ell} \bar{D}^2(\bar{r})(\bar{e}_i, \bar{e}_i)$$

with $\{e_1, \ldots e_\ell\}$ being any orthonormal basis of V and $\{\bar{e}_1, \ldots \bar{e}_\ell\}$ being an orthonormal basis of $T_{\bar{p}}M_K^{\ell+1}$ with $\bar{e}_i \perp \bar{\nabla} r$.

Proof. For $x \in \exp_p(\Sigma(p)) \setminus \{p\}$, let γ be the minimal normal geodesic joining p to x. At x, we choose an orthonormal frame $\{e_1, \ldots, e_n\}$, such that $e_1 = \nabla r$. By parallel translating the frame $\{e_i\}$, we obtain an orthonormal frame along γ also denoted by $\{e_i\}_{i=1}^n$ with the property that $e_1 = \nabla r$. Since $|\nabla r|^2 = 1$ on $\exp_p(\Sigma(p)) \setminus \{p\}$, by taking covariant derivative of this equation, we obtain

$$\begin{aligned} 0 &= (|\nabla r|^2)_{\alpha\alpha} \\ &= 2\sum_{i=1}^{n} r_{i\alpha} r_{i\alpha} + 2\sum_{i=1}^{n} r_i r_{i\alpha\alpha}, \end{aligned} \tag{1.1}$$

for each $2 \leq \alpha \leq n$. Since γ is a geodesic and each e_i is parallel along γ, each term on the right-hand side of (1.1) can be interpreted as covariant derivatives. The commutation formula for covariant derivative then implies

$$\sum_{i=1}^{n} r_i\, r_{i\alpha\alpha} = \sum_{i=1}^{n} r_i\, r_{\alpha\alpha i} + \sum_{i,j=1}^{n} R_{i\alpha j\alpha}\, r_i\, r_j.$$

Substituting into (1.1) and using the fact that $|\nabla r| = 1 = r_1$, we obtain

$$0 \geq 2r_{\alpha\alpha}^2 + 2\frac{\partial(r_{\alpha\alpha})}{\partial r} + 2K_M(e_1, e_\alpha). \tag{1.2}$$

Suppose $V \subset T_xM$ is spanned by $\{e_2, \ldots, e_{\ell+1}\}$, then summing over $\alpha = 2, \ldots, \ell+1$, (1.2) becomes

$$0 \geq \sum_{\alpha=2}^{\ell+1} r_{\alpha\alpha}^2 + \frac{\partial}{\partial r}\left(\sum_{\alpha=2}^{\ell+1} r_{\alpha\alpha}\right) + K_M^\ell(e_1, V). \tag{1.3}$$

Using the lower bound of the ℓ-sectional curvature, the inequality

$$\sum_{\alpha=2}^{\ell+1} r_{\alpha\alpha}^2 \geq \frac{1}{\ell}\left(\sum_{\alpha=2}^{\ell+1} r_{\alpha\alpha}\right)^2,$$

and by setting $f(t) = \sum_{\alpha=2}^{\ell+1} r_{\alpha\alpha}(\gamma(t))$, (1.3) can be expressed as

$$0 \geq \frac{1}{\ell} f^2(t) + f'(t) + \ell K. \tag{1.4}$$

Note that since a smooth Riemannian metric is locally Euclidean,

$$\lim_{t \to 0} t\, f(t) = \ell.$$

We will now consider the three separate cases when $K = 0$, $K > 0$, and $K < 0$.

Case 1. When $K = 0$, inequality (1.4) becomes

$$f'(t) + \frac{1}{\ell} f^2(t) \leq 0.$$

This implies that $f'(t) \leq 0$ and $f(t)$ is a decreasing function. Let $(0, T)$ be the largest interval such that $f(t) > 0$, then we have

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2} \geq \frac{1}{\ell}$$

and $f(t) \leq \frac{\ell}{t}$ on $(0, T)$. Since $f(t) \leq 0$ for $t \geq T$, we can still conclude that $f(t) \leq \frac{\ell}{t}$ on $(0, \rho(\theta))$.

Case 2. When $K > 0$, inequality (1.4) can be written as

$$\frac{\ell f'(t)}{f^2(t) + \ell^2 K} \leq -1.$$

This implies that

$$\frac{d}{dt} \tan^{-1}\left(\frac{f}{\ell\sqrt{K}}\right) \leq -\sqrt{K}.$$

Integrating from 0 to t, we have

$$\tan^{-1}\left(\frac{f}{\ell\sqrt{K}}\right) \leq \frac{\pi}{2} - \sqrt{K}t,$$

implying that

$$f(t) \leq \ell\sqrt{K}\cot\left(\sqrt{K}t\right).$$

Case 3. When $K < 0$, let T be the first time such that

$$f^2(t) + \ell^2 K = 0.$$

Then, on $(0, T)$, we have $f^2(t) + \ell^2 K > 0$ and

$$\frac{\ell f'(t)}{f^2(t) + \ell^2 K} \leq -1.$$

This implies that

$$\frac{d}{dt} \coth^{-1}\left(\frac{f}{\ell\sqrt{|K|}}\right) \geq \sqrt{|K|}$$

and

$$f(t) \leq \ell\sqrt{|K|}\coth\left(\sqrt{|K|}t\right)$$

on $(0, T)$. For $t \geq T$, we claim that $f(t) \leq \ell\sqrt{|K|}$. Indeed, if $f(t_1) > \ell\sqrt{|K|}$ for $t_1 > T$, then there exists $t_2 \in (T, t_1)$ such that $f'(t_2) \geq 0$ and $f(t_2) > \ell\sqrt{|K|}$. In this case,

$$f'(t_2) + \frac{1}{\ell} f^2(t_2) + \ell K > 0,$$

which is a contradiction. Thus,

$$f(t) \leq \ell\sqrt{|K|}$$

for $T \leq t < \rho(\theta)$, and we conclude that

$$f(t) \leq \ell\sqrt{|K|}\coth\left(\sqrt{|K|}t\right)$$

for $0 < t < \rho(\theta)$.

The Theorem follows by observing that $r_{11} = 0$ and that the above inequalities become equalities on a simply connected space form with constant sectional curvature. q.e.d.

Observe that the standard Laplacian comparison theorem and the Hessian comparison theorem follow from Theorem 1.2 by setting $\ell = n-1$ and $\ell = 1$, respectively. Moreover, the Bishop comparison theorem is also a corollary. Indeed, if we consider the polar coordinate system (r, θ), Gauss lemma implies that

$$ds_M^2 = dr^2 + g_{\alpha\beta}(r,\theta)\, d\theta^\alpha\, d\theta^\beta \quad , \ \alpha, \beta = 2, \ldots, n.$$

If we denote

$$J(r,\theta) = \sqrt{\det(g_{\alpha\beta})}$$

to be the area element of the geodesic sphere $\partial B_p(r)$, then

$$\Delta_M = \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}(\ln J)\frac{\partial}{\partial r} + \Delta_{\partial B_p(r)}.$$

Thus,

$$\Delta_M r = \frac{\partial}{\partial r}(\ln J)$$

on $\exp_p(\Sigma(p)) \setminus \{p\}$.

Corollary 1.3 (Bishop)**.** *If* $\mathrm{Ric}_M \geq (n-1)K$*, then*

$$\frac{J(r,\theta)}{J_K(r)}$$

is a non-increasing function of r*, where* $J_K(r)$ *is the area element of the geodesic sphere of radius* r *in the space form* M_K^n *given by*

$$J_K(r) = \begin{cases} \sin^{n-1}\left(\sqrt{K} r\right), & \text{if} \quad K > 0 \\ r^{n-1}, & \text{if} \quad K = 0 \\ \sinh^{n-1}\left(\sqrt{|K|} r\right), & \text{if} \quad K < 0. \end{cases}$$

Moreover, if $A_p(r)$ *and* $V_p(r)$ *denote the area of* $\partial B_p(r)$ *and the volume of* $B_p(r)$*, respectively, then*

$$\frac{A_p(r_2)}{A_p(r_1)} \leq \frac{J_K(r_2)}{J_K(r_1)}$$

and

$$\frac{V_p(r_2)}{V_p(r_1)} \leq \frac{\int_0^{r_2} J_K(r)dr}{\int_0^{r_1} J_K(r)dr}$$

for $r_1 \leq r_2$*.*

The following theorem is a global version of the Laplacian comparison theorem. For a proof, we refer to [7].

