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COMPARISON THEOREMS FOR DELAY DIFFERENTIAL EQUATIONS

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Comparison Theorems. This paper presents a technique for extending some oscillation results from ordinary to delay differential equations. According to the technique, if an oscillation result is known for the ordinary differential equation

$$x^{(n)} + f(t, x) = 0 ,$$

the corresponding result for the delay differential equation

$$x^{(n)}(t) + f(t, x(q(t))) = 0$$

may be obtained by a simple change of variables.

The conditions on q and f are mild and in the case of a bounded delay it is shown that the above equations have the same oscillatory behavior.

1. Introduction.

We consider the equation

$$x^{(n)}(t) + f(t, x(q(t))) = 0 \tag{1}$$

where $n \geqq 2$. We let $R^+ = [0, +\infty)$ and $R = (-\infty, +\infty)$ and assume throughout this paper that q and f satisfy the following conditions. $q: R^+ \to R$ and $f: R^+ \times R \to R$ are continuous, $q(t) \leqq t$ for $t \geqq 0$, $q(t) \to \infty$ as $t \to \infty$, $f(t, x)$ in nondecreasing in x, and $xf(t, x) > 0$ if $x \neq 0$.

We label the above conditions on q and f as Hypothesis (E) for a future reference.

For any $t_0 \geqq 0$, we let $E_{t_0} = \{s \mid s = q(t) \leqq t_0 \text{ for some } t \geqq t_0\} \cup \{t_0\}$. By a solution of (1) at t_0 is meant a function $x: E_{t_0} \cup [t_0, t_1) \to R$, $t_1 > t_0$, which satisfies (1) for all $t \in [t_0, t_1)$. All solutions of (1) at t_0 are assumed to exist on $[t_0, \infty)$ for every $t_0 \geqq 0$. As in the case of ordinary differential equations, the existence of solutions on (1) on $[t_0, \infty)$ are usually guaranteed by requiring some growth conditions on f. For details, see [3].

A solution $x(t)$ of (1) at t_0 is said to be oscillatory if $x(t)$ has zeros for arbitrarily large t and nonoscillatory if there exists $t^* \geqq t_0$ such that $x(t) \neq 0$ for all $t \geqq t^*$. Equation (1) is said to be oscillatory if every solution of (1) is oscillatory.

Most of the oscillation results which have appeared in the literature for delay differential equations are generalizations of known results for ordinary differential equations. Very often the method of proof of a generalized result is the same as that of the original result, but sometimes requiring a severe restriction on the delay. We propose in this paper to solve such generalized problems by reducing the study of the oscillatory properties of solutions of Equation (1) to that of an ordinary differential equation so that desirable generalizations of some oscillation criteria from ordinary to delay equations of the same types become immediate. In §3 we give three illustrative applications of our technique by deriving some results in [7], [9], and [12] from the corresponding ones in ordinary differential equations. Consequently we improve the results in [9] by relaxing the upper bound restriction on $q(t)$ and show that for a bounded delay Equation (1) and the corresponding ordinary equation

$$x^{(n)}(t) + f(t, x) = 0 \tag{2}$$

have the same oscillatory behavior.

2. Main results.

We need the following two lemmas; the first one is essentially Kiguradze's lemma [6]. For a proof, see [10].

LEMMA 1. *Suppose $x(t)$ is a solution of* (1) *which is of constant sign on $[t_0, \infty)$, $t_0 \geqq 0$. Then there exists $t_1 \geqq t_0$ such that on $[t_1, \infty)$ we have*

(i) *$x^{(k)}(t)x(t) > 0$ whenever $k + n$ is odd and $0 \leqq k \leqq n - 1$, and*

(ii) *there exists an integer l, $0 \leqq l \leqq n - 1$, $n + l$ is odd, such that $x^{(k)}(t)x(t) > 0$ for $k = 0, 1, \cdots, l$, $(-1)^{n+k-1}x^{(k)}(t)x(t) > 0$ for $k = l + 1, \cdots, n - 1$, and $x^{(n)}(t)x(t) \leqq 0$.*

