Comparison theorems for a generalized modulus of continuity

JAN BOMAN and HAROLD S. SHAPIRO

University of Stockholm, Sweden and University of Michigan, Ann Arbor, Mich., U.S.A.

1. Introduction

In previous work [18], [19], [20], one of the authors introduced a generalized modulus of continuity of a function $f \in L^p$. Like the usual L^p modulus of continuity it is a function of a positive variable a, but depends also upon a measure σ. By suitable specialization of σ this generalized modulus (written $\omega_{\sigma,p}(f\,;a)$) can serve as a measure either of the smoothness of a function or of the degree to which the function is approximable in L^p norm by its convolution with $(1/a)k(t/a)$, k being a given integrable function. *Comparison theorems* were proved enabling the τ modulus to be estimated in terms of the σ modulus under certain conditions, and this enabled several questions concerning so-called direct and inverse theorems of approximation theory to be studied from a unified viewpoint.

The main reason for writing a new paper on the subject is as follows. In the cited work, only sup norm estimates (i.e. $p = \infty$) were treated in detail, apart from a remark in [20] that identical inequalities were valid when all norms were interpreted in the L^p sense. While this is correct, examination showed that the results so obtained were unsharp for values of p other than 1 and ∞, in many typical cases where one would like to apply the method. Thus, for example, although the theory yielded the sharp Marchaud estimates (see [13], p. 48) for the (sup norm) modulus of continuity in terms of the second order modulus of smoothness, it yielded the identical estimate for all values of p. But it is known from work of A. F. Timan ($p = 2$) and Zygmund (general p) that sharper estimates are valid when $1 < p < \infty$ (more details below in § 6, see also [26], p. 121 and [30]).

The clue to overcoming the difficulty was provided by Zygmund's paper [30]. In this paper (seldom quoted in the literature, although it pioneered a technique which has since found wide application) Zygmund employed a characteristic method based upon the decomposition of the Fourier series into blocks of the type $\sum c_k e^{ikt}$, the summation being from 2^n to $2^{n+1} - 1$, to which he then applied a rather deep inequality due to Littlewood and Paley.

In order to apply the Zygmund method to our problems, however, some technical difficulties had to be overcome, since a version of the Littlewood-Paley theorem valid for non-periodic functions, and in many variables, was needed. This material is found in Section 3 of the present paper (with some historical comments in 3.7). Although the results there are not really new, we could not find just the versions which we needed, with complete proofs, in the literature. Therefore, after some hesitation, we decided (in view of the decisive importance of these results for our work) that we had no choice but to include complete proofs, which explains the length of that preparatory section. The variation we have selected, based upon a partition of unity, was suggested to us by J. Peetre's treatment [14] of Besov spaces.

The (sharp) generalizations to L^p of the two basic comparison theorems of [20] are given in § 4. The proofs are arranged so as to obtain the results of [20] too, somewhat more easily than in the former paper.

In § 5 a result of somewhat different character is proved (and this is another reason for the present paper), most nearly related to »embedding theorems» of Sobolev and others. Actually, our interest in these questions was partly inspired by a paper of M. Weiss and Zygmund [28] dealing with conditions on the second order modulus of smoothness sufficient to force absolute continuity of a function. The technique employed there is nearly identical to that of [30]. Our Theorem 5.3 yields the theorem of Weiss and Zygmund, as well as its higher-dimensional generalization due to John and Nirenberg [6]. The latter is, by the way, also a special case of the embedding theorem for Besov spaces (see further discussion in § 5, also [25]).

In § 6, we discuss applications and several counterexamples. We have not striven for completeness, and have concentrated on applications which specifically require the theorems of this paper (*i.e.* are not obtainable from [20]). We also remark here that, although we work in $L^p(R^m)$, *all theorems of this paper remain valid (mutatis mutandis) in the corresponding spaces of functions periodic in each variable.*

2. Definitions and notation

2.1. By $R^m (m \geq 1)$ we denote real Euclidean m-space. We shall always use the letters $t = (t_1, \ldots, t_m)$ and $u = (u_1, \ldots, u_m)$ to denote points of R^m. In the context of Fourier analysis we shall sometimes prefer, for conceptual clarity, to speak of another copy $\hat{R}^m$ of Euclidean m-space, thought of as the dual group of R^m. We shall use the letters x, y to denote points of $\hat{R}^m$. We write tu for $\sum_{j=1}^m t_j u_j$ (similarly tx, etc.), $\|t\| = (tt)^{1/2}$, and dt (similarly du, dx, etc.) denotes m-dimensional Lebesgue measure.

2.2. By p we mean a positive number (or $+\infty$), *where always* $1 \leq p \leq \infty$. $L^p = L^p(R^m)$ shall have its customary meaning, and $\|f\|_p$ denotes the L^p norm of f. *By* $\bar{p}$ *we always mean the number*

$$\tilde{p} = \min(p, 2), \quad \text{if} \quad 1 \leq p < \infty,$$
$$\tilde{p} = 1, \qquad\qquad \text{if} \quad p = \infty.$$

2.3. $M = M(R^m)$ denotes the set of bounded complex measures on the Lebesgue measurable sets of R^m. We use Greek letters (especially $\lambda, \mu, \nu, \varrho, \sigma, \tau$) to denotes elements of M, and $V(\sigma)$ denotes the total variation of σ. As usual, we use $\hat{\sigma}$ to denote the Fourier (- Stieltjes) transform of σ, defined by

$$\hat{\sigma}(x) = \int e^{-ixt} d\sigma_t .$$

$W = W(\hat{R}^m)$ denotes the Banach algebra of Fourier transforms of elements of M, where $\|\hat{\sigma}\|_W = V(\sigma)$ and »multiplication» is ordinary (pointwise) multiplication, corresponding to the convolution »product» (written $*$) in the isomorphic ring M. By abuse of language we also write, for $f \in L^1$, $\hat{f}$ to denote the usual Fourier transform (i.e. the Fourier transform of the measure $f\,dt$). We also write $f * \sigma$, where $f \in L^p$ and $\sigma \in M$, to denote the function whose value at t is $\int f(t-u)d\sigma_u$. It is defined a.e. and satisfies

$$\|f * \sigma\|_p \leq V(\sigma)\|f\|_p .$$

Occasionally, tempered distributions will enter into the discussion, and we follow the standard notational conventions of the L. Schwartz theory (i.e. $*$ denotes convolution and $\hat{T}$ the Fourier transform of T).

2.4. For $\sigma \in M$ and $a > 0$, $\sigma_{(a)}$ denotes the measure defined by $\sigma_{(a)}(E) = \sigma(a^{-1}E)$ for all measurable sets E. This is equivalent to $\int f\,d\sigma_{(a)} = \int (S_a f) d\sigma$ for all bounded continuous f, where $(S_a f)(t) = f(at)$, and also to $\hat{\sigma}_{(a)}(x) = \hat{\sigma}(ax)$. For a function f we shall also write $f_{(a)}$ to denote the function whose value at t is $a^{-m}f(t/a)$. This is in conformity with the preceding definition, since for $f \in L^1$ we have $\hat{f}_{(a)}(x) = \hat{f}(ax)$. For a distribution T, $T_{(a)}$ is defined in the same way, i.e. $T_{(a)} = TS_a$.

2.5. For $f \in L^p$, $\sigma \in M$ and $a > 0$ we define

$$D_{\sigma,p}(f\,;a) = \|f * \sigma_{(a)}\| \qquad \textit{(the } \sigma,p \textit{ deviation)}$$
$$\omega_{\sigma,p}(f\,;a) = \sup_{0<b\leq a} D_{\sigma,p}(f\,;b) \quad \textit{(the } \sigma,p \textit{ modulus)}.$$

It is not hard to prove that for fixed σ, p and f, with $1 \leq p < \infty$, $\omega_{\sigma,p}(f\,;a)$ is a non-decreasing and uniformly continuous function of a, bounded by $V(\sigma)\|f\|_p$. In most cases of interest $\sigma(R^m) = 0$, and then $\omega_{\sigma,p}(f\,;a) \to 0$ as $a \to 0$. These conclusions are valid also for $p = \infty$, provided f is uniformly continuous on R^m.

3. Inequalities »of Littlewood-Paley type»

3.1. *A special partition of unity.*

LEMMA. *There exists a function* Φ *on* $\hat{R}^m$ *which is infinitely differentiable at all points, and such that further*

(i) $\Phi(x) > 0$ *for* $1/2 < |x| < 2$, *and* Φ *vanishes elsewhere,*

(ii) $\sum_{j=-\infty}^{\infty} \Phi(2^j x) = 1$ *if* $x \neq 0$.

Proof. Let $h(a)$ denote a function defined for $0 \leq a < \infty$ and equal to one for $0 \leq a \leq 1$, to zero for $a \geq 2$, strictly decreasing on [1, 2], and infinitely differentiable. Then $\Phi(x) = h(|x|) - h(2|x|)$ satisfies the requirements. Observe that in the series (ii) at most two terms are different from zero, for each $x \neq 0$.

Φ is the Fourier transform of a certain function φ which is infinitely differentiable, and φ and all its partial derivatives tend to zero at infinity more rapidly than any negative power of $|t|$.

We shall, throughout this paper, write φ_j to denote $\varphi_{(2^j)}$, i.e.

$$\varphi_j(t) = \varphi_{(2^j)}(t) = 2^{-mj}\varphi(2^{-j}t)\,, \quad j = 0\,, \pm 1\,, \ldots \tag{1}$$

Observe that

$$\hat{\varphi}_j(x) = \Phi(2^j x)\,, \quad j = 0\,, \pm 1\,, \ldots \tag{2}$$

We shall also require the relation, for positive integral r,

$$\sum_{j=-r}^{r} \Phi(2^j x) = 1 \quad \text{for} \quad 2^{-r} \leq |x| \leq 2^r\,, \tag{3}$$

which follows from (ii), since for x in this range, and $|j| > r$, $\Phi(2^j x) = 0$.