Corollary 1.4. *If* $\mathrm{Ric}_M \geq (n-1)K$*, then*

$$\Delta r(x) \leq \bar{\Delta}\bar{r}\left(r(x)\right)$$

in the sense of distributions, where $\bar{\Delta}$ *is the Laplacian on the space form* M^n_K *and* $\bar{r}$ *is the distance function of* M^n_K *with respect to a fixed point. That is to say, for any* $\varphi \in C_0^\infty(M)$ *with* $\varphi \geq 0$*, we have*

$$\int_M r(x)\Delta\varphi(x) \leq \int_M \left(\bar{\Delta}\bar{r}\right)\varphi.$$

We are now ready to prove the comparison for Kähler manifolds. Recall that if the Kähler metric of M is given by $ds^2 = h_{\alpha\bar{\beta}}\, dz^\alpha\, d\bar{z}^\beta$, then the gradient and the Laplacian is given by

$$\langle \nabla f, \nabla g\rangle = 2h^{\alpha\bar{\beta}}(f_\alpha g_{\bar{\beta}} + f_{\bar{\beta}} g_\alpha)$$

and

$$\Delta f = 4h^{\alpha\bar{\beta}}\,\frac{\partial^2 f}{\partial z^\alpha \partial \bar{z}^\beta}.$$

Theorem 1.5. *Let* M^m *be a complete Kähler manifold such that its bisectional curvature* $BK_M \geq 0$*. Then, on* $\exp_p\left(\Sigma(p)\right) \setminus \{p\}$*, we have*

$$(r^2)_{\alpha\bar{\alpha}} \leq 1.$$

Proof. For any $x \in M$, we choose a unitary frame $\{e_1, \cdots, e_m\}$ at x and parallel translate each e_α along the minimizing geodesic γ. We also parallel translate each e_α so that they are defined on a neighborhood of γ. Setting $u = r^2$, a similar calculation as in the Riemannian case above shows that

$$\begin{aligned}
|\partial u|^2_{\alpha\bar{\alpha}} &= \sum_\delta \left(u_\delta u_{\bar{\delta}}\right)_{\alpha\bar{\alpha}} \\
&= \sum_\delta \left(|u_{\delta\bar{\alpha}}|^2 + |u_{\alpha\delta}|^2\right) + u_{\alpha\bar{\alpha}\delta} u_{\bar{\delta}} + u_{\alpha\bar{\alpha}\bar{\delta}} u_\delta + R_{\alpha\bar{\alpha}\delta\bar{\eta}} u_{\bar{\delta}} u_\eta \\
&\geq \frac{1}{2}\langle \nabla u_{\alpha\bar{\alpha}}, \nabla u\rangle + |u_{\alpha\bar{\alpha}}|^2.
\end{aligned}$$

Let $f(t) = u_{\alpha\bar{\alpha}}\left(\gamma(t)\right)$. Then, we have

$$f^2(t) + tf'(t) \leq f(t)$$

and $f(0) = 1$. If there exists $t > 0$ such that $f(t) \geq 1 + \epsilon$ for some $\epsilon > 0$, then using the initial condition

$$\lim_{r\to 0}(r^2)_{\alpha\bar{\alpha}} = 1$$

there must be a $0 < t_1 \leq t$ such that $f'(t_1) \geq 0$ and $f(t_1) = 1 + \epsilon$. This contradicts the differential inequality above and the theorem follows.

q.e.d.

Theorem 1.6. *Let M^m be a complete Kähler manifold with its holomorphic bisectional curvature satisfying the bound $BK_M \geq -1$. Then, on $\exp_p(\Sigma(p)) \setminus \{p\}$, we have*

$$\begin{aligned}\Delta r(x) &\leq 2(m-1)\,\coth(r(x)) + 2\,\coth(2r(x))\\ &= \bar{\Delta}\bar{r}(r(x)),\end{aligned}$$

where $\bar{\Delta}$ and $\bar{r}$ are the Laplacian and the distance function of the model manifold $\mathbb{CH}^m$.

Proof. For any x, we choose a unitary frame $\{e_1, \cdots, e_m\}$ at point x such that

$$e_1 = \frac{1}{2}\,(\nabla r - \sqrt{-1}\,J\,\nabla r).$$

We parallel translate each e_α along the minimizing geodesic γ between p and x and then to a neighborhood of γ. Along γ, one easily checks that the Hessian of r must satisfy $r_{11} = -r_{1\bar{1}}$. Therefore,

$$\begin{aligned}0 &= |\nabla r|^2_{1\bar{1}}\\ &= 2\,|r_{1\bar{1}}|^2 + 2\,|r_{11}|^2 + \langle \nabla r_{1\bar{1}}, \nabla r\rangle + 2\,R_{1\bar{1}1\bar{1}}r_1 r_{\bar{1}}\\ &\geq 4\,|r_{1\bar{1}}|^2 + \frac{\partial}{\partial r}(r_{1\bar{1}}) - 1.\end{aligned}$$

Let $f(t) = r_{1\bar{1}}\,(\gamma(t))$. Then, we have

$$4\,f^2(t) + f'(t) \leq 1 \tag{1.5}$$

and $\lim_{t\to 0} t\,f(t) = \frac{1}{4}$. It is then not difficult to see that

$$f(t) \leq \frac{1}{2}\,\coth(2t). \tag{1.6}$$

For $\alpha \neq 1$, we have

$$\begin{aligned}0 &= |\nabla r|^2_{\alpha\bar{\alpha}}\\ &\geq 2\,|r_{\alpha\bar{\alpha}}|^2 + \langle \nabla r_{\alpha\bar{\alpha}}, \nabla r\rangle + 2\,R_{1\bar{1}\alpha\bar{\alpha}}r_1 r_{\bar{1}}\\ &\geq 2\,|r_{\alpha\bar{\alpha}}|^2 + \frac{\partial}{\partial r}(r_{\alpha\bar{\alpha}}) - \frac{1}{2}.\end{aligned}$$

Let $w(t) = r_{\alpha\bar{\alpha}}\,(\gamma(t))$. Then, we have

$$4\,w^2(t) + 2\,w'(t) \leq \frac{1}{2} \tag{1.7}$$

and $\lim_{t\to 0} t\,w(t) = 1$, hence

$$w(t) \leq \frac{1}{2}\,\coth(t). \tag{1.8}$$

Finally, we have

$$\begin{aligned}\Delta r &= 4\sum_{\alpha=1}^{m} r_{\alpha\bar{\alpha}} \\ &\leq 2(m-1)\,\coth(r) + 2\,\coth(2r).\end{aligned}$$

Equivalently, this can be written as

$$\Delta(\ln(\cosh(r))) \geq 2m.$$

One computes readily that equality is achieved on $\mathbb{CH}^m$. q.e.d.

Corollary 1.7. *Let M^m be a complete Kähler manifold with its holomorphic bisectional curvature satisfying the bound $BK_M \geq -1$. Then, for any $x \in M$ and $0 \leq r \leq R$, the volume of the geodesic balls satisfy*

$$\frac{V_x(R)}{V_x(r)} \leq \frac{V_{\mathbb{CH}^m}(R)}{V_{\mathbb{CH}^m}(r)},$$

where $V_{\mathbb{CH}^m}(r)$ denotes the volume of the geodesic ball of radius r in $\mathbb{CH}^m$. In particular, the bottom of the spectrum of M has an upper bound given by $\lambda_1(M) \leq m^2$.

Proof. The volume comparison theorem follows similar to the Riemannian case by applying Theorem 1.6. Taking $r = 1$ in the volume comparison inequality, we have

$$\begin{aligned}V_p(R) &\leq C\, V_{\mathbb{CH}^m}(R) \\ &\leq C\, e^{2mR}\end{aligned}$$

for all $R \geq 1$. However, in [**11**], we have proved that

$$V_p(R) \geq C\, \exp(2\sqrt{\lambda_1(M)}R).$$

Combining with the upper bound, we conclude that $\lambda_1(M) \leq m^2$ as claimed. q.e.d.

Theorem 1.8. *Let M^m be a complete Kähler manifold with its holomorphic bisectional curvature satisfying the bound $BK_M \geq 1$. Then, on $\exp_p(\Sigma(p)) \setminus \{p\}$, we have*

$$\begin{aligned}\Delta r(x) &\leq 2(m-1)\,\cot(r(x)) + 2\,\cot(2r(x)) \\ &= \bar{\Delta}\bar{r}(r(x)),\end{aligned}$$

where $\bar{\Delta}$ and $\bar{r}$ are the Laplacian and the distance function of the model manifold $\mathbb{CP}^m$.