LEMMA 2. *Suppose q and f satisfy Hypothesis* (E). *If the differential inequality*

$$z'(t) - f(t, z(q(t))) \geqq 0 \tag{3}$$

has a positive solution on $[\alpha, \infty)$, for some $\alpha > 0$, so does the equation

$$z'(t) - f(t, z(q(t))) = 0 . \tag{4}$$

Proof. Let $z(t)$ be a positive solution of (3) on $[\alpha, \infty)$. Choose $t_1 \geqq \alpha$ so that $q(t) \geqq \alpha$ for $t \geqq t_1$. Then $z(t)$ satisfies the inequality

$$z(t) \geqq z(t_1) + \int_{t_1}^{t} f(u, z(q(u)))du .$$

Let

$$
\begin{aligned}
& y_1(t) = z(t) \quad \text{for} \quad t \geqq \alpha \quad \text{and} \\
(5) \qquad & y_n(t) = z(t) \quad \text{for} \quad t \in [\alpha, t_1] \\
& \quad = z(t_1) + \int_{t_1}^{t} f(u, y_{n-1}(q(u)))du \quad \text{for} \quad t \geqq t_1
\end{aligned}
$$

and $n = 2, 3, \cdots$.

It follows from the definition of y_n and (5) that the sequence $\{y_n\}$ satisfies the property that $z(t) = y_1(t) \geqq y_2(t) \geqq \cdots \geqq z(t_1)$ for all $t \geqq t_1$. Hence $\{y_n\}$ converges pointwise to a function $y(t)$ where $z(t) \geqq y(t) \geqq z(t_1)$ for all $t \geqq t_1$. Let $f_n(t) = f(t, y_n(q(t)))$, $n = 1, 2, \cdots$. Then $f_1(t) \geqq f_2(t) \geqq \cdots \geqq 0$. Since f_1 is integrable on $[t_1, t]$ for any $t \geqq t_1$ and $\lim_{n\to\infty} f_n(u) = f(u, y(q(u)))$ for any $u \in [t_1, t]$, then, by the monotone convergence theorem, we have

$$y(t) = z(t_1) + \int_{t_1}^{t} f(u, y(q(u)))du \quad \text{for} \quad t \geqq t_1 .$$

Hence $y(t)$ satisfies (4) and the proof is complete.

In connection with the study of solutions of Equation (1), we consider solutions of the equation

$$x^{(n)}(t) + f^*(t, x(q(t))) = 0 \tag{6}$$

where f^* is defined by

$$
\begin{aligned}
f^*(t, x) &= f(t, x) \qquad \text{if} \quad x \leqq 0 \\
&= -f(t, -x) \quad \text{if} \quad x \geqq 0 .
\end{aligned}
$$

It is clear from the definition of f^* that $f^*(t, -x) = -f^*(t, x)$ and that $xf^*(t, x) > 0$ if $x \neq 0$. Also, if $x(t)$ is a solution of (6), so is $-x(t)$; furthermore, $y(t) < 0$ is a solution of (6) if and only if $y(t)$ is a solution of (1).

THEOREM 1. *Suppose q is continuously differentiable on $[\alpha, \infty)$, $\alpha \geqq 0$, and $q'(t) > 0$.*

If, for n even, Equation (1) has a nonoscillatory solution, then the equation

$$\frac{d^n y}{ds^n} + \frac{1}{q'(q^{-1}(s))} f(q^{-1}(s), y) = 0 \tag{7}$$

has a nonoscillatory solution.

If, for n odd, Equation (1) has an unbounded nonoscillatory solution, so does Equation (7).

Proof. Let $x(t)$ be a nonoscillatory solution of (1) and assume

$x(t) > 0$ for $t \geqq t_0$, $t_0 \geqq 0$. By Lemma 1, there exists $t_1 \geqq t_0$ and an integer l, $0 \leqq l \leqq n-1$, $n+l$ is odd, such that $x^{(k)}(t) > 0$, $k = 0, 1, \cdots, l$, $(-1)^{n+k-1}x^{(k)}(t) > 0$, $k = l+1, \cdots, n-1$, and $x^{(n)}(t) \leqq 0$ for $t \geqq t_1$. Thus, if n is even, or n is odd and $x(t)$ is unbounded, then $l \geqq 1$. Choose $t_1 \geqq \alpha$ and $t_2 \geqq t_1$ so that $q(t) \geqq t_1$ for $t \geqq t_2$ and integrate (1) from s to τ, $\tau \geqq s \geqq t_2$, to obtain

$$x^{(n-1)}(\tau) - x^{(n-1)}(s) + \int_s^\tau f(u, x(q(u)))du = 0$$

and hence

$$x^{(n-1)}(s) \geqq \int_s^\infty f(u, x(q(u)))du \quad \text{for all} \quad s \geqq t_2 \, .$$