In the analysis which follows, we shall always suppose the dimension m and the choice of a particular Φ with the properties enumerated in the lemma to have been fixed, and treat as »constants» numbers which depend only on m and Φ.

The reader wishing to move on to § 4 need only consult sections 3.2 and 3.3.4 for the essential preparatory material.

3.2. THEOREM. *Let* T *be a tempered distribution in* R^m, *and let* φ_j *be the functions defined in* 3.1. *Suppose moreover, for some* p $(1 \leq p \leq \infty)$ $T * \varphi_j$ *belongs to* L^p *for each* j *and*

$$\sum_{j=-\infty}^{\infty} \left(\int |(T * \varphi_j)(t)|^p\, dt\right)^{\bar{p}/p} = \sum_{j=-\infty}^{\infty} \|T * \varphi_j\|_p^{\bar{p}} = B < \infty\,, \tag{1}$$

where $\bar{p}$ *is defined by*

$$\begin{aligned} \bar{p} &= \min(p\,, 2)\,, \quad && \textit{if} \;\; 1 \leq p < \infty\,, \\ \bar{p} &= 1\,, && \textit{if} \;\; p = \infty\,. \end{aligned} \tag{2}$$

Then T is a function, which is representable in the form $f + P$, where P is a polynomial and $f \in L^p$, moreover $\|f\|_p^{\bar{p}} \leq C_p B$, where C_p depends only on p.

Remarks. 1. Note that $T * \varphi_j$ is meaningful, and is a function of class C^∞, because φ_j is the Fourier transform of an infinitely differentiable function with compact support.

2. Observe that any polynomial P satisfies $P * \varphi_j \equiv 0$ for all j.

The proof of the above theorem is rather long, and will occupy most of this section. The essential step is the following inequality.

3.3. LEMMA. Let $f \in L^p(R^m)$, *where* $1 < p < \infty$ *and suppose moreover that the spectrum of f (i.e. the support of its distributional Fourier transform) is compact and does not contain the origin. Then, if φ_j denote the functions defined in* 3.1,

$$\int |f|^p \, dt \leq A_p^p \int \Big(\sum_{-\infty}^{\infty} |(f * \varphi_j)(t)|^2\Big)^{p/2} dt \,, \tag{1}$$

where A_p is a constant depending only on p.

It is sometimes convenient to write the inequality in the form

$$\|f\|_p \leq A_p \|F\|_p \,, \tag{2}$$

where

$$F(t) = \Big(\sum_{-\infty}^{\infty} |(f * \varphi_j)(t)|^2\Big)^{1/2} . \tag{3}$$

We emphasize that, in contradistinction to Theorem 3.2, in the present lemma we presuppose $1 < p < \infty$.

3.3.1. Let us define

$$G(x) = \sum_{-\infty}^{\infty} \hat{\varphi}(2^j x)^2 \,, \quad x \in \hat{R}^m \,.$$

Then, for $x \neq 0$, $G(x) \geq b^2$ where b is the (positive) minimum of $\hat{\varphi}(x)$ for $3/4 \leq |x| \leq 3/2$. Hence $\hat{\varphi}/G$ vanishes outside the »spherical shell» $\{1/2 < |x| < 2\}$ and is infinitely differentiable on $\hat{R}^m$, therefore it is the Fourier transform of a certain function $\psi \in L^1(R^m)$. One readily verifies, using the fact that $G(2^j x) = G(x)$ for all integers j,

$$\sum_{j=-\infty}^{\infty} \hat{\varphi}(2^j x)\hat{\psi}(2^j x) = 1 \,, \quad x \neq 0 \,. \tag{1}$$

It is easily seen from (1) that if r is a positive integer

$$\sum_{j=-r}^{r} \hat{\varphi}(2^j x)\hat{\psi}(2^j x) = 1 \,, \quad \text{for } \ 2^{-r} \leq |x| \leq 2^r \,. \tag{2}$$

We shall consistently write ψ_j to denote the function $\psi_{(2^j)}$.

3.3.2. Fix a positive integer r, and let $E = E_r$ denote the $(2r+1)$-dimensional Hilbert space of complex sequences $c = (c_{-r}, c_{-r+1}, \ldots c_r)$ with the usual l^2 norm

$$\|c\|_E = (\sum_{-r}^{r} |c_j|^2)^{1/2}.$$

By $L^p(R^m, E)$ we denote the Banach space of (vector-valued) functions F, defined in R^m, taking values in E, measurable, and such that

$$\|F\| = \|F\|_{L^p(R^m, E)} = \left(\int_{R^m} \|F(t)\|_E^p\, dt\right)^{1/p}$$

is finite. We denote by $F_j(t)$ the j^{th} component of $F(t)$.

We now define a function K whose domain is R^m and whose value at the point t is the linear map (functional) from E to the complex numbers defined by

$$K(t)c = \sum_{j=-r}^{r} \psi_j(t)c_j\,, \quad c \in E\,. \tag{1}$$

Consider now, for $F \in L^p(R^m, E)$ the map

$$F \to \int_{R^m} K(t-u)F(u)du\,. \tag{2}$$

We propose to show that *this is a bounded map from* $L^p(R^m, E)$ *to* $L^p(R^m)$ *if* $1 < p < \infty$, *i.e.*

$$\left(\int \left| \int K(t-u)F(u)du \right|^p dt\right)^{1/p} \le A_p \|F\|_{L^p(R^m, E)}\,, \tag{3}$$

where A_p depends on p, but not on F nor r. To establish this it is enough, by virtue of Theorem 2 of [1], to verify: (i) that (3) holds for some *particular* choice of $p > 1$ (we shall make the choice $p = 2$, as is usual when employing this method), and (ii) the estimate

$$\int_{|t| \ge 4|u|} \|K(t-u) - K(t)\| dt \le C \tag{4}$$

for all $u \in R^m$. Here $\|\cdot\|$ denotes the operator norm, and C must be independent of u. The constant A_p in (3) can be determined in terms of C, A_2, and p. As for (i): for $p = 2$, the left side of (3) is

$$\left\{\int \left| \int [\sum_{j=-r}^{r} \psi_j(t-u)F_j(u)]du \right|^2 dt\right\}^{1/2},$$

and by Parseval's identity the square of this expression equals (we assume here the Fourier transform suitably normalized, so we may suppress factors of $(2\pi)^m$):

$$\int |\sum_{j=-r}^{r} \hat{\varphi}(2^j x)\hat{F}_j(x)|^2\, dx \leq \int (\sum_{-r}^{r} |\hat{\varphi}(2^j x)|^2)(\sum_{-r}^{r} |\hat{F}_j(x)|^2)\, dx \leq$$
$$\leq 2M^2 \int (\sum_{-r}^{r} |\hat{F}_j(x)|^2)\, dx = 2M^2 \sum_{-r}^{r} \int |F_j(t)|^2\, dt = 2M^2 \int \|F(t)\|_E^2\, dt\,,$$

where $M = \max |\hat{\varphi}(x)|$, since for any x there are at most two values of j such that $\hat{\varphi}(2^j x) \neq 0$. Taking square roots gives (3), with $p = 2$. As for (ii) (i.e. the proof of (4)) this is harder. We have

$$\|K(t-u) - K(t)\|^2 = \sum_{j=-r}^{r} |\varphi_j(t-u) - \varphi_j(t)|^2 \leq \tag{5}$$
$$\leq \sum_{j=-\infty}^{\infty} 2^{-2mj} |\varphi(2^{-j}(t-u)) - \varphi(2^{-j}t)|^2 \leq \sum_{j=-\infty}^{\infty} 2^{-2mj} \cdot 2^{-2j} |u|^2 Q(2^{-j}|t|)^2$$

if $|t| \geq 4|u|$, where $Q(\alpha)$ denotes the maximum of $|\text{grad}\, \varphi(t)|$ in the spherical shell $\{3\alpha/4 \leq |t| \leq 5\alpha/4\}$. Since φ and all its partial derivatives vanish at infinity faster than any negative power of $|t|$, Q is bounded and, for a suitable constant C_1

$$Q(\alpha) \leq C\alpha^{-(m+2)}\,, \quad \alpha \geq 1\,.$$

Therefore, writing

$$R(t) = \{\sum_{j=-\infty}^{\infty} 2^{-2(m+1)j}\, Q(2^{-j}|t|)^2\}^{1/2} \tag{6}$$

the series on the right converges for $|t| \neq 0$. Now, in order to prove (4) we have only, in view of (5), to establish, for $a > 0$

$$\int_{|t| \geq a} R(t)\, dt \leq Ca^{-1} \tag{7}$$

with C independent of a. Now, replacing t by $2t$ in (6) and making the change of variable $j \to j + 1$ we see that

$$R(2t) = 2^{-(m+1)} R(t)\,, \quad 0 \neq t \in R^m\,,$$

so that $S(t) = |t|^{m+1} R(t)$ satisfies $S(t) = S(2t)$ and since $S(t)$ is bounded on $1 \leq |t| \leq 2$ it is bounded on $|t| > 0$, in other words

$$R(t) \leq C_2 |t|^{-(m+1)}\,, \quad 0 \neq t \in R^m\,.$$

This implies (7), and thus we can now assert that (3) holds.