Proof. Following the argument as in the proof of Theorem 1.6, except (1.5) and (1.6) become

$$4\,f^2(t) + f'(t) \le -1$$

and

$$f(t) \le \frac{1}{2}\,\cot(2t),$$

respectively. Also (1.7) and (1.8) become

$$4\,w^2(t) + 2\,w'(t) \le -\frac{1}{2}$$

and

$$w(t) \le \frac{1}{2}\,\cot(t),$$

respectively. The theorem now follows as claimed. q.e.d.

Corollary 1.9. *Let M^m be a complete Kähler manifold with $BK_M \ge 1$. Then, the diameter $d(M)$ of M is bounded above by*

$$d(M) \le \frac{\pi}{2},$$

which is the diameter of the model space $\mathbb{CP}^m$. Moreover, the volume of M is bounded by

$$\begin{aligned} V(M) &\le \frac{1}{m}\sin^{2m}(d(M)) \\ &\le V(\mathbb{CP}^m). \end{aligned}$$

Proof. Suppose the diameter of M is greater than $\frac{\pi}{2}$. Then, there exists a pair of points $p, x \in M$ such that $r(x) > \frac{\pi}{2}$ and $x \in \exp_p(\Sigma(p)) \setminus \{p\}$. Using the fact that $\Delta r(x)$ is given by the mean curvature $H(x)$ of the geodesic sphere of radius $r(x)$ at x, the bound given by Theorem 1.8 asserts that the function r cannot be smooth since the upper bound

$$2(m-1)\,\cot(r) + 2\,\cot(2r)$$

becomes $-\infty$ at $r = \frac{\pi}{2}$. This contradicts the assumption that $d(M) > \frac{\pi}{2}$, and the first part of the theorem follows.

If we write $x = (r, \theta)$ in polar coordinates and let $A(r, \theta)$ be the area element of the sphere of radius r centered at p, then $\frac{\partial}{\partial r}A(r,\theta) = H(r,\theta)\,A(r,\theta)$. The comparison theorem then asserts that

$$A^{-1}(t,\theta)\,\frac{\partial A(t,\theta)}{\partial r} \le 2(m-1)\cot(t) + 2\cot(2t).$$

Integrating over the interval $0 \le t \le r$, we obtain

$$\begin{aligned} A(r,\theta) &\le \sin^{2(m-1)} r \, \sin 2r \\ &= 2 \sin^{2m-1} r \, \cos r. \end{aligned}$$

Following the same argument as in the Riemannian case (see [7]), we conclude the volume comparison

$$\begin{aligned} V(M) &= V_p(d(M)) \\ &\le 2 \int_0^{d(M)} \sin^{2m-1} r \, \cos r \, dr \\ &= \frac{1}{m} \sin^{2m}(d(M)). \end{aligned}$$

q.e.d.

Following Cheng's argument [1], one can also conclude the following eigenvalue comparison theorem.

Corollary 1.10. *Let M^m be a complete Kähler manifold with holomorphic bisectional curvature bounded from below by K, where K is either 1, 0, or -1. Then, the first Dirichlet eigenvalue, $\lambda_1(B_p(r))$, of the geodesic ball of radius r centered at $p \in M$ must be bounded from above by*

$$\lambda_1(B_p(r)) \le \lambda_1(B_{\bar{M}}(r)),$$

where $\lambda_1(B_{\bar{M}}(r))$ is the first Dirichlet eigenvalue of the geodesic ball of radius r on the model manifold $\bar{M}^m$. The model is taken to be $\mathbb{CP}^m$, $\mathbb{C}^m$, or $\mathbb{CH}^m$ for K being 1, 0, or -1, respectively.

2. Estimates for harmonic functions

Throughout this section, we assume M^m is a complete Kähler manifold of complex dimension m with holomorphic bisectional curvature bounded by

$$BK_M \ge -1. \tag{2.1}$$

We also assume that the bottom spectrum of M satisfies

$$\lambda_1(M) \ge m^2. \tag{2.2}$$

The first step is to give precise estimates on the volume growth or volume decay of an end of M. The volume estimates will then be used to derive sharp estimates for the barrier harmonic functions on the corresponding end.

Recall that an end E is defined to be an unbounded component of $M \setminus D$ for some compact set D. Without loss of generality, we may assume that $D = B_p(R_0)$ is a geodesic ball centered at some fixed point $p \in M$ with radius $R_0 > 0$. We will denote $V_E(R)$ to be the volume of the set $B_p(R) \cap E$, and $V_E(\infty)$ is simply the volume of E. Also, we recall (see [**8**] and [**9**]) that an end E is said to be a non-parabolic (or parabolic) end if it admits (or does not admit) a positive Green's function for the Laplacian on E with Neumann boundary condition on ∂E.

Let us first recall Theorem 1.4 of [**11**] stated for the class of manifolds being considered.

Theorem 2.1. *Let E be an end of a complete Kähler manifold M satifying* (2.2)*. Then, either*

(1) *E is a parabolic end with finite volume, and it must have exponential volume decay given by*

$$V_E(\infty) - V_E(R) \le C_1 \exp(-2mR)$$

for $R \ge R_0 + 1$ and some constant $C_1 > 0$ depending only on E; or

(2) *E is a non-parabolic end with infinite volume, and it must have exponential volume growth given by*

$$V_E(R) \ge C_2 \exp(2mR)$$

for $R \ge R_0 + 1$ and some constant $C_2 > 0$ depending only on E.

On the other hand, if M satisfies (2.1), then by setting $r = 1$ in Corollary 1.7, we conclude that for any $x \in M$,

$$V_p(R) \le C_3 \exp(2mR) \tag{2.3}$$

for sufficiently large R. On the other hand, if we let $x \in \partial B_p(R_1)$, $r = 1$ and $R = R_1 + 1$ in Corollary 1.7, then we have

$$\begin{aligned} V_x(1) &\ge C_4\, V_x(R_1 + 1) \exp(-2m(R_1 + 1)) \\ &\ge C_4\, V_p(1) \exp(-2m(R_1 + 1)). \end{aligned}$$

Since $B_x(1) \subset B_p(R_1 + 1)$, this can be rewritten as

$$V_p(R) \ge C_5 \exp(-2mR) \tag{2.4}$$

for $x \in \partial B_p(R)$. Combining (2.3), (2.4) with Theorem 2.1, we obtain the following corollary.

Corollary 2.2. *Let M^m be a complete Kähler manifold satisfying* (2.1) *and* (2.2)*. Let $p \in M$ be a fixed point and E be an end of M given by an unbounded component of $M \setminus B_p(R_0)$. Then, either*

(1) *E is a parabolic end with finite volume, and it must have exponential volume decay given by*

$$C_1 \exp(-2mR) \le V_E(\infty) - V_E(R) \le C_2 \exp(-2mR)$$

for $R \geq R_0 + 1$ and some constants $0 < C_1 < C_2$ depending only on E; or

(2) *E is a non-parabolic end with infinite volume, and it must have exponential volume growth given by*

$$C_3 \exp(2mR) \leq V_E(R) \leq C_4 \exp(2mR)$$

for $R \geq R_0 + 1$ and some constants $0 < C_3 \leq C_4$ depending only on E.

According to Theorem 0.1 in [**11**], the condition $\lambda_1(M) > 0$ implies that M must have infinite volume. Hence, we may assume that M has a non-parabolic end E_1. In the following discussion, we assume that M also has a finite volume, parabolic end E_2.

Recall that the theory of Li–Tam [**9**] (also see [**8**]) asserts that there exists a positive harmonic function f satisfying the following properties:

(1) $\inf_{\partial B_p(r) \cap E_1} f(x) \to 0$ as $r \to \infty$;
(2) $\sup_{\partial B_p(r) \cap E_2} f(x) \to \infty$ as $r \to \infty$; and
(3) f is bounded and has finite Dirichlet integral on $M \setminus E_2$.

In order to obtain the appropriate estimates on f, we will give an outline of the construction. Let us consider the sequence of harmonic functions v_R satisfying

$$\begin{aligned} \Delta v_R = 0 \qquad &\text{on} \qquad E_1(R), \\ v_R = 1 \qquad &\text{on} \qquad \partial E_1, \end{aligned}$$

and

$$v_R = 0 \qquad \text{on} \qquad \partial B_p(R) \cap E_1.$$

The assumption that E_1 is non-parabolic implies that v_R converges uniformly on compact subsets of E_1 to a non-constant harmonic function v.