Let $V = q(u)$; then the above inequality yields

$$x^{(n-1)}(s) \geqq \int_{q(s)}^\infty [f(q^{-1}(v), x(v))/q'(q^{-1}(v))]dv \, , \quad s \geqq t_2 \, .$$

Since $q(t) \leqq t$, then

$$x^{(n-1)}(s) \geqq \int_s^\infty [f(q^{-1}(v), x(v))/q'(q^{-1}(v))]dv \, , \quad s \geqq t_2 \, . \tag{8}$$

Define

$$F_1x(t) = \int_t^\infty [f(q^{-1}(v), x(v))/q'(q^{-1}(v))]dv \, .$$

And

$$F_ix(t) = \int_t^\infty F_{i-1}x(s)ds \, , \quad i = 2, 3, \cdots, n-l \, .$$

Then it follows from (8) that the operators F_i, $i = 1, \cdots, n-l$ are well-defined and that $F_ix(t) \geqq F_iy(t)$, $i = 1, \cdots, n-l$, whenever $x(t) \geqq y(t)$ and $t \geqq t_2$. Furthermore, $F_ix(t) > 0$ and

$$\frac{d}{dt}F_ix(t) = -F_{i-1}x(t) \, , \quad t \geqq t_2 \quad \text{and} \quad i = 2, \cdots, n-l \, . \tag{9}$$

Thus (8) reduces to

$$x^{(n-1)}(s) \geqq F_1x(s) \, , \quad s \geqq t_2 \, .$$

By successive integrations of this inequality from s to τ, $\tau \geqq s \geqq t_2$, discarding positive terms, we obtain

$$(-1)^{i+1}x^{(n-i)}(s) \geqq F_ix(s) \, , \quad s \geqq t_2 \quad \text{and} \quad i = 1, \cdots, n-l \, .$$

In particular,

$$x^{(l)}(s) \geqq F_{n-l}x(s) \, , \quad s \geqq t_2 \, . \tag{10}$$

If we let $T_0x(t) = F_{n-l}x(t)$ and

$$T_ix(t) = \int_{t_2}^{t} T_{i-1}x(s)ds\ , \quad t \geqq t_2 \quad \text{and} \quad i = 1, \cdots, l\ ,$$

then $T_ix(t) \geqq T_iy(t)$, $i = 1, \cdots, l$, whenever $x(t) \geqq y(t)$ and $t \geqq t_2$. Furthermore, $T_ix(t) > 0$ and

$$\frac{d}{dt}T_ix(t) = T_{i-1}x(t)\ , \quad t \geqq t_2 \quad \text{and} \quad i = 1, \cdots, l\ . \tag{11}$$

By successive integrations of (10) from t_2 to s, discarding positive constants, we obtain

$$x^{(l-i)}(s) \geqq T_ix(s)\ , \quad s \geqq t_2 \quad \text{and} \quad i = 0, 1, \cdots, l-1\ .$$

In particular,

$$x'(s) \geqq T_{l-1}x(s)\ , \quad s \geqq t_2\ .$$

By Lemma 2, the equation $x'(s) = T_{l-1}x(s)$ has a positive solution $y(s)$ on $[t_2, \infty)$ such that $y'(s) > 0$ for all $s \geqq t_2$. By successive differentiations of the equation $y'(s) = T_{l-1}y(s)$ using (9) and (11), we obtain

$$y^{(l)}(s) = T_0y(s) = F_{n-l}y(s)$$

and hence

$$y^{(n-1)}(s) = F_1y(s) = \int_s^\infty [f(q^{-1}(v), y(v))/q'(q^{-1}(v))]dv\ .$$

Thus $y(s)$ satisfies Equation (7). If n is odd, then $y''(s) > 0$ and hence $y(s)$ is unbounded.

Now, assume $x(t) < 0$ for $t \geqq t_0$ and let $u(t) = -x(t)$; then $u(t)$ is a positive solution of Equation (6) which is unbounded if n is odd and $x(t)$ is unbounded. By the conclusion above, the equation

$$y^{(n)} + [1/q'(q^{-1}(s))]f^*(q^{-1}(s), y) = 0 \tag{12}$$

has a positive solution $v(s)$. Let $y(s) = -v(s)$; then $y(s)$ is a solution of (12) and hence of (7) and which is unbounded when n is odd. The proof is now complete.