3.3.3. We can now easily prove Lemma 3.3. Indeed, suppose $f \in L^p(R^m)$ and the spectrum of f omits neighborhoods of zero and infinity. Fix an integer r so large that the spectrum lies in the open set $\{2^{-r} < |x| < 2^r\}$. We now apply the result of the previous paragraph to the (vector valued) function F whose value at the point $t \in R^m$ is $\{(f * \varphi_j)(t)\}_{j=-r}^{r}$. Observe that $\int K(t-u)F(u)du$ is just the convolution of f with $\sum_{-r}^{r} \varphi_j * \psi_j$, whose Fourier transform is

$$\sum_{j=-r}^{r} \hat{\varphi}(2^j x)\hat{\psi}(2^j x)\,,$$

and this equals 1 for $2^{-r} \leq |x| \leq 2^r$, as remarked in 3.3.1. Since the spectrum of f is contained in the interior of this set, we conclude that

$$\int K(t-u)F(u)du = f(t) \quad \text{a.e.}$$

Applying 3.3.2(3) we get

$$\|f\|_p \leq A_p \left\{ \int \Big(\sum_{j=-r}^{r} |(f * \varphi_j)(t)|^2\Big)^{p/2} dt \right\}^{1/p},$$

completing the proof of Lemma 3.3. (It is essential to observe here that the bound A_p obtained in the previous paragraph did not depend upon the choice of the integer r).

3.3.4. Actually we require not Lemma 3.3 but a corollary of it, as follows: *Under the hypotheses of Lemma* 3.3, *we have*

$$\int |f|^p dt \leq A_p^p \left\{ \sum_{-\infty}^{\infty} \left(\int |(f * \varphi_j)(t)|^p dt \right)^{\bar{p}/p} \right\}^{p/\bar{p}} \tag{1}$$

where A_p *depends only on* p.

Indeed, writing f_j for $f * \varphi_j$, we observe, using the elementary inequality

$$\Big(\sum x_j\Big)^\alpha \leq \sum x_j^\alpha\,, \quad x_j \geq 0\,,\ 0 < \alpha \leq 1\,,$$

that $(\sum |f_j(t)|^2)^{p/2} \leq \sum |f_j(t)|^p$ for $p \leq 2$. Therefore, the right side of (1) majorizes that of 3.3(1) when $p \leq 2$.

On the other hand, when $p \geq 2$ we have, applying Minkowski's inequality with exponent $p/2$

$$\left(\int \Big(\sum |f_j|^2\Big)^{p/2} dt \right)^{2/p} \leq \sum \left(\int (|f_j|^2)^{p/2} dt \right)^{2/p} = \sum \left(\int |f_j|^p dt \right)^{2/p}$$

so that, also in the case $p \geq 2$, (1) is a consequence of 3.3(1).

Observe that any choice of A_p which works in 3.3(1) also renders (1) valid. Also, (1) can be written more compactly as

$$\|f\|_p^{\bar{p}} \leq A_p \sum_{-\infty}^{\infty} \|f * \varphi_j\|_p^{\bar{p}}\,. \tag{2}$$

So far, (2) is established only under the restriction $1 < p < \infty$. However, since $f = \sum f * \varphi_j$ (the series containing only finitely many non-zero terms, under the stated hypotheses about f), (2) is valid trivially for $p = 1$ and $p = \infty$ (recall that we interpret $\bar{p}$ as 1 in the latter case). More generally, Minkowski's inequality gives

$$\|f\|_p \leq \sum_{-\infty}^{\infty} \|f * \varphi_j\|_p \,. \tag{3}$$

It is worth while to compare (2) and (3): for $1 < p < \infty$, (2) is stronger, and is the key to the »sharp» results which we shall obtain in the present paper, whereas an analogous use of (3) would lead only to the results in [20] (see § 4.3.1 below).

3.4. We can now complete the proof of Theorem 3.2. By hypothesis, we have $T * \varphi_j \in L^p$ for each j. We claim that for arbitrary k and n, $k \leq n$,

$$\|\sum_{k}^{n} T * \varphi_j\|_p^{\bar{p}} \leq A_p \sum_{k-1}^{n+1} \|T * \varphi_j\|_p^{\bar{p}} \,. \tag{1}$$

To prove (1), let $g = \sum_{j=k}^{n} T * \varphi_j$, and note that $g \in L^p$ and that the spectrum of g is compact and does not contain the origin. Hence we can apply 3.3.4 (2) to g and obtain

$$\|g\|_p^{\bar{p}} \leq A_p \sum_{-\infty}^{\infty} \|g * \varphi_i\|_p^{\bar{p}} = A_p \sum_{k-1}^{n+1} \|g * \varphi_i\|_p^{\bar{p}} \,. \tag{2}$$

The last equality holds because $\varphi_i * \varphi_j = 0$ for $|i - j| > 1$. Similarly, for each i the sum

$$\sum_{j=k}^{n} T * \varphi_j * \varphi_i = g * \varphi_i$$

can contain at most three non-vanishing terms. Hence

$$\|g * \varphi_i\|_p \leq 3C \, \|T * \varphi_i\|_p \,, \quad i = 0 \,, \pm 1 \,, \ldots \,, \tag{3}$$

where $C = \|\varphi\|_1$. Combination of (2) and (3) gives (1) (with a new constant A_p).

Using (1) we see that

$$\sum_{-n}^{n} T * \varphi_j$$

is a Cauchy sequence in L^p if the hypothesis of Theorem 3.2 is fulfilled. Let $f \in L^p$ be the limit of this sequence. Clearly

$$\|f\|_p^{\bar{p}} \leq A_p \sum_{-\infty}^{\infty} \|T * \varphi_j\|_p^{\bar{p}} \,.$$

From the definition of f it follows that $(T - f) * \varphi_j = 0$ for all j. This shows that the (distributional) Fourier transform of $T - f$ is supported at the origin. But a distribution with support at the origin must be a finite linear combination of the Dirac functional and its derivatives ([16], p. 100), hence $T - f$ is a polynomial.

Remark. It is evident that T cannot have two distinct representations of the form $f+P$, except in the case $p=\infty$ when the representation is unique only modulo an additive constant.

3.5. We wish here to make an observation that is important for applications later. The constant A_p in Lemma 3.3 and C_p in Theorem 3.2 depend upon the initial choice of Φ (and so of the φ_j). However *if we replace $\Phi(x)$ by $\Phi(ax)$ (where $a>0$), so that the system $\{\varphi_{(2^j)}\}$ becomes replaced by $\{\varphi_{(a2^j)}\}$, Lemma 3.3 and Theorem 3.2 remain valid, with the same constants.* This is seen by simply replacing f by $f_{(1/a)}$ (resp. T by $T_{(1/a)}$) and making the change of variable $t \to t/a$.

3.6. In the case $1 \leq p \leq 2$ one could prove 3.3.4(2) using the Riesz-Thorin interpolation theorem instead of the Marcinkiewicz interpolation theorem on which the proof above is based (see below). The Riesz-Thorin theorem ([31], p. 95) implies that the set of values of p for which a linear operator is continuous from $L^p(X)$ to $L^p(Y)$ (X and Y are arbitrary measure spaces) is an interval. To deduce 3.3.4(2) from this theorem we observe that

$$\left(\sum \|f * \varphi_j\|_p^p\right)^{1/p}$$

is the L^p-norm of the function

$$F(j,t) = f * \varphi_j(t)$$

defined in $Z \times R^m$, where Z denotes the set of integers. Now it has been proved above that the operator U defined by

$$U(f) = f * \varphi_j(t)$$

is a bounded operator from $L^p(R^m)$ to $L^p(Z \times R^m)$ for $p=1$ and 2. Hence, by the Riesz-Thorin theorem, 3.3.4(2) holds for $1 \leq p \leq 2$, and in fact with a constant A_p independent of p. On the other hand, the proof given above, based on the Marcinkiewicz interpolation theorem shows only that A_p is bounded in each compact subinterval of $1 < p < \infty$.

3.7. *Historical comments concerning Theorem 3.2.* The prototype of Theorem 3.2 (or Lemma 3.3), as well as their converses, is found in Littlewood and Paley [10], and is in terms of Fourier *series* rather than integrals. The original proof utilized analytic functions in a way that made extensions to several variables seem quite difficult.

The first to prove an inequality of this type by real variable methods was E. M. Stein [23]. A simplification and extension of Stein's result was later given by Hörmander [5]. Hörmander's method was based on a combination of the Calderón-Zygmund technique for estimating singular integrals and the Marcinkiewicz interpolation theorem. It would have been possible for us to deduce Theorem 3.2 from

Theorem 3.5 of Hörmander's paper dealing with so-called »mixed L^2 estimates» (he employs a continuous, rather than a discrete parameter), or else to paraphrase the proof of the latter theorem in our context. However, we followed a somewhat different path. Namely, as remarked by J. Schwartz [17], and independently Benedek, Calderón and Panzone [1], (L^p, L^2) mixed norm estimates involving a parameter can often be most conveniently dealt with by proving an L^p estimate for *vector-valued* functions (with values in a suitable L^2 space), acted on by convolution with *operator-valued* kernels. They observed that the theorems of Marcinkiewicz and Calderón-Zygmund, the basis of Hörmander's method, extend to this more general context. Our Theorem 3.2 is essentially taken from Peetre's lecture notes [14]. Peetre based his proof on an operator-valued »Mihlin-type theorem» which we could not locate in the literature; its verification would involve an analysis similar to that given above.

There is by now a rather considerable literature dealing with extensions of the Littlewood-Paley theorem, and the interested reader is referred to [9], [11], [14] where also further references may be found.

4. Comparison theorems

4.1. We begin by recapitulating and extending some basic lemmas from [20], contenting ourselves in part with references to that paper and others for proofs.

4.1.1. We recall that if $\sigma \in M(R^m)$, then the map $f \to \sigma * f$ from L^p to L^p is continuous, with bound not exceeding $V(\sigma)$.