Similarly, let u_R be a sequence of harmonic functions satisfying

$$\begin{aligned} \Delta u_R = 0 \qquad &\text{on} \qquad E_2(R), \\ u_R = 0 \qquad &\text{on} \qquad \partial E_2, \end{aligned}$$

and

$$u_R = c_R \qquad \text{on} \qquad \partial B_p(R) \cap E_2.$$

The assumption that E_2 is parabolic implies that there exist a subsequence $R_i \to \infty$ and a sequence of constants $c_i = c_{R_i} \to \infty$ such that the sequence of functions

$$u_i = u_{R_i}$$

converges uniformly on compact subsets of E_2 to a harmonic function u. Multiplying u by a constant if necessary, we may assume that

$$\int_{\partial E_1} \frac{\partial v}{\partial \nu} = -\int_{\partial E_2} \frac{\partial u}{\partial \nu}.$$

After this normalization, it was proved (also see [15]) that there exists a harmonic function f defined on M which is bounded distance from v and u on the corresponding ends E_1 and E_2. Moreover, f will satisfy the properties stated above.

It was proved in Lemma 1.2 of [11] that on $M \setminus E_2$, the Dirichlet integral of the function f must satisfy the decay estimate

$$\int_{(B_p(R+1)\setminus B_p(R))\setminus E_2} |\nabla f|^2 \leq C \exp(-2mR) \tag{2.5}$$

for R sufficiently large.

Theorem 2.3. *Let M be a complete Kähler manifold satisfying* (2.1) *and* (2.2)*. On the parabolic end E_2, the function f satisfies the gradient estimate*

$$|\nabla f|(x) \leq C \exp(2mr(x))$$

as $x \to \infty$ and $x \in E_2$ with $r(x)$ being the distance from x to the fixed point $p \in M$.

Proof. Let u be the harmonic function defined on E_2 obtained from the above construction. Observe that since the Ricci curvature is bounded from below, the gradient estimate of Cheng–Yau [2] (also see [13]) implies that

$$|\nabla(\log u)|^2 \leq C \tag{2.6}$$

on $E_2 \setminus E_2(R_0+1)$. Integrating along a geodesic joining from $x \in E_2 \setminus E_2(R_0+1)$ to $\partial B_p(R_0+1) \cap E_2$, this implies that

$$u(x) \leq C_6 \exp(C\, r(x)).$$

Applying the gradient estimate again, this yields the estimate

$$|\nabla u|(x) \leq C\, u(x) \leq C_7 \exp(C\, r(x)) \tag{2.7}$$

for some constant $C_7 > 0$.

For $R > 0$, let us denote

$$s_i(R) = \sup_{x \in \partial B_p(R) \cap E_2} |\nabla u_i|.$$

Since u_i is harmonic, the Ricci curvature bound implies that $|\nabla u_i|$ satisfies the Bochner formula

$$\Delta |\nabla u_i| \geq -2(m+1)\, |\nabla u_i|.$$

If $x \in \partial B_p(R)$ such that $s_i(R) = |\nabla u_i|(x)$, then the mean value inequality of Li–Tam [10] implies that

$$|\nabla u_i|^2(x)\, V_x(1) \leq C \int_{B_x(1)} |\nabla u_i|^2.$$

Combining with the volume lower bound (2.4), we have

$$s_i^2(R) \leq C\,\exp(2mR) \int_{B_x(1)} |\nabla u_i|^2.$$

On the other hand, if we let $a = \inf_{B_x(1)} u_i$ and $b = \sup_{B_x(1)} u_i$, then

$$\int_{B_x(1)} |\nabla u_i|^2 \leq \int_{\Omega_b \setminus \Omega_a} |\nabla u_i|^2,$$

where $\Omega_a = \{x \,|\, u_i(x) \leq a\}$. Note that by the maximum principle, if $x \in E_2 \setminus E_2(R_0 + 2)$ and for i sufficiently large, then $0 < a < b < c_i$ and the set $\Omega_b \setminus \Omega_a$ is bounded. Hence, the quantity on the right-hand side is finite. However, Stoke's theorem yields that

$$\int_{\Omega_b \setminus \Omega_a} |\nabla u_i|^2 = b \int_{\partial \Omega_b} \frac{\partial u_i}{\partial \nu} - a \int_{\partial \Omega_a} \frac{\partial u_i}{\partial \nu},$$

where ν is the outward unit normal to the sets $\partial\Omega_a$ and $\partial\Omega_b$. On the other hand, we also have

$$\begin{aligned} 0 &= \int_{\Omega_a} \Delta u_i \\ &= \int_{\partial \Omega_a} \frac{\partial u_i}{\partial \nu} - \int_{\partial B_p(R_0) \cap E_2} \frac{\partial u_i}{\partial \nu} \end{aligned}$$

for any $a > 0$. Therefore, we conclude that

$$\begin{aligned} s_i^2(R) &\leq C\,\exp(2mR) \left(\sup_{B_x(1)} u_i - \inf_{B_x(1)} u_i \right) \int_{\partial B_p(R_0) \cap E_2} \frac{\partial u_i}{\partial \nu} \\ &\leq C\,\exp(2mR) \sup_{B_x(1)} |\nabla u_i| \int_{\partial B_p(R_0) \cap E_2} \frac{\partial u_i}{\partial \nu} \\ &\leq C\,\exp(2mR)\, s_i(R+1) \int_{\partial B_p(R_0) \cap E_2} \frac{\partial u_i}{\partial \nu}. \end{aligned}$$

Setting

$$A = \left(C \int_{\partial B_p(R_0) \cap E_2} \frac{\partial u_i}{\partial \nu} \right)^{\frac{1}{2}},$$

we can rewrite the above inequality as

$$s_i(R) \leq A\,\exp(mR)\, s_i^{\frac{1}{2}}(R+1).$$

Iterating this inequality k times, we conclude that

$$s_i(R) \leq A^{\sum_{j=0}^{k-1} 2^{-j}}\,\exp\left(\sum_{j=0}^{k-1} 2^{-j}\, m\,(R+j) \right)\, s_i^{2^{-k}}(R+k).$$

Letting $i \to \infty$, this implies that

$$s(R) \le A^{\sum_{j=0}^{k-1} 2^{-j}} \exp\left(\sum_{j=0}^{k-1} 2^{-j}\, m\,(R+j)\right) s^{2^{-k}}(R+k) \tag{2.8}$$

where $s(R) = \sup_{\partial E_2(R)} |\nabla u|$. Note that since

$$\sum_{j=0}^{\infty} 2^{-j} = 2,$$

$$\begin{aligned}\exp\left(\sum_{j=0}^{\infty} 2^{-j}\, m\,(R+j)\right) &= \exp\left(\sum_{j=0}^{\infty} j\, 2^{-j}\right) \exp(2mR)\\ &= C\, \exp(2mR),\end{aligned}$$

and by (2.7)

$$\begin{aligned}\lim_{k\to\infty} s^{2^{-k}}(R+k) &\le \lim_{k\to\infty} C_7^{2^{-k}}\, \exp\left(C\,(R+k)2^{-k}\right)\\ &= 1,\end{aligned}$$

after letting $k \to \infty$ in (2.8), we obtain

$$\begin{aligned}s(R) &\le C\, \exp(2mR)\\ &= C\, \exp(2mR).\end{aligned}$$

Integrating along geodesics, this gives the estimate

$$u(x) \le C\, \exp(2mr(x))$$

as $x \to \infty$ and $x \in E_2$. Since $f - u$ is bounded on E_2, the same upper bound is valid on f. Applying the gradient estimate (2.6) on f, we obtain the growth estimate as claimed. q.e.d.

Corollary 2.4. *Let M be a complete Kähler manifold satisfying* (2.1) *and* (2.2)*. There exists a constant $C > 0$ such that the complex Hessian of f satisfies the growth estimate*

$$\int_{B_p(R)} |f_{\alpha\bar{\beta}}| \le C\, R$$

for all $R \ge 1$.

Proof. Using the fact f is harmonic and the Ricci curvature of M has uniform lower bound, from the Bochner formula and a standard cut-off argument, we have

$$\begin{aligned}\int_{(B_p(R+2)\setminus B_p(R+1))\setminus E_2} |f_{\alpha\bar\beta}|^2 &\leq C \int_{(B_p(R+3)\setminus B_p(R))\setminus E_2} |\nabla f|^2 \\ &\leq C \exp(-2mR),\end{aligned}$$

where we used (2.5) for the last inequality.