The following theorem is an extension of [9, Theorems 6 and 15] to Equation (1). The result we obtain is essentially a comparison result between Equation (1) and the delay equation

$$x^{(n)}(t) + f(t, x(Q(t))) = 0 \tag{13}$$

where $Q: R^+ \to R$ is continuous, $Q(t) \leqq t$ for $t \geqq 0$, and $Q(t) \to \infty$ as $t \to \infty$. The purpose of this extension is, on one hand, to relax the conditions of smoothness and monotonicity on $q(t)$ and, on the other hand, to show that for a bounded delay Equation (1) and Equation (2) have the same oscillatory behavior.

THEOREM 2. *Suppose* $q(t) \geqq Q(t)$, $t \geqq 0$.

If, for n even, Equation (1) has nonoscillatory solution, so does Equation (13).

If, for n odd, Equation (1) has an unbounded nonoscillatory solution, so does Equation (13).

Proof. Let $x(t)$ be a nonoscillatory solution of (1) and assume $x(t) > 0$ for $t \geqq t_0$, $t_0 \geqq 0$. It follows as in the proof of Theorem 1 that there exists $t_2 \geqq t_0$ such that

$$x^{(n-1)}(s) \geqq \int_s^\infty f(u, x(q(u)))du , \quad s \geqq t_2 .$$

Define the sequence of operators $F_i(x, q)$ and $T_i(x, q)$ respectively by

$$\begin{aligned} F_1(x, q)(t) &= \int_t^\infty f(u, x(q(u)))du \\ F_i(x, q)(t) &= \int_t^\infty F_{i-1}(x, q)(u)du , \quad i = 2, \cdots, n - l , \end{aligned}$$

and

$$\begin{aligned} T_0(x, q)(t) &= F_{n-l}(x, q)(t) \\ T_i(x, q)(t) &= \int_{t_2}^t T_{i-1}(x, q)(u)du , \quad i = 1, \cdots, l . \end{aligned}$$

It is clear that these functions satisfy the differentiation properties of (9) and (11) as well as the monotonicity property in both arguments x and q. Hence, by successive integrations as in the proof of Theorem 1, we obtain

$$x'(t) \geqq T_{l-1}(x, q)(t) , \quad t \geqq t_2 .$$

Since $q(t) \geqq Q(t)$, then $x'(t) \geqq T_{l-1}(x, Q)(t)$ and hence, by Lemma 2, the equation

$$x'(t) = T_{l-1}(x, Q)(t)$$

has a nonoscillatory solution $y(t)$ which is unbounded when n is odd. By successive differentiations of the equation $y'(t) = T_{l-1}(y, Q)(t)$, we conclude that $y(t)$ satisfies Equation (13).

Now, if we assume $x(t) < 0$ for $t \geqq t_0$, then the equation

$$x^{(n)}(t) + f^*(t, x(Q(t))) = 0 \tag{14}$$

has a positive solution $v(t)$ which is unbounded when n is odd and $x(t)$ is unbounded. Hence $y(t) = -v(t)$ is a solution of (14) which satisfies (13). The proof is now complete.

Results similar to Theorem 2 have also been obtained in [5] and [11].

THEOREM 3. *Suppose* $q(t) \geqq Q(t)$, Q *is continuously differentiable, and* $Q'(t) > 0$ *for* $t \geqq \alpha$, $\alpha \geqq 0$.

If, for n even, the equation

$$\frac{d^n y}{ds^n} + \frac{1}{Q'(Q^{-1}(s))} f(Q^{-1}(s), y) = 0 \tag{15}$$

is oscillatory, so is Equation (1).

If, for n odd, Equation (15) has no bounded nonoscillatory solutions, neither does Equation (1).

Proof. It follows from Theorems 1 and 2.

COROLLARY. *Suppose* $ct \leqq q(t) \leqq t$ *for some* $c \in (0, 1]$.

If, for n even, the equation

$$x^{(n)} + c^{n-1} f(t, x) = 0 \tag{16}$$

is oscillatory, so is Equation (1).

If, for n odd, Equation (16) has no bounded nonoscillatory solutions, neither does Equation (1).

Proof. Take $Q(t) = ct$ and set $s = Q(t)$ and $y(s) = x(t)$ in (15). Then $d^n y/ds^n = 1/c^n \, d^n x/dt^n$ and hence (15) reduces to (16).

If we let $\tau(t) = t - q(t)$, then Equation (1) may be written as

$$x^{(n)}(t) + f(t, x(t - \tau(t))) = 0 . \tag{17}$$

The following result is concerned with bounded delays.