4.1.2. If we consider a tempered distribution σ which is not necessarily a bounded measure, the inequality $\|\sigma * f\|_p \leq A\|f\|_p$ with $A = A_p$ independent of f, may hold for all functions $f \in L^p$. For a discussion of such distributions see Hörmander [5]. The Fourier transform $\hat{\sigma}(x)$ is then necessarily a bounded measurable function, called a *Fourier multiplier* (associated with the exponent p). It is known (see [5]) that the class M_p of Fourier multipliers associated with the exponent p is identical with $M_{p'}$ $(p' = p/(p-1))$, moreover $M_p \subset M_q$ for $1 \leq p \leq q \leq 2$. The remark in the preceding paragraph implies $W \subset M_p$ for all $p \geq 1$. It is known, moreover, that $M_1 = M_\infty = W$, and $M_2 = L^\infty$. It is the fact that the elements of W are in all the M_p that explains the important role which bounded measures and their Fourier transforms play in the present investigation, as well as in studies of L^p inequalities relating differential operators ([3], [4], [12]). The function $\operatorname{sgn} x$ on $\hat{R}^1$ is the classic example of a function not in W which is nevertheless in M_p for $1 < p < \infty$, the corresponding convolution operator being the Hilbert transform.

4.1.3. If $\sigma, \tau \in M(R^m)$ and $\hat{\sigma}$ divides $\hat{\tau}$ in W, then for $1 \leq p \leq \infty$

$$D_{\tau,p}(f;a) \leq C D_{\sigma,p}(f;a) \tag{1}$$

$$\omega_{\tau,p}(f;a) \leq C \omega_{\sigma,p}(f;a) \tag{2}$$

for all $f \in L^p$. Here C can be taken as the W-norm of any element $F \in W$ such that $\hat{\tau} = \hat{\sigma}F$ (F is not always uniquely determined). These inequalities follow easily from 4.1.1 (cf. [20]). In view of the discussion in 4.1.2 they remain valid even if $\hat{\sigma}$ divides $\hat{\tau}$ in M_p. In this case C must be taken to be the M_p-norm (see [5]) of the function F. We remark, too, that if σ is a tempered distribution such that $\hat{\sigma} \in M_p$, then also $\hat{\sigma}(ax) \in M_p$, and so $D_{\sigma,p}(f;a)$ and $\omega_{\sigma,p}(f;a)$ can still be defined in the obvious way for f belonging to L^p; likewise for τ, and (1), (2) are then valid providing $\hat{\sigma}$ divides $\hat{\tau}$ in M_p.

For conciseness we shall state our theorems below in terms of elements of W rather than M_p, but shall indicate some results which are valid in the wider context, as these are useful in some applications to approximation theory.

Finally, the analogues of (1), (2) hold when $\hat{\tau}$ has a representation $\sum_{i=1}^{n} \hat{\varrho}_i \hat{\sigma}_i$ where $\hat{\varrho}_i$ are elements of W (or, more generally, of M_p); in this case one has n summands on the right side of the inequality (cf. [20], p. 286).

4.1.4. A continuous function F on $\hat{R}^m$ is said to *satisfy the Tauberian condition* if, for every x with $|x| = 1$, there exists $c \geq 0$ such that $F(cx) \neq 0$, in other words, if F takes a non-zero value on every closed half-ray. (If, in particular, $F(0) \neq 0$, F satisfies the Tauberian condition trivially.) By way of orientation we may remark that, for $f \in L^1$, the Tauberian condition on $\hat{f}(x)$ is necessary and sufficient that every $g \in L^1$ with $\hat{g}(0) = 0$ be approximable (in L^1 norm) by finite linear combinations of the functions $f(ct + u)$, where $c > 0$ and $u \in R^m$. (This is a simple consequence of Wiener's Theorem.)

4.1.5. LEMMA. *If F is continuous on $\hat{R}^m$ and satisfies the Tauberian condition, $\delta > 0$ is arbitrary, there exist positive numbers $d_1 < d_2 \ldots < d_r$ such that*

$$\sum_{j=1}^{r} |F(d_j x)| > 0\,, \quad \delta \leq |x| \leq 1/\delta\,.$$

Proof. Let S denote $\{x : |x| = 1\}$ and F_c, for $c > 0$, the function on S defined by $F_c(x) = F(cx)$. The hypotheses imply $\{F_c\}_{c>0}$ have no common zero on S, i.e. the closed subsets E_c of S defined by $E_c = \{x | F_c(x) = 0\}$ have an empty intersection. Therefore, since S is compact, there is some finite subset $E_{c_1}, \ldots, E_{c_r}$ of the E_c whose intersection is empty. This means that $G(x) = \sum_{i=1}^{r} |F(c_i x)|$ is positive on S, and hence by continuity remains positive in the spherical shell $b \leq |x| \leq 1/b$ if b is chosen sufficiently close to 1. Therefore, if k is chosen large enough, $\sum_{j=-k}^{k} G(b^j x)$ is positive for $\delta \leq |x| \leq 1/\delta$, which implies the assertion in the lemma.

4.1.6. Lemma. *If* $\sigma, \tau \in M(R^m)$, $\hat{\sigma}$ *satisfies the Tauberian condition and* $\hat{\tau}$ *vanishes in neighborhoods of the origin and infinity, then for* $f \in L^p$ $(1 \leq p \leq \infty)$ *and* $a > 0$,

$$\omega_{\tau,p}(f\,;a) \leq A\omega_{\sigma,p}(f\,;Ba) \tag{1}$$

where A, B *depend only on* σ *and* τ.

Proof. By Lemma 4.1.5, there exist positive d_j such that $\sum_{j=1}^r |\hat{\sigma}(d_j x)|$ is positive on the (compact) support of $\hat{\tau}$, therefore (cf. [20], p. 282) τ belongs to the ideal in M generated by $\sigma_{(d_1)}, \dots, \sigma_{(d_r)}$. Therefore

$$D_{\tau,p}(f\,;a) \leq \sum_{j=1}^{r} A_j D_{\sigma,p}(f\,;d_j a) \leq \sum_{j=1}^{r} A_j \omega_{\sigma,p}(f\,;d_j a) \leq A\omega_{\sigma,p}(f\,;Ba)\,,$$

where $B = \max d_j$ and $A = \sum A_j$ depend only on σ and τ. For $0 < b \leq a$ we have, therefore,

$$D_{\tau,p}(f\,;b) \leq A\omega_{\sigma,p}(f\,;Bb) \leq A\omega_{\sigma,p}(f\,;Ba)$$

which implies (1).

4.2. As in [20], we wish now to remove the restriction that $\hat{\tau}$ in Lemma 4.1.6 vanishes in a neighborhood of infinity. The method used in [20] leads to an estimate of the form

$$\omega_{\tau,p}(f\,;a) \leq A \int_0^{Ba} \omega_{\sigma,p}(f\,;v)\,\frac{dv}{v}\,. \tag{1}$$

(The formulation in [20] was in terms of D rather than ω, and infinite sums rather than integrals were employed, but these differences are not essential.)

The essential novelty of the present paper, the technical backbone of which is the following lemma, is to replace (1) by a finer estimate which takes better account of the concrete choice of p.

Lemma. *If* $\sigma, \tau \in M(R^m)$, $\hat{\sigma}$ *satisfies the Tauberian condition, and* $\hat{\tau}$ *vanishes in a neighborhood of the origin, then for* $f \in L^p$ $(1 \leq p \leq \infty)$ *and* $a > 0$

$$\omega_{\tau,p}(f\,;a)^{\bar{p}} \leq A \int_0^{Ba} \omega_{\sigma,p}(f\,;v)^{\bar{p}}\,\frac{dv}{v}\,, \tag{2}$$

where A, B *depend only on* σ, τ *and* p.

Proof. Observe that $f * \tau_{(a)}$ is in L^p. By 3.3.4(2) in conjunction with the remark in § 3.5, we have, for any $b > 0$

$$\|f * \tau_{(a)}\|_p^{\bar{p}} \leq A_p \sum_{j=-\infty}^{\infty} \|f * \tau_{(a)} * \varphi_{(2^j b)}\|_p^{\bar{p}}\,, \tag{3}$$

where A_p depends on p only (we emphasize the fact that it does not depend on b). Now, the Fourier transform of $\varphi_{(2^j b)}$ is supported in $2^{-j-1}b^{-1} \leq |x| \leq 2^{-j+1}b^{-1}$. By hypothesis $\hat{\tau}$ vanishes in a neighborhood of the origin, say for $|x| \leq c$, hence $\hat{\tau}_{(a)}$ vanishes for $|x| \leq c/a$ and so $\tau_{(a)} * \varphi_{(2^j b)} = 0$ if $2^{-j+1}b^{-1} \leq c/a$. Let us make the choice $b = 2a/c$, so that the latter condition is fulfilled for $j \geq 0$. Then, on the right side of (3) we need consider only negative values of j, and the sum becomes

$$\sum_{j=-\infty}^{-1} \|f * \tau_{(a)} * \varphi_{(2^j b)}\|_p^{\bar{p}} \leq V(\tau)^{\bar{p}} \sum_{j=-\infty}^{-1} \|f * \varphi_{(2^j b)}\|_p^{\bar{p}} . \tag{4}$$

Now, the summands on the right can be estimated by Lemma 4.1.6, since the measure φdt (which here plays the role of τ in Lemma 4.1.6) has a Fourier transform which vanishes in neighborhoods of the origin and infinity. We get

$$\|f * \varphi_{(2^j b)}\|_p \leq A\omega_{\sigma,p}(f\,;\, B2^j b) \tag{5}$$

where A , B depend only on σ (φ being considered fixed once for all). Finally, from (3), (4), (5), and recalling that $b = 2a/c$,

$$\|f * \tau_{(a)}\|_p^{\bar{p}} \leq A_p V(\tau)^{\bar{p}} \sum_{n=1}^{\infty} A^{\bar{p}} \omega_{\sigma,p}(f\,;\, Bc^{-1}2^{-n+1}a)^{\bar{p}} = C_1 \sum_{n=1}^{\infty} \omega_{\sigma,p}(f\,;\, C_2 2^{-n}a)^{\bar{p}} ,$$

where C_1 and C_2 depend only on σ , τ and p. This completes the proof, once we take account of the elementary inequality

$$\sum_{n=1}^{\infty} \psi(n) \leq \int \psi(t)dt$$

for functions ψ continuous and decreasing for $0 \leq \lambda < \infty$, and apply it to $\psi(\lambda) = \omega_{\sigma,p}(f\,;\, C_2 2^{-\lambda}a)^{\bar{p}}$.