On the end E_2, Theorem 2.3 and the volume decay estimate in Corollary 2.2 imply that

$$\begin{aligned}\int_{(B_p(R+2)\setminus B_p(R+1))\cap E_2} |f_{\alpha\bar\beta}|^2 &\leq C \int_{(B_p(R+3)\setminus B_p(R))\cap E_2} |\nabla f|^2 \\ &\leq C \exp(2mR).\end{aligned}$$

Combining these two estimates, we conclude that

$$\begin{aligned}&\int_{B_p(R+2)\setminus B_p(R+1)} |f_{\alpha\bar\beta}| \\ &\quad = \int_{(B_p(R+2)\setminus B_p(R+1))\setminus E_2} |f_{\alpha\bar\beta}| + \int_{(B_p(R+2)\setminus B_p(R+1))\cap E_2} |f_{\alpha\bar\beta}| \\ &\quad \leq \left(\int_{(B_p(R+2)\setminus B_p(R+1))\setminus E_2} |f_{\alpha\bar\beta}|^2\right)^{\frac{1}{2}} V_p^{\frac{1}{2}}(R+2) \\ &\quad + \left(\int_{(B_p(R+2)\setminus B_p(R+1))\cap E_2} |f_{\alpha\bar\beta}|^2\right)^{\frac{1}{2}} (V_{E_2}(R+2) - V_{E_2}(R+1))^{\frac{1}{2}} \\ &\quad \leq C.\end{aligned}$$

The corollary now follows by iterating and summing over this estimate. q.e.d.

3. Infinite volume ends

In this section, we will prove that for a broad class of Kähler manifolds, there are only one end with infinite volume. A version of this theorem was first proved in [**11**] where the authors assumed a lower bound on the Ricci curvature. In the following theorem, we will present a version which a lower bound of the holomorphic bisectional curvature is assumed.

Theorem 3.1. *Let M be a complete Kähler manifold of complex dimension m. Suppose the holomorphic bisectional curvature of M is* *bounded below by $BK_M \geq -1$ and the bottom of the spectrum $\lambda_1(M)$ of M satisfies*

$$\lambda_1(M)^{\frac{3}{2}} + m\lambda_1(M) - m(m+1) > 0.$$

Then, M must have only one end with infinite volume. In particular, if

$$\lambda_1(M) \geq m,$$

then M must have only one end with infinite volume.

Proof. Following the proof of Theorem 2.1 in [**11**], if M has more than one infinite volume ends, then there exists a harmonic function f with finite Dirichlet integral. It follows from Lemma 3.1 of [**8**] that it must be pluriharmonic. On the other hand, if we set $h = |\nabla f|$, then the Bochner formula for pluriharmonic function (see [**8**]) becomes

$$\Delta h \geq -2(m+1)\, h + \frac{|\nabla h|^2}{h}, \tag{3.1}$$

since the assumptions on the holomorphic bisectional curvature imply that the Ricci curvature of M is bounded by

$$\mathrm{Ric}_M \geq -2(m+1).$$

If we let $g = h^p$, $0 < p < 1$, then by an argument similar to (2.5) of [**11**], and the volume estimate of Corollary 1.7, we have

$$\begin{aligned}&\int_{B_p(2R)\setminus B_p(R)} g^2 \\ &\leq C\, R^p \left(\int_R^{2R} \exp\left(-\frac{p}{1-p}\, 2\sqrt{\lambda_1(M)}r \right) \exp(2mr)\, dr \right)^{1-p}.\end{aligned}$$

Choosing p to satisfy

$$p\sqrt{\lambda_1(M)} = (1-p)m, \tag{3.2}$$

we conclude that

$$\int_{B_p(R)} g^2 = O(R).$$

Moreover, since (3.1) implies that $g = h^p$ satisfies

$$\Delta g \geq -2p(m+1)g + \frac{|\nabla g|^2}{g},$$

by Lemma 4.1 of [**11**], we obtain

$$\left(\lambda_1(M) - \frac{2p(m+1)(1+\delta)}{1+2\,\delta} \right) \int_M \phi^2\, g^2 \leq \left(1 + \frac{\delta^2}{1+2\delta} \right) \int_M |\nabla \phi|^2\, g^2$$

for all $\delta > 0$. Now, if

$$\lambda_1(M) > p(m+1),$$

then there exists a sufficiently large δ such that

$$\lambda_1(M) - \frac{2p(m+1)(1+\delta)}{1+2\,\delta} > 0.$$

Arguing as in Theorem 4.2 of [11], we conclude that $g = 0$ and M has only one infinite volume end. However, condition (3.2) for p asserts that

$$p = \frac{m}{m + \sqrt{\lambda_1(M)}},$$

hence, we need

$$\lambda_1(M)(m + \sqrt{\lambda_1(M)}) - m(m+1) > 0. \tag{3.3}$$

This proves the first part of the theorem.

Note that since the function

$$q(x) = x^3 + mx^2 - m(m+1)$$

is strictly increasing when $x > 0$ with $q(0) < 0$, (3.3) will be fulfilled as long as $\lambda_1(M) > x_0^2$, where $x_0 > 0$ is the positive solution to the cubic

$$x^3 + mx^2 - m(m+1) = 0.$$

The second part follows by observing that $q(\sqrt{m}) = m(\sqrt{m} - 1) \geq 0$. Hence, the assumption that $\lambda_1(M) \geq m$ implies that $\lambda_1(M) > x_0^2$ for $m > 1$. q.e.d.

Following the argument in [11], one can also prove the following finiteness theorem.

Theorem 3.2. *Let M be a complete Kähler manifold of complex dimension m. Let x_0 be the unique positive solution to the cubic*

$$x^3 + mx^2 - m(m+1) = 0.$$

Suppose there exists a geodesic ball $B_p(R_0) \subset M$ such that $\lambda_1(M \setminus B_p(R_0)) \geq x_0^2 + \epsilon$ for some $\epsilon > 0$. Also assume that

$$BK_M \geq -1$$

on $M \setminus B_p(R_0)$. Then, M must have finitely many ends with infinite volume. In particular, there exists a constant $C(m, R_0, \alpha, v, \epsilon) > 0$ depending on the quantities n, R_0, ϵ, $\alpha = \inf_{B_p(3R_0)} Ric_M$, and $v = \inf_{x \in B_p(2R_0)} V_x(R_0)$, such that the number of infinite volume ends of M is at most C.

4. Finite volume ends

To deal with finite volume ends, since the constructed harmonic function may not be pluriharmonic, we will utilize a Bochner type formula for the Laplacian of the length of the complex Hessian.

Lemma 4.1. *Let M be a Kähler manifold with complex dimension m. If f is a harmonic function on M, then*

$$\Delta|f_{\alpha\bar{\beta}}|^2 \geq -8m\,\rho\,|f_{\alpha\bar{\beta}}|^2 + \frac{m+1}{2m}\,|f_{\alpha\bar{\beta}}|^{-2}\,|\nabla|f_{\alpha\bar{\beta}}|^2|^2,$$

where $f_{\alpha\bar{\beta}}$ denotes the complex Hessian of f and $BK_M(x) \geq -\rho(x)$ is the pointwise lower bound of the holomorphic bisectional curvature of M.