THEOREM 4. *Suppose* $\tau(t)$ *is bounded. Then, for* n *even, Equation* (17) *is oscillatory if and only if Equation* (2) *is oscillatory. For* n *odd, every nonoscillatory solution of* (17) *is bounded if and only if every nonoscillatory solution of* (2) *is bounded.*

Proof. Suppose $\tau(t) \leqq M$ for some $M > 0$. Let $Q(t) = t - M$, $s = Q(t)$, and $y(s) = x(t)$; then $t \geqq q(t) \geqq Q(t)$ and $d^n y/ds^n = d^n x/dt^n$. Hence, the result follows from Theorems 2 and 3.

REMARK. For n odd, the terminology "strongly decreasing" has been used in [8, 9] to describe a solution $x(t)$ of (1) which satisfies $(-1)^k x^{(k)}(t) > 0$, $k = 0, 1, \cdots, n-1$. It is then obvious that a nonoscillatory solution $x(t)$ of (1) is strongly decreasing if and only if it is bounded.

Brands [2] obtained Theorem 4 for $n = 2$.

3. Applications.

(a) We first consider the nonlinear delay equation

$$x^{(n)}(t) + a(t)f(x(q(t))) = 0 \tag{18}$$

where $n \geqq 2$, n is even, a is continuous with $a(t) \geqq 0$, f is continuously differentiable with $f'(x) \geqq 0$, and $xf(x) > 0$ if $x \neq 0$.

Kamenev [4] gave the following oscillation criterion for Equation (18) when $q(t) = t$.

THEOREM A. *Suppose there exists a nondecreasing continuously differentiable function* $\phi\colon (0, \infty) \to (0, \infty)$ *such that*

(i) $\int^{\pm\infty} [\phi(|x|^{1/n-1})f(x)]^{-1} dx < \infty$ *and*

(ii) $\int^{\infty} [t^{n-1}a(t)/\phi(t)]dt = \infty$.

Then the equation $x^{(n)} + a(t)f(x) = 0$ *is oscillatory.*

This result has been generalized by Kusano and Onose [7] to Equation (18) for any $q(t) \leqq t$ with $q'(t) \geqq 0$. We show below that their generalization follows from Theorem 3. Indeed, by Theorem 3, Equation (18) is oscillatory if the corresponding ordinary equation

$$\frac{d^n x}{ds^n} + [a(q^{-1}(s))/q'(q^{-1}(s))]f(x) = 0 \tag{19}$$

is oscillatory. By Theorem A, Equation (19) is oscillatory if conditions

(i) and

(ii)′ $\int^{\infty} [s^{n-1}a(q^{-1}(s))/(q'(q^{-1}(s))\phi(s))]ds = \infty$

are satisfied. If we now let $s = q(t)$, then (ii)′ reduces to

(ii)″ $\int^{\infty} [q^{n-1}(t)a(t)/\phi(q(t))]dt = \infty$.

Thus (i) and (ii)″ imply that Equation (19) and hence Equation (18) are oscillatory. This is precisely the result obtained in [7]. However, Theorem 3 requires that $q'(t) > 0$ while the condition on $q(t)$ in [7] is $q'(t) \geqq 0$. This difference is insignificant especially when oscillation is described by divergent integrals such as the one in (ii)″. We show below that our result which is obtained for $q'(t) > 0$ extends easily to the case $q'(t) \geqq 0$. To see this, we assume that $q'(t) \geqq 0$ and that (i) and (ii)″ are satisfied. We let $q_1(t) = (1 - e^{-t})^{1/n-1}q(t)$ and consider the equation

$$x^{(n)}(t) + a(t)f(x(q_1(t))) = 0 . \tag{20}$$

It is clear that $q_1(t) \leqq q(t)$, $q_1(t) \to \infty$ as $t \to \infty$, and $q_1'(t) > 0$. Hence, by Theorem 2, oscillation of Equation (18) follows from oscillation of Equation (20). Since $q_1'(t) > 0$, then, by the above result, Equation (20) is oscillatory if (i) and

$$\text{(ii)}''' \quad \int^{\infty} [q_1^{n-1}(t)a(t)/\phi(q_1(t))]dt = \infty$$

are satisfied. It remains only to show that (ii)″ implies (ii)‴. Since ϕ is nondecreasing, then $\phi(q_1(t)) \leqq \phi(q(t))$. From the definition of $q_1(t)$ we have eventually $q_1^{n-1}(t) \geqq q^{n-1}(t)/2$ and hence $q_1^{n-1}(t)a(t)/\phi(q_1(t)) \geqq q^{n-1}(t)a(t)/[2\phi(q(t))]$. Thus (ii)″ implies (ii)‴ and the proof is complete.