Remark. Observe that, using 3.3.4(3) in the above argument in place of 3.3.4(2), we obtain the weaker estimate (1) instead of (2). This remark has some methodological interest, since (1) is for many purposes as useful as (5), but does not require for its proof the deep inequality 3.3.4(2). Moreover, the same method of proof establishes (1) not only for $L^p(R^n)$, but for a large class of Banach spaces with translation-invariant norm (cf. Shapiro [22], Chapter 9).

4.3. THEOREM. *Suppose σ , $\tau \in M(R^m)$, $\hat{\sigma}$ satisfies the Tauberian condition, and there exists $F \in W$ such that $\hat{\tau}(x) = \hat{\sigma}(x)F(x)$ for x in some neighborhood of the origin. Then for $f \in L^p$ $(1 \leq p \leq \infty)$ and $a > 0$*

$$\omega_{\tau,p}(f\,;\, a)^{\bar{p}} \leq A \int_0^{Ba} \omega_{\sigma,p}(f\,;\, v)^{\bar{p}} \frac{dv}{v} \tag{1}$$

where A and B depend only on σ, τ, and p.

Proof. By hypothesis $\hat{\tau} = \hat{\sigma}\hat{\mu} + \hat{\nu}$ where $\mu, \nu \in M$ and $\hat{\nu}$ vanishes in a neighborhood of the origin. Then for $f \in L^p$ and $a > 0$

$$D_{\tau,p}(f\,;a) \leq D_{\sigma*\mu,p}(f\,;a) + D_{\nu,p}(f\,;a) \leq V(\mu)D_{\sigma,p}(f\,;a) + D_{\nu,p}(f\,;a) \leq$$
$$\leq V(\mu)\omega_{\sigma,p}(f\,;a) + \omega_{\nu,p}(f\,;a)\,.$$

Estimating $\omega_{\nu,p}(f\,;a)$ by the previous lemma, and observing that

$$\omega_{\sigma,p}(f\,;a) \leq \int_a^{ea} \omega_{\sigma,p}(f\,;v)\,\frac{dv}{v}$$

the proof is complete. (Note that B is actually independent of p.)

Remarks. Since the restriction to any compact set of a Fourier-Stieltjes transform coincides with the restriction to that set of the Fourier transform of an integrable function, it would involve no loss of generality in Theorem 4.3 to require F to be a function of the latter type. On the other hand, as an examination of the proof shows, Theorem 4.3 remains valid if we only require that F be an element of M_p (see 4.1.2). We will use this in 6.4.

4.3.1. In view of the remark at the end of 4.2, a correct theorem is obtained when the exponent $\bar{p}$ is dropped on both sides of 4.3(1). This is the comparison theorem of [20]; its proof along the lines of the present paper (i.e. based upon a partition of unity) is perhaps simpler, or at least more instructive.

To see that the present theorem is stronger we observe that for $r > 1$ (write $\omega(a) = \omega_{\sigma,p}(f\,;a)$)

$$\left(\int_0^a \omega(v)^r\,\frac{dv}{v}\right)^{1/r} \leq \omega(a)^{(r-1)/r}\left(\int_0^a \omega(v)\,\frac{dv}{v}\right)^{1/r},$$

and since $\omega(a)$ is increasing

$$\omega(a)\log 2 \leq \int_a^{2a} \omega(v)\,\frac{dv}{v} \leq \int_0^{2a} \omega(v)\,\frac{dv}{v}\,.$$

Hence

$$\left(\int_0^a \omega(v)^r\,\frac{dv}{v}\right)^{1/r} \leq (\log 2)^{-\frac{r-1}{r}} \int_0^{2a} \omega(v)\,\frac{dv}{v}\,,$$

which, with $r = \bar{p}$, proves the assertion. It is easy to see by examples that except for $\bar{p} = 1$ (i.e. $p = 1$ or ∞) an inequality in the reverse direction cannot hold, so that Theorem 4.3 is stronger than the comparison theorem in [20].

4.3.2. Just as in [20], we may state in place of 4.3 a more general theorem in which the role of σ is taken over by a set $\sigma_1, \ldots, \sigma_n$ of measures such that $\sum |\hat{\sigma}_i(x)|$ satisfies the Tauberian condition, and $\hat{\tau}$ agrees in a neighborhood of the origin with an element of the ideal in W generated by $\hat{\sigma}_1, \ldots, \hat{\sigma}_n$. On the right side of 4.3(1) we then get an estimate with $\sum_{i=1}^n \omega_{\sigma_i,p}(f;v)^{\bar{p}}$ in place of $\omega_{\sigma,p}(f;v)^{\bar{p}}$. We suppress the details, as the necessary modifications in the above argument are similar to what was done in [20]. This kind of generalization, important for certain applications, applies equally well to the remaining theorems in this paper, and shall henceforth be taken for granted.

4.4. THEOREM. *Suppose* $\sigma, \tau \in M(R^m)$ *and* $\hat{\sigma}$ *satisfies the Tauberian condition. Let* P *be a function which is positive-homogeneous of degree* $r > 0$ *(i.e.* $P(bx) = b^r P(x)$ *for* $b \geq 0$*), and suppose there exist* F, $G \in W$ *such that* $F(x) = P(x)$ *and* $G(x)P(x) = \hat{\tau}(x)$ *for all* x *in some neighborhood of the origin. Then, for* $f \in L^p$ $(1 \leq p \leq \infty)$ *and* $a > 0$

$$\omega_{\tau,p}(f;a)^{\bar{p}} \leq A \int_0^\infty [\min(1, (a/v)^r)\, \omega_{\sigma,p}(f;v)]^{\bar{p}} \frac{dv}{v} \tag{1}$$

where A *depends only on* σ, τ *and* p.

Proof. Write $\tau = \mu + \nu$ and $\hat{\mu}(x) = P(x)\hat{\theta}(x)$, where μ, ν, $\theta \in M(R^m)$, and $\hat{\nu}$ and $\hat{\theta}$ vanish in neighborhoods of the origin and infinity respectively. According to Lemma 4.2, $\omega_{\nu,p}(f, a)$ can be estimated by the integral in (1) taken only from 0 to Ba. To estimate $\omega_{\mu,p}(f, a)$ we use Theorem 3.2 and obtain

$$\|f * \mu_{(a)}\|_p^{\bar{p}} \leq A_p \sum_{-\infty}^{\infty} \|f * \mu_{(a)} * \varphi_{(2^j ca)}\|_p^{\bar{p}}, \tag{2}$$

where c is at our disposal. Since $\hat{\mu}(x) = 0$ for large x, we may choose c such that

$$\hat{\mu}_{(a)}(x) \cdot \varphi_{(2^j ca)}(x) = \hat{\mu}(ax)\hat{\varphi}(2^j cax)$$

is identically zero for $j \leq 0$. This choice of c depends (ultimately) only on the measure τ. For $j > 0$ we rewrite the last expression as follows:

$$\hat{\mu}_{(a)}(x)\, \hat{\varphi}_{(2^j ca)}(x) = P(ax)\, \hat{\theta}(ax)\, \hat{\varphi}(2^j cax) = C_1\, 2^{-jr}\, \hat{\theta}(ax)\, \hat{g}(2^j cax),$$

where $g \in L^1(R^m)$ is defined by

$$\hat{g}(x) = \hat{\varphi}(x) P(x).$$

Hence

$$\|f * \mu_{(a)} * \varphi_{(2^j ca)}\|_p \leq C_1\, 2^{-jr} \|f * \theta_{(a)} * g_{(2^j ca)}\|_p \leq C_1\, V(\theta)\, 2^{-jr} \|f * g_{(2^j ca)}\|_p \leq$$
$$\leq C_2\, V(\theta)\, 2^{-jr}\, \omega_{\sigma,p}(f; 2^j C_3 a),$$

by Lemma 4.1.6, since $\hat{g}$ vanishes in neighborhoods of zero and infinity (here and in the following all constants depend (ultimately) only on σ, τ and p). Using this estimate in (2) we get

$$\|f * \mu_{(a)}\|_p^{\bar{p}} \leq C_4 \sum_{j=1}^{\infty} 2^{-jr\bar{p}} \, \omega_{\sigma,p}(f\,;\, 2^j C_3 a)^{\bar{p}} \,. \tag{3}$$

To estimate the last sum by an integral we use the fact that for any increasing function $w(v)$ and $A > 0$

$$2^{-jA} \, w(2^j) \leq (\log 2)^{-1} \, 2^A \int_{2^j}^{2^{j+1}} w(v) \, v^{-A-1} \, dv \,,$$

and hence

$$\sum_{j=1}^{\infty} 2^{-jA} \, w(2^j) \leq C_6 \int_{2}^{\infty} w(v) \, v^{-A-1} \, dv \,.$$