Proof. Let $\{z^1, \cdots, z^m\}$ be complex normal coordinates at a point $z \in M$. The Hermitian metric can be written in the form

$$ds^2 = h_{\alpha\bar{\beta}}\,dz^\alpha\,\bar{z}^\beta$$

where $1 \leq \alpha, \beta \leq m$. Using the Kähler condition

$$\begin{aligned}
(4.1)\quad \frac{1}{4}\Delta|f_{\alpha\bar{\beta}}|^2 &= \Box|f_{\alpha\bar{\beta}}|^2 \\
&= h^{\theta\bar{\eta}}\,\partial_\theta\partial_{\bar{\eta}}\left(h^{\alpha\bar{\gamma}}\,h^{\tau\bar{\beta}}\,f_{\alpha\bar{\beta}}\,f_{\tau\bar{\gamma}}\right) \\
&= h^{\theta\bar{\eta}}\Big((\partial_\theta\partial_{\bar{\eta}}h^{\alpha\bar{\gamma}})\,h^{\tau\bar{\beta}}\,f_{\alpha\bar{\beta}}\,f_{\tau\bar{\gamma}} + h^{\alpha\bar{\gamma}}(\partial_\theta\partial_{\bar{\eta}}h^{\tau\bar{\beta}})\,f_{\alpha\bar{\beta}}\,f_{\tau\bar{\beta}} \\
&\quad + h^{\alpha\bar{\gamma}}\,h^{\tau\bar{\beta}}(\partial_\theta f_{\alpha\bar{\beta}})(\partial_{\bar{\eta}}f_{\tau\bar{\gamma}}) + h^{\alpha\bar{\gamma}}\,h^{\tau\bar{\beta}}(\partial_{\bar{\eta}}f_{\alpha\bar{\beta}})(\partial_\theta f_{\tau\bar{\gamma}}) \\
&\quad + h^{\alpha\bar{\gamma}}\,h^{\tau\bar{\beta}}\,(\partial_\theta\partial_{\bar{\eta}}f_{\alpha\bar{\beta}})\,f_{\tau\bar{\gamma}} + h^{\alpha\bar{\gamma}}\,h^{\tau\bar{\beta}}\,f_{\alpha\bar{\beta}}\,(\partial_\theta\partial_{\bar{\eta}}f_{\tau\bar{\gamma}})\Big)
\end{aligned}$$

at the point z, where we have used the assumption that $\partial_\theta h^{\alpha\bar{\gamma}}(z) = 0 = \partial_{\bar{\eta}}h_{\alpha\bar{\gamma}}(z)$. Using the assumption that $\Delta f = 0$, we have

$$\begin{aligned}
(4.2)\qquad 0 &= \partial_\alpha\partial_{\bar{\beta}}\,(h^{\theta\bar{\eta}}f_{\theta\bar{\eta}}) \\
&= (\partial_\alpha\partial_{\bar{\beta}}h^{\theta\bar{\eta}})f_{\theta\bar{\eta}} + h^{\theta\bar{\eta}}\,\partial_\alpha\partial_{\bar{\beta}}(f_{\theta\bar{\eta}}) \\
&= (\partial_\alpha\partial_{\bar{\beta}}h^{\theta\bar{\eta}})f_{\theta\bar{\eta}} + h^{\theta\bar{\eta}}\,\partial_\theta\partial_{\bar{\eta}}(f_{\alpha\bar{\beta}}).
\end{aligned}$$

The Kähler condition also implies that

$$(4.3)\qquad \partial_\theta\partial_{\bar{\eta}}h^{\alpha\bar{\gamma}} = R_{\bar{\eta}}{}^{\bar{\gamma}}{}_{\bar{\mu}\theta}\,h^{\alpha\bar{\mu}}.$$

Substituting (4.2) and (4.3) this into (4.1) yields

(4.4)
$$
\begin{aligned}
\frac{1}{4}\Delta|f_{\alpha\bar\beta}|^2 = h^{\theta\bar\eta}\, h^{\alpha\bar\mu}\, h^{\tau\bar\beta}\, R_{\bar\eta}{}^{\bar\gamma}{}_{\bar\mu\theta}\, f_{\alpha\bar\beta}\, f_{\tau\bar\gamma} \\
&+ h^{\theta\bar\eta}\, h^{\alpha\bar\gamma}\, h^{\tau\bar\mu}\, R_{\bar\eta}{}^{\bar\beta}{}_{\bar\mu\theta}\, f_{\alpha\bar\beta}\, f_{\tau\bar\gamma} + 2|\partial_\theta f_{\alpha\bar\beta}|^2 \\
&- h^{\alpha\bar\gamma}\, h^{\tau\bar\beta}\, h^{\theta\bar\delta}\, R_{\bar\beta}{}^{\bar\eta}{}_{\bar\delta\alpha}\, f_{\theta\bar\eta}\, f_{\tau\bar\gamma} - h^{\alpha\bar\gamma}\, h^{\tau\bar\beta}\, h^{\theta\bar\delta}\, R_{\bar\gamma}{}^{\bar\eta}{}_{\bar\delta\tau}\, f_{\theta\bar\eta}\, f_{\alpha\bar\beta} \\
= 2h^{\theta\bar\eta}\, h^{\alpha\bar\mu}\, h^{\tau\bar\beta}\, R_{\bar\eta}{}^{\bar\gamma}{}_{\bar\mu\theta} f_{\alpha\bar\beta}\, f_{\tau\bar\gamma} + 2|\partial_\theta f_{\alpha\bar\beta}|^2 \\
&- 2h^{\alpha\bar\gamma}\, h^{\tau\bar\beta}\, h^{\theta\bar\delta}\, R_{\bar\beta}{}^{\bar\eta}{}_{\bar\delta\alpha}\, f_{\theta\bar\eta}\, f_{\tau\bar\gamma}.
\end{aligned}
$$

At a fixed point $z \in M$, let us choose normal coordinates so that

$$f_{\alpha\bar\beta} = \lambda_\alpha\, \delta_{\alpha\bar\beta}$$

and

$$h_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}.$$

This implies that

$$2h^{\theta\bar\eta}\, h^{\alpha\bar\mu}\, h^{\tau\bar\beta}\, R_{\bar\eta}{}^{\bar\gamma}{}_{\bar\mu\theta}\, f_{\alpha\bar\beta}\, f_{\tau\bar\gamma} = 2R_{\bar\eta}{}^{\bar\alpha}{}_{\bar\alpha\eta}\, \lambda_\alpha^2$$

and

$$2h^{\alpha\bar\gamma}\, h^{\tau\bar\beta}\, h^{\theta\bar\delta}\, R_{\bar\beta}{}^{\bar\eta}{}_{\bar\delta\alpha}\, f_{\theta\bar\eta}\, f_{\tau\bar\gamma} = 2R_{\bar\tau}{}^{\bar\theta}{}_{\bar\theta\tau}\, \lambda_\theta\, \lambda_\tau.$$

Hence, the assumption on the bisectional curvature and two curvature terms in (4.4) combine to become

$$
\begin{aligned}
2R_{\bar\alpha}{}^{\bar\beta}{}_{\bar\beta\alpha}\, \lambda_\beta^2 - 2R_{\bar\alpha}{}^{\bar\beta}{}_{\bar\beta\alpha}\, \lambda_\alpha\, \lambda_\beta &= R_{\bar\alpha}{}^{\bar\beta}{}_{\bar\beta\alpha}\, (\lambda_\alpha - \lambda_\beta)^2 \\
&\geq -\rho(\lambda_\alpha - \lambda_\beta)^2.
\end{aligned}
$$

However, f is harmonic implies that

$$\sum_\alpha \lambda_\alpha = 0.$$

Hence,

$$
\begin{aligned}
2R_{\bar\alpha}{}^{\bar\beta}{}_{\bar\beta\alpha}\, \lambda_\beta^2 - 2R_{\bar\alpha}{}^{\bar\beta}{}_{\bar\beta\alpha}\, \lambda_\alpha\, \lambda_\beta &\geq -2m\rho \sum_\alpha \lambda_\alpha^2 - 2\rho \sum_{\alpha,\beta} \lambda_\alpha\, \lambda_\beta \\
&= -2m\rho\, |f_{\alpha\bar\beta}|^2.
\end{aligned}
$$

Substituting this estimate into (4.4), we conclude that

$$\frac{1}{4}\Delta|f_{\alpha\bar\beta}|^2 \geq 2|\partial_\theta f_{\alpha\bar\beta}|^2 - 2m\rho\, |f_{\alpha\bar\beta}|^2. \tag{4.5}$$

On the other hand, let us now consider the term

$$\begin{aligned}\frac{1}{4}|\nabla|f_{\alpha\bar\beta}|^2|^2 &= h^{\theta\bar\eta}\,\partial_\theta|f_{\alpha\bar\beta}|^2\,\partial_{\bar\eta}|f_{\tau\bar\gamma}|^2\\ &= h^{\theta\bar\eta}(f_{\bar\alpha\beta}\,\partial_\theta f_{\alpha\bar\beta}+f_{\alpha\bar\beta}\,\partial_\theta f_{\bar\alpha\beta})(f_{\bar\tau\gamma}\,\partial_{\bar\eta}f_{\tau\bar\gamma}+f_{\tau\bar\gamma}\,\partial_{\bar\eta}f_{\bar\tau\gamma}).\end{aligned}$$

At the point $z\in M$, this can be written as

$$\begin{aligned}|\nabla|f_{\alpha\bar\beta}|^2|^2 &= 16h^{\theta\bar\eta}\,(\lambda_\alpha\,\partial_\theta f_{\alpha\bar\alpha})(\lambda_\gamma\,\partial_{\bar\eta}f_{\gamma\bar\gamma})\\ &\le 16|f_{\alpha\bar\beta}|^2\,|\partial_\theta f_{\alpha\bar\alpha}|^2.\end{aligned}\tag{4.6}$$