(b) We now consider the linear delay equation

$$x^{(n)}(t) + a(t)x(q(t)) = 0 \tag{21}$$

where $n \geqq 2$, n is odd, a is continuous with $a(t) \geqq 0$, and $q(t)$ satisfies the hypothesis of Theorem 3.

In [1] G. V. Aman'eva and V. I. Balaganskii gave a sufficient condition for the nonoscillatory solutions of Equation (21) to be bounded when $q(t) = t$. This result has been extended by Lovelady [8] so that the combined results state as follows

THEOREM B. *If*

$$\text{(i)} \quad \int^{\infty} t^{n-2}a(t)dt = \infty$$

or (i) *fails and the second order equation*

$$w''(t) + \frac{1}{(n-3)!}\left(\int_t^{\infty} (s-t)^{n-3}q(s)ds\right)w(t) = 0$$

is oscillatory, then every nonoscillatory solution of Equation (21) *is bounded when* $q(t) = t$.

Recently Lovelady [9] generalized Theorem B to Equation (21) for any $q(t) \leqq t$ provided $q'(t) \geqq 0$ and either $q'(t) \leqq 1$ or $q(t) - q(s) \leqq t - s$. We will show that his generalization follows immediately from Theorem 3 without the upper bound restriction on $q(t)$ or its derivative. Indeed, by Theorem 3, boundedness of the nonoscillatory solutions of Equation (21) follows from the boundedness of the nonoscillatory solution of the ordinary equation

$$\frac{d^n x}{ds^n} + [a(Q^{-1}(s))/Q'(Q^{-1}(s))]x = 0 \,. \tag{22}$$

By Theorem B, every nonoscillatory solution of Equation (22) is bounded if either

$$\int_0^\infty [s^{n-2}a(Q^{-1}(s))/Q'(Q^{-1}(s))]ds = \infty \tag{23}$$

or (23) fails and the second order equation

$$\frac{d^2 z}{ds^2} + \frac{1}{(n-3)!}\left(\int_s^\infty [(u-s)^{n-3}a(Q^{-1}(u))/Q'(Q^{-1}(u))]du\right)z = 0 \tag{24}$$

is oscillatory.

If we let $s = Q(t)$, then (23) and (24) reduce respectively to

$$\int^\infty Q^{n-2}(t)a(t)dt = \infty \tag{25}$$

and

$$\begin{aligned}&[z'/Q'(t)]' \\ &\quad + \frac{Q'(t)}{(n-3)!}\left(\int_{Q(t)}^\infty [(u-Q(t))^{n-3}a(Q^{-1}(u))/Q'(Q^{-1}(u))]du\right)z = 0 \,.\end{aligned} \tag{26}$$

Let $v = Q^{-1}(u)$; then (26) reduces to

$$[z'/Q'(t)]' + \frac{Q'(t)}{(n-3)!}\left(\int_t^\infty [Q(v) - Q(t)]^{n-3}a(v)dv\right)z = 0 \,. \tag{27}$$

Thus either condition (25) or the oscillation of Equation (27) when (25) fails implies that every nonoscillatory solution of Equation (21) is bounded. This is [9, Theorem 16] obtained in the form of [9, Corollary 3] without the condition $q'(t) \leqq 1$ required in [9].

(b) Finally, we consider the second order delay equation

$$x''(t) + a(t)x(q(t)) = 0 \tag{28}$$

where $a(t)$ is continuous with $a(t) \geqq 0$.

In [12] Wong obtained the following oscillation result

THEOREM C. *If* $ct \leqq q(t) \leqq t$ *for some* $c \in (0, 1]$ *and if* $a(t) \geqq (1 + \varepsilon)/(4ct^2)$, *then Equation* (28) *is oscillatory.*

We observe that for $c = 1$, Theorem C reduces to a well-known condition for oscillation of the ordinary equation

$$x'' + a(t)x = 0 \,.$$

Thus, as a corresponding result for delay equations, Theorem C can be obtained immediately from Theorem 3. In fact, by the Corollary of Theorem 3, Equation (28) is oscillatory if the ordinary equation

$$x'' + ca(t)x = 0$$

is oscillatory. But Equation (29) is oscillatory if $ca(t) \geqq (1 + \varepsilon)/(4t^2)$. Hence Theorem C follows.

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