Applying this to the right side of (3), with $A = r\bar{p}$ and $w(v) = \omega_{\sigma,p}(f\,;\, C_3 a v)^{\bar{p}}$ gives

$$\|f * \mu_{(a)}\|_p^{\bar{p}} \leq C_7 \int_{2}^{\infty} v^{-r\bar{p}-1} \, \omega_{\sigma,p}(f\,;\, C_3 a v)^{\bar{p}} \, dv = C_8 \int_{C_9 a}^{\infty} \left(\frac{a}{v}\right)^{r\bar{p}} \omega_{\sigma,p}(f\,;\, v)^{\bar{p}} \, \frac{dv}{v} \leq$$

$$\leq C_{10} \int_{0}^{\infty} \left[\min\left(1\,,\, \left(\frac{a}{v}\right)^r\right) \omega_{\sigma,p}(f\,;\, v)\right]^{\bar{p}} \frac{dv}{v} \,.$$

Since the last expression is increasing in a, it majorizes $\omega_{\mu,p}(f\,;\, a)^{\bar{p}}$. Combining the estimates of $\omega_{\nu,p}(f\,;\, a)$ and $\omega_{\mu,p}(f\,;\, a)$ gives the result, since

$$\omega_{\tau,p}(f\,;\, a) \leq \omega_{\mu,p}(f\,;\, a) + \omega_{\nu,p}(f\,;\, a) \,.$$

Remark. In applications, it is often convenient to write the estimate for $\omega_{\tau,p}$ in the more extended form

$$\omega_{\tau,p}(f\,;\, a) \leq A^{\bar{p}} \int_{0}^{a} \omega_{\sigma,p}(f\,;\, v)^{\bar{p}} \, \frac{dv}{v} + A a^{r\bar{p}} \int_{a}^{\infty} \omega_{\sigma,p}(f\,;\, v)^{\bar{p}} \, v^{-r\bar{p}-1} \, dv \,. \tag{4}$$

Observe that, since $\omega_{\sigma,p}(f\,;\, v)$ is bounded, the upper limit ∞ in the last integral can be replaced by 1 if we add on a term $Ca^{r\bar{p}}$ — here C will depend of course on f (more precisely, on $\|f\|_p$) but this is of little consequence in most applications.

It is instructive to compare (4) and 4.3(1). The first term on the right of (4) is (apart from a constant factor) identical with the right side of 4.3(1), so that in a situation where Theorems 4.3 and 4.4 are both applicable, the latter can never yield a better estimate for $\omega_{\tau,p}$ than the former. Of the two terms on the right of (4), either one may (under appropriate circumstances) have a larger order of magnitude than the other as $a \to 0$.

5. An »embedding theorem» for distributions

5.1. In order to state our next theorem we introduce some further notation. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ denote a multi-index whose coordinates α_i are non-negative integers, and write $|\alpha|$ for $\sum |\alpha_i|$. By D^α we mean the differential operator $(\partial/\partial t_1)^{\alpha_1} \ldots (\partial/\partial t_m)^{\alpha_m}$. We recall that the *Sobolev space* $W_p^{(s)}(R^m)$ is the set of functions f in $L^p(R^m)$ such that all the derivatives $D^\alpha f$ with $|\alpha| \leq s$ (understood in the distribution sense) are functions of class L^p. If $1 \leq p < \infty$, $W_p^{(s)}$ can also be characterized as the Banach space obtained by forming the completion of the set of functions g which are infinitely differentiable and have compact support, with respect to the norm

$$\|g\| = \left\{ \int \Big(\sum_{|\alpha| \leq s} |D^\alpha g|^p \Big) \, dt \right\}^{1/p}$$

(see, for example, [8] or [29]).

The prototypical »embedding theorem», due to Sobolev, states that $W_p^{(s)}$ is contained in $W_q^{(r)}$, the injection map being continuous, providing

$$\frac{1}{q} = \frac{1}{p} - \frac{s-r}{m}$$

(the right side being assumed positive). An important corollary is that an element of $W_p^{(s)}$, after correction on a set of measure zero, has r continuous derivatives in the ordinary sense, providing $r < s - m/p$. Results of this kind are important in applying the methods of functional analysis to the solution of e.g. elliptic boundary value problems. A vast literature has sprung up generalizing in numerous ways Sobolev's results; for our purposes here the most relevant work is that of Besov [2], in which partial derivatives are replaced by finite differences, and Peetre [15].

5.2. From the point of view of this literature, our two theorems in § 4 are of the nature of »embedding theorems» in the sense that for measures σ,τ with $\sigma(0) = \tau(0) = 0$ the σ,p modulus and the τ,p modulus of a function are (as reflected in their behavior for small values of the parameter a) in some general sense measures of the »smoothness» of a function f in L^p.

5.3. THEOREM. *Suppose $\sigma \in M(R^m)$, $\hat{\sigma}$ satisfies the Tauberian condition and s is a positive integer. Let $1 \leq p \leq \infty$, and suppose $f \in L^p$ and*

$$\int_0^1 [a^{-s} \omega_{\sigma,p}(f; a)]^{\bar{p}} \frac{da}{a} < \infty . \tag{1}$$

Then the distribution $D^\alpha f$ is a function of class L^p for each α such that $|\alpha| \leq s$. In other words, f belongs to the Sobolev space $W_p^{(s)}$.

Remarks. This theorem is very similar to a case of the »embedding theorem» for Besov spaces, namely in the usual notation $B^s_{p,\bar{p}} \subset W^{(s)}_p$. In fact the latter result is a consequence of our theorem, providing we use measures $\sigma_1, \ldots, \sigma_m$ in place of σ, as discussed in 4.3.2. The embedding theorem for Besov spaces is proved in Besov's paper [2] as well as in [14], [15], [25]. Peetre's proof in [14] has served as our model in the following proof of Theorem 5.3. We remark that it would be easy to extend the formulation and proof to cover non-integral values of s; in this respect we do not strive for maximum generality.

Proof of Theorem 5.3. Let $K(x)$ be infinitely differentiable on $\hat{R}^m$, equal to one for $|x| \leq 3$ and zero outside a compact set. Define $k \in L^1(R^m)$ by $\hat{k} = K$, so that k has integrable derivatives of all orders. Then $f * k \in C^\infty$ and $D^\alpha(f * k) = f * (D^\alpha k) \in L^p(R^m)$ for all α. Therefore, it is sufficient to prove that the derivatives up to order s of $g = f - (f * k)$ are in L^p. Now, g is in L^p and the support of its (distributional) Fourier transform lies in $|x| > 2$. Let now $|\alpha| \leq s$, and write $T = D^\alpha g$. We propose to apply Theorem 3.2, and to this end wish to show

$$\sum_{j=-\infty}^{\infty} \|T * \varphi_j\|_p^{\bar{p}} < \infty . \tag{2}$$

Now, $T * \varphi_j = g * (D^\alpha \varphi_j)$, and

$$D^\alpha \varphi_j = 2^{-j|\alpha|}\theta_{(2^j)}$$

where $\theta = D^\alpha \varphi$. Therefore

$$\|T * \varphi_j\|_p = 2^{-j|\alpha|}\|g * \theta_{(2^j)}\|_p \leq A 2^{-j|\alpha|}\, \omega_{\sigma,p}(g\,;\,2^j B)$$

by Lemma 4.1.6, since the Fourier transform of the measure $d\tau = \theta(t)dt$ vanishes in neighborhoods of zero and infinity. Moreover, the Fourier transform of $\theta_{(2^j)}$ vanishes for $|x| \geq 2$ when $j = 0, 1, \ldots$ and so $g * \theta_{(2^j)}$ is zero for these values of j. Hence

$$\sum_{j=-\infty}^{\infty} \|T * \varphi_j\|_p^{\bar{p}} \leq A^{\bar{p}} \sum_{i=1}^{\infty} 2^{\bar{p}i|\alpha|}\, \omega_{\sigma,p}(g\,;\,2^{-i}B)^{\bar{p}} . \tag{3}$$

Using the elementary inequality

$$2^{ir}\, w(2^{-i}) \leq \left(\frac{2^r}{\log 2}\right) \int_{2^{-i}}^{2^{-i+1}} w(v) v^{-r-1}\, dv\,, \quad i = 1, 2, \ldots,$$

valid for an increasing function w, and $r > 0$, and recalling that $|\alpha| \leq s$, we see that the right hand side of (3) is bounded by a constant times

$$\int_0^1 \omega_{\sigma,p}(g\,;\,Bv)^{\bar{p}}\, v^{-\bar{p}s-1}\, dv .$$

Finally, since $g = f - (f * k)$, we have

$$\omega_{\sigma,p}(g\,;a) \le \omega_{\sigma,p}(f\,;a)(1 + \|k\|_1)\,.$$

Thus, in view of the hypothesis (1), (2) holds.

Therefore, by Theorem 3.2, $T = D^\alpha g \in L^p$ (observe that the polynomial part of $D^\alpha g$ must be zero, since $\hat{g}(x) = 0$ in a neighborhood of the origin). This implies the conclusion of Theorem 5.3.

6. Examples, applications and remarks

6.1. To simplify the computation of the Fourier transforms involved, we illustrate our theorems (4.3, 4.4 and 5.3) mainly in R^1 and for measures of very simple structure. Our emphasis will be on clarification of the previous theorems, rather than novelty as such. The measures we shall principally employ are the following.

(a) α_k, where $k \in L^1$ and $\int k(t)dt = 1$, is defined by $d\alpha_k = \delta - k(t)dt$ where δ is the »Dirac measure». The α_k, p deviation of $f \in L^p$ is then the L^p norm of

$$f(t) - \int f(t - au)k(u)du = f(t) - \int f(t - u)\lambda k(\lambda u)du$$

where $\lambda = 1/a$ is a »large» parameter. Hence the α_k, p modulus of f measures the error of approximation to f by a standard type of convolution integral with »kernel» k, depending on a parameter λ. It will be convenient below to write $k_\lambda(t)$ for $\lambda k(\lambda t)$.