Also note that

$$\begin{aligned}|\partial_\theta f_{\alpha\bar\beta}|^2 &= |\partial_\theta f_{\alpha\bar\alpha}|^2+\sum_{\bar\beta\neq\bar\alpha}|\partial_\theta f_{\alpha\bar\beta}|^2\\ &\ge |\partial_\theta f_{\alpha\bar\alpha}|^2+\sum_{\bar\theta\neq\bar\alpha}|\partial_\theta f_{\alpha\bar\theta}|^2\\ &= |\partial_\theta f_{\alpha\bar\alpha}|^2+\sum_{\bar\theta\neq\bar\alpha}|\partial_\alpha f_{\theta\bar\theta}|^2.\end{aligned}\tag{4.7}$$

Using the fact that f is harmonic yields

$$\begin{aligned}(m-1)\sum_{\bar\theta\neq\bar\alpha}|\partial_\alpha f_{\theta\bar\theta}|^2 &\ge \left|\sum_{\bar\theta\neq\bar\alpha}\partial_\alpha f_{\theta\bar\theta}\right|^2\\ &= |\partial_\alpha f_{\alpha\bar\alpha}|^2,\end{aligned}\tag{4.8}$$

hence,

$$m\sum_{\bar\theta\neq\bar\alpha}|\partial_\alpha f_{\theta\bar\theta}|^2\ge|\partial_\theta f_{\alpha\bar\alpha}|^2.$$

Combining this with (4.7), we concude that

$$|\partial_\theta f_{\alpha\bar\beta}|^2\ge\frac{m+1}{m}\,|\partial_\theta f_{\alpha\bar\alpha}|^2.$$

Substituting this estimate into (4.6) gives

$$|\nabla|f_{\alpha\bar\beta}|^2|^2\le\frac{16m}{m+1}\,|f_{\alpha\bar\beta}|^2\,|\partial_\theta f_{\alpha\bar\beta}|^2.$$

The lemma follows by combining this with (4.5). q.e.d.

We now restrict our attention to the case $m=2$.

Lemma 4.2. *Let M be a complete Kähler manifold of complex dimension 2. Suppose the holomorphic bisectional curvature of M is bounded from below by $BK_M\ge-1$ and the bottom of the spectrum of* *the Laplacian for functions is bounded by $\lambda_1(M) \geq 4$. If f is a harmonic function on M whose complex Hessian satisfies the growth estimate*

$$\int_{B_x(r)} |f_{\alpha\bar{\beta}}| \leq o(r^2)$$

as $r \to \infty$, then f must either be pluriharmonic, or the function

$$g = |f_{\alpha\bar{\beta}}|^{\frac{1}{2}}$$

must be positive and satisfy the equation

$$\Delta g = -4g.$$

Proof. Let $g = |f_{\alpha\bar{\beta}}|^{\frac{1}{2}}$. Lemma 4.1 asserts that

$$\Delta g \geq -2mg - \frac{m-2}{m}\, g^{-1}\, |\nabla g|^2.$$

When $m = 2$, this becomes

$$\Delta g \geq -4g.$$

For a non-negative compactly supported function ϕ defined on M, applying the assumption on the spectrum, then

$$\begin{aligned} 4\int_M (\phi\, g)^2 &\leq \int_M |\nabla(\phi\, g)|^2 \\ &= \int_M |\nabla\phi|^2\, g^2 - \int_M \phi^2\, g\, \Delta g. \end{aligned} \tag{4.9}$$

So, we have

$$\int_M \phi^2\, g\, (\Delta g + 4g) \leq \int_M |\nabla\phi|^2\, g^2. \tag{4.10}$$

If we choose the function

$$\phi(x) = \begin{cases} 1 & \text{on} \quad B_p(R) \\ \dfrac{2R - r(x)}{R} & \text{on} \quad B_p(2R) \setminus B_p(R) \\ 0 & \text{on} \quad M \setminus B_p(2R), \end{cases}$$

then

$$\int_M |\nabla\phi|^2\, g^2 \leq R^{-2} \int_{B_p(2R)\setminus B_p(R)} g^2.$$

Since the right-hand side tends to 0 as $R \to \infty$ due to the growth assumption on g, we conclude that all the inequalities used in the proof and Lemma 4.1 must be equalities.

If $f_{\alpha\bar{\beta}}$ is identitally 0, then this implies that f is pluriharmonic. Otherwise, $g = |f_{\alpha\bar{\beta}}|^{\frac{1}{2}}$ must satisfy

$$\Delta g = -4g. \tag{4.11}$$

We now claim that $g > 0$. Indeed, if $g = 0$ at some point, then by regularity of the equation (4.11), g must change sign. However, since $g \geq 0$, this is impossible.

Since inequality (4.8) becomes equality, we have

$$\partial_\alpha f_{\theta\bar{\theta}} = \partial_\alpha f_{\eta\bar{\eta}}$$

for all θ, $\eta \neq \alpha$. In particular, this implies that

$$\partial_\alpha f_{\alpha\bar{\alpha}} = (m-1)\partial_\alpha f_{\theta\bar{\theta}}$$

for all $\theta \neq \alpha$. Also, the fact that inequality (4.7) becomes equality implies that

$$\partial_\theta f_{\alpha\bar{\beta}} = 0$$

for all $\theta \neq \beta$ and $\beta \neq \alpha$. q.e.d.

Theorem 4.3. *Let M^2 be a complete Kähler manifold of complex dimension 2. Suppose the holomorphic bisectional curvature of M is bounded by $BK_M \geq -1$. If the bottom of the spectrum of the Laplacian, $\lambda_1(M)$, is bounded from below by $\lambda_1(M) \geq 4$, then M must have at most four ends. Moreover, exactly one of its ends must have infinite volume, while the rest of the ends have finite volume.*

Proof. As discussed earlier, the assumption $\lambda_1(M) \geq 4$ implies that M must have exponential volume growth. In particular, one of the ends of M must have infinite volume. Combining with Theorem 3.1, we see that M has exactly one infinite volume end E_1.

Let us now assume that M has at least three ends. By the above discussion, other than E_1, all the other ends must have finite volume. For each finite volume end E_2, following the construction in [**9**] and [**15**], there exists a positive harmonic function f satisfying the following properties:

(1) $\sup_{\partial B_p(R)\cap E_2} f(x) \to \infty$ as $R \to \infty$;
(2) $\inf_{\partial B_p(R)\cap E_1} f(x) \to 0$ as $R \to \infty$; and
(3) f is bounded on all other ends.

Moreover, it also follows that f has finite Dirichlet integral on E_1. In fact, we also derive the growth estimate in Corollary 2.4 that

$$\int_{B_p(R)} |f_{\alpha\bar{\beta}}| \leq C\, R.$$

Applying Lemma 4.2, we conclude that either f is pluriharmonic, or that $g = |f_{\alpha\bar{\beta}}|^{\frac{1}{2}}$ is positive satisfying the equation

$$\Delta g = -4g. \tag{4.12}$$

If f is pluriharmonic, then using the argument in Theorem 5.1 of [11] or Theorem 3.1, we conclude that this is impossible. Hence, g must satisfy (4.12). Since there are more than one finite volume end, we can find at least two linearly independent harmonic functions f and $\tilde{f}$, constructed using two small ends E_2 and E_3, such that

$$g = |f_{\alpha\bar{\beta}}|^{\frac{1}{2}}$$

and

$$\tilde{g} = |\tilde{f}_{\alpha\bar{\beta}}|^{\frac{1}{2}}$$

both satisfy (4.12). If g is not a scalar multiple of $\tilde{g}$, then we can find a linear combination $G = ag + b\tilde{g}$ such that G must change sign. Moreover, G also satisfies

$$\Delta G = -4G.$$

The function $|G|$ will then satisfy

$$\Delta|G| \geq -4|G|$$

in the weak sense and must vanish somewhere. However, since g and $\tilde{g}$ have L^2 norms satisfying the growth condition

$$\int_{B_p(R)} g^2 \leq C\,R$$

and

$$\int_{B_p(R)} \tilde{g}^2 \leq C\,R,$$

the function $|G|$ will also satisfy the growth condition

$$\int_{B_p(R)} |G|^2 \leq C\,R.$$

Applying the cut-off argument on $|G|$ as in the proof of Lemma 4.2, we conclude that

$$\Delta|G| = -4|G|.$$

The regularity argument of Lemma 4.2 implies that this is impossible. Hence, g must be a scalar multiple of $\tilde{g}$. In particular, after a rescaling of $\tilde{f}$, we may assume that $g = \tilde{g}$.