(b) β_n, where n is a positive integer, is the purely atomic »binomial measure», with »mass» $(-1)^j\binom{n}{j}$ at the point j $(j = 0, 1, \ldots, n)$. The β_n, p modulus of $f \in L^p$ is then the »modulus of smoothness» of f of order n, relative to the L^p metric. In particular, the β_1, p modulus is the usual L^p modulus of continuity, and shall be denoted in this section simply by $\omega_p(f\,;a)$, whereas the β_2, p modulus shall be denoted by $\omega_p^*(f\,;a)$. Finally, observe that all the measures α_k and β_n satisfy the Tauberian condition.

We turn first to the deduction of the results mentioned in § 1.

6.2. *Estimation of* ω_p *from* ω_p^*. We apply Theorem 4.4 with $\sigma = \beta_2$, $\tau = \beta_1$. Here $\hat{\tau}(x) = 1 - e^{-ix}$ and the theorem is applicable with $P(x) = x$, $r = 1$. From 4.4(4) we get

$$\omega_p(f\,;a)^{\bar{p}} \le A_1 \int_0^a \omega_p^*(f\,;v)^{\bar{p}} \frac{dv}{v} + A_2 a^{\bar{p}} \int_a^1 \omega_p^*(f\,;v)^{\bar{p}} v^{-\bar{p}-1} dv + A_3 a^{\bar{p}}$$

where the various constants A_i are independent of a. In particular, if $\omega_p^*(f\,;a) = O(a)$ as $a \to 0$, we see that $\omega_p(f\,;a) = O(a(\log 1/a)^{1/\bar{p}})$. This is the theorem of Zygmund referred to in the introduction. Zygmund [30] constructed examples to show that the exponent $1/\bar{p}$ of the logarithmic factor cannot here be replaced by any smaller number. Thus, indirectly, we have evidence of the »sharpness» of Theorem 4.4, and of Theorem 4.3 upon which it is based.

6.3. *Relation of* ω_p^* *to the derivative.*

Let us apply Theorem 5.3 with $\sigma = \beta_2$ and $s = 1$. We conclude that if, for some $f \in L^p$,

$$\int_0^1 a^{-\bar{p}-1}\omega_p^*(f\,;a)^{\bar{p}}\,da < \infty\,, \tag{1}$$

then f has a distributional derivative which is a function of class L^p. Since this in turn implies that f is (after correction on a set of measure zero) absolutely continuous and f' belongs to L^p, we have obtained the theorem of M. Weiss and Zygmund [28] referred to in the introduction. Counter-examples in [28] show that for each p, (1) is essentially the weakest hypothesis which will force this conclusion. For example, consider the case $2 \le p < \infty$, so that $\bar{p} = 2$. Then one can construct (modifying slightly the Weiss-Zygmund construction, since those authors work with periodic functions) a continuous function f of compact support such that its uniform (and *a fortiori* its L^p) second order modulus of smoothness is $O(a(\log 1/a)^{-1/2})$, so that the integral in (1) »just barely» diverges, yet f fails to be absolutely continuous. In the Weiss-Zygmund example, f is a.e. non-differentiable. (Other counter-examples in which f has bounded variation, but is purely singular, follow from constructions in [7], [21].) Further examples constructed in [28] show that for $1 \le p < 2$, $\omega_p^*(f\,;a)$ may be $O(a(\log 1/a)^{-1/p})$, with f absolutely continuous and yet not locally in L^p. Here again, the integral in (1) »just barely» diverges (now $\bar{p} = p$). These examples support the view that Theorem 5.3 is »sharp». The John-Nirenberg generalisation [6] to several variables of the Weiss-Zygmund theorem also follows from an earlier mentioned generalisation of Theorem 5.3, taking in place of σ an m-tuple $\sigma_1, \ldots, \sigma_m$ of measures, each of the type of β_2 with respect to a different coordinate, so that $\sum_1^m |\hat{\sigma}_j(x)|$ satisfies the Tauberian condition on $\hat{R}^m$.

6.4. *Approximation generated by a Fejér kernel.* Let $k(t)$ denote the function $\sin^2 t/(\pi t^2)$, the so-called »Fejér-de la Vallée Poussin kernel» (the entire discussion in the present paragraph applies equally well to the »Cauchy kernel» $(1/\pi)(1+t^2)^{-1}$, in which case the results may be interpreted in terms of the boundary behaviour of a harmonic function in a half-plane, cf. [18], Chapter 5). Let us first apply Theorem 4.3 with $\hat{\sigma}(x) = 1 - e^{-ix}$, and $\hat{\tau}(x) = 1 - \hat{k}(x)$ which equals $|x|$ for $|x| \le 1$.

In this case $\hat{\tau}/\hat{\sigma}$ does not coincide with an element of W near $x = 0$, in fact this ratio has a jump discontinuity at $x = 0$. It does, however, coincide with a Fourier multiplier of class M_p, if $1 < p < \infty$ (cf. 4.1.2). We get therefore, setting $k_\lambda(t) = \lambda k(\lambda t)$:

$$\|f - (f * k_\lambda)\|_p \leq A_p \left(\int_0^{B/\lambda} \omega_p(f; v)^{\bar{p}} \frac{dv}{v} \right)^{1/\bar{p}}. \tag{1}$$

This holds for all $f \in L^p$, provided $1 < p < \infty$. Here B is an absolute constant, and A_p depends on p only. (1) is a so-called »direct» theorem of approximation theory, being of the form »smoothness implies approximability». For example, if $\omega_p(f; v) = O(v)$ as $v \to 0$, (1) says that the »approximation error» represented by the expression on the left must be $O(1/\lambda)$ as $\lambda \to \infty$. Observe that we get non-trivial information from (1) in case $\omega_p(f; v)$ is $O((\log 1/v)^{-c})$ with $c > 1/\bar{p}$, what we might call a »threshhold» phenomenon.

The above analysis is entirely symmetrical so far as σ and τ are concerned, and therefore we also have the »inverse» theorem:

$$\omega_p(f; a) \leq A_p \left(\int^{Ba} \Psi(v)^{\bar{p}} \frac{dv}{v} \right)^{1/\bar{p}} \tag{2}$$

valid for $1 < p < \infty$, where

$$\Psi(v) = \sup_{\lambda \geq 1/v} \|f - (f * k_\lambda)\|_p .$$

The above results are false in the limiting cases $p = 1$ and $p = \infty$, as is known e.g. from the »saturation theory» of the kernel k. We can get valid analogues of (1) and (2) in this case by applying Theorem 4.4. For instance, applying Theorem 4.4 with $p = \infty$, $r = 1$, $P(x) = |x|$ and σ and τ as above gives the following estimate in place of (1):

$$\|f - (f * k_\lambda)\|_\infty \leq A \int_0^\infty \min\,(1\,,\,(\lambda v)^{-r})\,\omega_\infty(f\,,\,v)\,\frac{dv}{v}\,. \tag{3}$$

If $\omega_\infty(f, v) = O(v^\alpha)$, where $0 < \alpha < 1$, then the right hand side of (3) is $O(\lambda^{-\alpha})$, so in this case (3) gives the same result as (1). However, if $\omega_\infty(f, v) = O(v)$, (3) gives something weaker than (1), i.e. the well known estimate

$$\|f - (f * k_\lambda)\|_\infty = O\,(\lambda^{-1} \log \lambda) \quad \text{as} \quad \lambda \to \infty\,,$$

which cannot be strengthened (see [26]).

6.5. »*Inverse*» *theorems for trigonometric approximation*. As remarked in the introduction, the results of this paper apply *mutatis mutandis* to *periodic* functions, and *in this section we shall mean by* L^p *the space of measurable* 2π*-periodic functions, with*

$$\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p \, dt\right)^{1/p} .$$

Let us consider the sequence

$$E_{n,p}(f) = \inf\{\|f - T\|_p \; ; \; T \in \mathcal{T}_{n-1}\}$$

measuring the goodness of approximation to $f \in L^p$ by elements of $\mathcal{T}_{n-1}$ (trigonometric polynomials of degree at most $n - 1$.) It is easy to show (cf. [19], p. 504) that if σ is any measure such that $\hat{\sigma}(x)$ vanishes for $|x| \leq 1$, then

$$\omega_{\sigma,p}(f\,;a) \leq V(\sigma)\, E_{n,p}(f)\,, \tag{1}$$

where n is the smallest integer such that $na \geq 1$. This inequality enables us to apply theorems 4.4 and 5.3 to the »inverse» problem of trigonometric approximation. For example, suppose

$$E_{n,p}(f) \leq \psi(1/n)\,, \tag{2}$$

where $\psi(v)$ is a continuous increasing function defined for $v \geq 0$, and $\psi(0) = 0$. Note that, by (1), $\omega_{\sigma,p}(f\,;a)$ is bounded by a constant times $\psi(a)$. If now σ satisfies the Tauberian condition, we get from 4.4(4) (with $\tau = \beta_1$, $r = 1$, $P(x) = x$)

$$\omega_p(f\,;a)^{\bar{p}} \leq A_p\left[\int_0^a \psi(v)^{\bar{p}}\,\frac{dv}{v} + a^{\bar{p}}\int_a^\infty \psi(v)^{\bar{p}}\, v^{-\bar{p}-1}\, dv\right], \tag{3}$$

where A_p depends on p only. For example, if $\psi(v) = O(v)$, we get (assuming, as we may, ψ bounded)

$$\omega_p(f\,;a) = O\left(a\left(\log\frac{1}{a}\right)^{1/\bar{p}}\right).$$

The estimate (3) gives non-trivial information for $\psi(v) = |\log v|^{-c}$ when $c > 1/\bar{p}$, but gives no information when $c \leq 1/\bar{p}$. On the other hand, from a theorem of A. F. and M. F. Timan (cf. Timan [26], p. 331), it is known that (2) implies

$$\omega_p(f\,;a) \leq Ca\int_a^\infty \psi(v)\, v^{-2}\, dv\,. \tag{4}$$

(Of course, (4) gives non-trivial information for arbitrary $\psi(v)$ tending to zero as $v \to 0$. Comparing (3) and (4) makes it natural to expect that an estimate

$$\omega_p(f\,;a)^{\bar{p}} \leq Ca^{\bar{p}}\int_a^\infty \psi(v)^{\bar{p}}\, v^{-\bar{p}-1}\, dv \tag{5}$$

should hold. It is clear that (5) cannot be deduced from any of the comparison theorems of this paper. However, one of the authors (J. B.) has recently proved that Theorems 4.3 and 4.4 can be sharpened to yield estimates like (5) for a large class of measures σ, including, in particular, any measure that is the sum of a discrete and an absolutely continuous measure. The sharpening consists in replacing the integral

$$\int_0^{Ba} \omega_{\sigma,p}(f\,;v)^{\bar{p}}\, v^{-1}\, dv$$

by $\omega_{\sigma,p}(f\,;Ba)^{\bar{p}}$. It can also be proved that this stronger conclusion is *not* valid for arbitrary measures σ satisfying the Tauberian condition.)

Similarly, applying Theorem 5.3 with $s = 1$, we get: *if* (2) *holds, where*

$$\int_0^1 a^{-\bar{p}-1}\, \psi(a)^{\bar{p}}\, da < \infty\,,$$

then f is equal a.e. to an absolutely continuous function, having a derivative of class L^p.

6.6. Corollary to Theorem 4.4. *If, under the hypotheses of Theorem* 4.4 *the integral*

$$\int_0^{\infty} \omega_{\sigma,p}(f\,;v)^{\bar{p}}\, v^{-r\bar{p}-1}\, dv \tag{1}$$

is finite, then $\omega_{\tau,p}(f\,;a) = O(a^r)$.

Indeed, this is an immediate consequence of 4.4(4).

Observe that the finiteness of (1) is implied by

$$\int_0^1 \omega_{\sigma,p}(f\,;v)^{\bar{p}}\, v^{-r\bar{p}-1}\, dv < \infty \tag{2}$$

which is precisely the hypothesis of Theorem 5.3.

6.7. *Mixed* (L^p, L^q) *estimates.* The »classical» embedding theorems (of Sobolev, Besov, etc.) are always carried out in the generality of mixed (L^p, L^q) estimates, whereas in each of our theorems the same value of p appears in both hypothesis and conclusion. By way of contrast, consider the following known results ([27], p. 677) valid for functions on $[0, 1]$.

(i) *For* $1 \le p < q < \infty$, *if* $f \in L^p$ *and*

$$\omega_p(f\,;a) = O\left(a^{\frac{1}{p}-\frac{1}{q}}\left(\log\frac{1}{a}\right)^{-c}\right)$$

where $c > 1/q$, *then* $f \in L^q$.

(ii) *For* $1 < p < \infty$, *if* $f \in L^p$ *and*

$$\omega_p(f\,; a) = O\left(a^{1/p}\left(\log \frac{1}{a}\right)^{-c}\right)$$

where $c > 1$, *then* f *coincides a.e. with a continuous function.*

Using general embedding theorems, e.g. as given by Peetre in [15], it is not hard to extend our theorems so as to encompass results like the cited theorem of Ulyanov. As an example we mention the following theorem.

THEOREM 6.1. *Suppose* σ, $\tau \in M(R^m)$, $\hat{\sigma}$ *satisfies the Tauberian condition, and there exists* $F \in W$ *such that* $\hat{\tau}(x) = \hat{\sigma}(x)F(x)$ *for* x *in some neighborhood of the origin. Let* $1 < p < q < \infty$, $s = m(1/p - 1/q)$, *and* $1 \le r \le q$. *Assume that* $f \in L^p(R^m)$ *and that the integral in the right hand side of* (1) *converges. Then* $f \in L^q(R^m)$ *and*

$$\omega_{\tau,q}(f\,, a)^r \le C \int_0^{Ba} [\omega_{\sigma,p}(f\,, v)\, v^{-s}]^r \frac{dv}{v}\,. \tag{1}$$

For the proof of this theorem one needs the inequality

$$\|f\|_q^r \le C \sum_{j=-\infty}^{\infty} (2^{-js}\|f * \varphi_j\|_p)^r \tag{2}$$

in place of 3.3.4(2). The inequality (2) is related to the embedding theorem (cf. [15])

$$\dot{W}_p^{s,r} \subset L^q \tag{3}$$

just as 3.3.4(2) is related to

$$\dot{W}_p^{0,\bar{p}} \subset L^p\,.$$

In fact, taking into account the definition of the spaces $\dot{W}_p^{s,r}$ and L^p one observes that (2) is just another way of stating (3). Using (2) one can prove Theorem 6.1 in the same way as Theorem 4.3.

References

1. BENEDEK, A., CALDÉRON, A. P. & PANZONE, R., Convolution operators on Banach space-valued functions. *Proc. Nat. Acad. Sci.*, 48 (1962), 356–365.
2. BESOV, O. V., Investigation of a family of function spaces in connection with theorems of embedding and continuation. *Trudy Mat. Inst. Steklova,* 60 (1961), 42–81 (Russian).
3. BOMAN, J., Supremum norm estimates for partial derivatives of functions of several variables. To appear.
4. DE LEEUW, K. & MIRKIL, H., A priori sup norm estimates for differential operators. *Ill. J. Math.* 8 (1964), 112–124.
5. HÖRMANDER, L., Estimates for translation invariant operators in L^p spaces. *Acta Math.*, 104 (1960), 93–140.

6. JOHN, F. & NIRENBERG, L., On functions of bounded mean oscillation. *Comm. Pure Appl. Math.*, 14 (1961), 415—426.
7. KAHANE, J.-P., Trois notes sur les ensembles parfaits linéaires. *Enseignement Math.*, 15 (1969), 185—192.
8. KANTOROVICH, L. V. & AKILOV, G. P., *Functional analysis in normed spaces.* Moscow, 1959 (Russian).
9. KRÉE, P., Sur les multiplicateurs dans FL^p. *Ann. Inst. Fourier Grenoble*, 16 (1966), 31—89.
10. LITTLEWOOD, J. E. & PALEY, R. E. A. C., Theorems on Fourier series and power series (II). *Proc. London Math. Soc.*, 42 (1937), 52—89.
11. LITTMAN, W., MCCARTHY, C. & RIVIÈRE, N., L^p-multiplier theorems. *Studia Math.*, 30 (1968), 193—217.
12. —»— —»— —»— The non-existence of L^p estimates for certain translation invariant operators. *Studia Math.*, 30 (1968), 220—229.
13. LORENTZ, G. G., *Approximation of functions.* Holt, Rinehart and Winston, 1966.
14. PEETRE, J., *Thoughts about Besov spaces.* Mimeographed lecture notes, Lund, 1966 (Swedish).
15. —»— Espaces d'interpolation et théorème de Soboleff. *Ann. Inst. Fourier*, 16 (1966), 279-317.
16. SCHWARTZ, L., *Théorie des distributions.* Hermann, Paris.
17. SCHWARTZ, J. T., A remark on inequalities of the Calderón-Zygmund type for vector-valued functions. *Comm. Pure Appl. Math.*, 14 (1961) 785—799.
18. SHAPIRO, H. S., *Smoothing and approximation of functions.* Van Nostrand, Reinhold, 1969.
19. —»— Some Tauberian theorems with applications to approximation theory. *Bull. Amer. Math. Soc.*, 74 (1968), 500—504.
20. —»— A Tauberian theorem related to approximation theory. *Acta Math.*, 120 (1968), 279—292.
21. —»— Monotonic singular functions of high smoothness. *Mich. Math. J.*, 15 (1968), 265—275.
22. —»— *Topics in approximation theory.* Series »Lecture Notes in Mathematics», Springer, Berlin & Heidelberg, 1971.
23. STEIN, E. M., On the functions of Littlewood-Paley, Lusin and Marcinkiewicz. *Trans. Amer. Math. Soc.*, 88 (1958), 430—466.
24. TAIBLESON, M. H., A note on indefinite integrals. *Proc. Amer. Math. Soc.*, 16 (1965), 923—925.
25. —»— On the theory of Lipschitz spaces of distributions on Euclidean n-space, I. Principal properties. *J. Math. Mech.*, 13 (1964), 407—480.
26. TIMAN, A. F., *Theory of approximation of functions of a real variable.* English edition, Pergamon, 1963.
27. ULYANOV, P. L., Embedding of certain function classes H_p^ω. *Izv. Akad. Nauk SSSR*, 32 (1968), 649—686 (Russian).
28. WEISS, M. & ZYGMUND, A., A note on smooth functions. *Indag. Math.*, 21 (1959), 52—58.
29. YOSIDA, K., *Functional analysis.* Springer-Verlag, 1966.
30. ZYGMUND, A., A remark on the integral modulus of continuity. *Revista Univ. Nac. Tucuman, A*, 7 (1950), 259—269.
31. —»— *Trigonometric Series, vol. 2.* Cambridge, 1959.

Received August 23, 1970

Jan Boman
Department of Mathematics
University of Stockholm
Box 6701
S—113 85 Stockholm, Sweden

Harold S. Shapiro
University of Michigan
Department of Mathematics
Ann Arbor, Mich. 48104
USA