Let us now choose a unitary frame $\{e_1, e_2, e_{\bar{1}}, e_{\bar{2}}\}$ such that

$$(f_{\alpha\bar{\beta}}) = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}.$$

Since

$$\begin{aligned} g^4 &= |f_{\alpha\bar\beta}|^2 \\ &= 2\mu^2 \end{aligned}$$

is positive, the unitary frame is, in fact, globally defined. With respect to this frame, the complex Hessian of $\tilde f$ can be written as

$$(\tilde f_{\alpha\bar\beta}) = \begin{pmatrix} \tilde\mu & \nu \\ \bar\nu & -\mu \end{pmatrix}.$$

Since

$$\begin{aligned} g^4 &= |\tilde f_{\alpha\bar\beta}|^2 \\ &= 2\tilde\mu^2 + 2|\nu|^2 \end{aligned}$$

we conclude that

$$\mu^2 = \tilde\mu^2 + |\nu|^2. \tag{4.13}$$

On the other hand, for any $0 \le t \le 1$, the function

$$h = t\, f + (1-t)\, \tilde f$$

is harmonic. Moreover, the above argument implies that there is a constant $\alpha > 0$, depending only on t, such that

$$|h_{\alpha\bar\beta}|^2 = \alpha\, g^2.$$

On the other hand, since

$$(h_{\alpha\bar\beta}) = \begin{pmatrix} t\,\mu + (1-t)\,\tilde\mu & (1-t)\,\nu \\ (1-t)\,\bar\nu & -t\,\mu - (1-t)\,\tilde\mu \end{pmatrix}$$

we have

$$2\alpha\mu^2 = 2(t\,\mu + (1-t)\,\tilde\mu)^2 + 2(1-t)^2|\nu|^2.$$

Using the identity (4.13), this implies that

$$(\alpha - 1)\mu = 2t(1-t)\tilde\mu.$$

However, since

$$\frac{\alpha - 1}{2t(t-1)}$$

is independent on the point in M and

$$\frac{\tilde\mu}{\mu}$$

is independent on t, we conclude that

$$\tilde\mu = \beta\,\mu$$

for some constant

$$\beta = \frac{\alpha - 1}{2t(t-1)}.$$

Using (4.13) again, we deduce that

$$(\tilde{f}_{\alpha\bar{\beta}}) = \begin{pmatrix} \beta\,\mu & \sqrt{1-\beta^2}\,\mu\,e^{i\theta} \\ \sqrt{1-\beta^2}\,\mu\,e^{-i\theta} & -\beta\,\mu \end{pmatrix}$$

for some constant $0 \le \theta < 2\pi$.

Note that if $\beta = 1, -1$, then either $f - \tilde{f}$ or $f + \tilde{f}$ will be a non-constant pluriharmonic function and the argument in [11] will give a contradiction. Hence, $-1 < \beta < 1$ because of (4.13). In the event if $\beta \neq 0$, then the linear combination

$$\frac{\tilde{f} - \beta f}{\sqrt{1-\beta^2}}$$

will have complex Hessian of the form

$$\begin{pmatrix} 0 & \mu\,e^{-i\theta} \\ \mu\,e^{-i\theta} & 0 \end{pmatrix}.$$

In any case, by possibly multiplying by -1, we may then assume

$$(\tilde{f}_{\alpha\bar{\beta}}) = \begin{pmatrix} 0 & \mu\,e^{-i\theta} \\ \mu\,e^{-i\theta} & 0 \end{pmatrix},$$

for some constant $0 \le \theta < \pi$.

Now, let us assume that M has at least 5 ends. Since exactly one end has infinite volume, there must be at least 4 finite volume ends. Each of the ends with finite volume will produce a positive harmonic function as discussed above. Let us denote these functions by $\{f_i\}_{i=1}^4$. Using f_1 to play the role of f and each f_j playing the role of $\tilde{f}$, we conclude that their complex Hessians are of the form

$$((f_1)_{\alpha\bar{\beta}}) = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix},$$

and

$$((f_j)_{\alpha\bar{\beta}}) = \begin{pmatrix} 0 & \mu\,e^{i\theta_j} \\ \mu\,e^{-i\theta_j} & 0 \end{pmatrix}$$

for $2 \le j \le 4$, where $0 \le \theta_j < \pi$. Obviously, by taking linear combinations of $\{f_j\}_{j=2}^4$, we may arrange an $\tilde{f}$ in the subspace spanned by the $\{f_j\}$ such that its complex Hessian is identically 0. Hence, $\tilde{f}$ is a non-constant pluriharmonic function, which gives a contradiction. This implies that M cannot have more than 4 ends. q.e.d.

5. Acknowledgements

The first author was partially supported by NSF grants DMS-0202508 and DMS-0112416. The second author was partially supported by NSF grant DMS-0072181.

References

[1] S.Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math. Z. **143** (1975) 289–297, MR 0378001, Zbl 0329.53035.

[2] S.Y. Cheng & S.-T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975) 333–354, MR 0385749, Zbl 0312.53031.

[3] M. Cai & G.J. Galloway, *Boundaries of zero scalar curvature in the ADS/CFT correspondence*, Adv. Theor. Math. Phys. **3** (1999) 1769–1783, MR 1812136, Zbl 0978.53084.

[4] H.-D. Cao & L. Ni, *Matrix Li-Yau-Hamilton estimates for the heat equation on Kähler manifolds*, to appear in Math. Ann..

[5] R.E. Greene & H. Wu, *On Kähler manifolds of positive bisectional curvature and a theorem of Hartogs*, Abh. Math. Sem. Univ. Hamburg **47** (1978) 171–185, MR 0499318.

[6] J. Lee, *The spectrum of an asymptotically hyperbolic Einstein manifold*, Comm. Anal. Geom. **3** (1995) 253–271, MR 1362652, Zbl 0934.58029.

[7] P. Li, *Lecture Notes on Geometric Analysis*, Lecture Notes Series, **6**, Research Institute of Mathematics and Global Analysis Research Center, Seoul National University, Seoul, 1993, MR 1320504, Zbl 0822.58001.

[8] P. Li, *Curvature and function theory on Riemannian manifolds*, in 'Survey in Differential Geometry "In Honor of Atiyah, Bott, Hirzebruch, and Singer",' International Press, Cambridge **VII** (2000) 71–111, MR 1919432.

[9] P. Li & L.F. Tam, *Harmonic functions and the structure of complete manifolds*, J. Differential Geometry **35** (1992) 359–383, MR 1158340, Zbl 0768.53018.

[10] P. Li & L.F. Tam, *The heat equation and harmonic maps of complete manifolds*, Invent. Math. **105** (1991) 1–46, MR 1109619, Zbl 0748.58006.

[11] P. Li & J. Wang, *Complete manifolds with positive spectrum*, J. Differential Geometry **58** (2001) 501–534, MR 1906784, Zbl 1032.58016.

[12] P. Li & J. Wang, *Complete manifolds with positive spectrum*, II, J. Differential Geometry **62** (2002) 143–162, MR 1987380.

[13] P. Li & S.-T. Yau, *On the Parabolic Kernel of the Schrödinger Operator*, Acta Math. **156** (1986) 153–201, MR 0834612, Zbl 0611.58045.

[14] D. Sullivan, *Related aspects of positivity in Riemannian geometry*, J. Differential Geometry **25** (1987) 327–351, MR 0882827, Zbl 0615.53029.

[15] C.J. Sung, L.F. Tam, & J. Wang, *Spaces of harmonic functions*, J. London Math. Soc. **61** (2000) 789–806, MR 1766105, Zbl 0963.31004.

[16] X. Wang, *On conformally compact Einstein manifolds*, Math. Res. Let. **8** (2001) 671–688, MR 1879811.

[17] E. Witten & S.-T. Yau, *Connectedness of the boundary in the AdS/CFT correspondence*, Adv. Theor. Math. Phys. **3** (1999) 1635–1655, MR 1812133, Zbl 0978.53085.

Department of Mathematics
University of California
Irvine, CA 92697-3875
E-mail address: pli@math.uci.edu

School of Mathematics
University of Minnesota
Minneapolis, MN 55455
E-mail address: jiaping@math.umn.edu