COMPARISON THEOREMS FOR SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS*

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Abstract. Comparison theorems for a nonlinear eigenvalue problem as well as a Lyapunov type of inequality are derived. They are used to establish upper and lower bounds for various integral functionals associated with real solutions of the nonlinear boundary value problem $y'' + p(x)y^{2n+1} = 0$, $y(a) = y'(b) = 0$, where $a < b$ are real, n is a positive integer and p is positive and continuous on $[a, b]$. Some of the results are analogues of a distance between zeros problem for the linear case of $n = 0$.

Characteristic value problems for various nonlinear differential equations have been studied in recent years; a good bit of work has been done by Nehari, Moore and Moroney, among others [8], [9], [12], [13]. Much of this work involves the study of oscillation and nonoscillation of solutions of these differential equations.

The linear characteristic value problem

$$y'' + \lambda p(x)y = 0, \qquad y(a) = y'(b) = 0,$$

where $a < b$ are finite reals and p is real and continuous on $[a, b]$, has been studied for the behavior of the least positive eigenvalue $\lambda_1(p)$ relative to the "shape" of p on $[a, b]$. This has been done for other boundary conditions as well as for other positive eigenvalues and this problem is related to the determination of the fundamental frequencies of oscillation of a vibrating string. See for example Nehari [11], St. Mary [14], Fink [6] and especially the bibliography in [6] for a background on this problem.

The purpose of this paper is to establish various extensions and analogues of the linear theory to nonlinear differential equations.

Moore and Nehari [8] consider the nonlinear second-order differential equation of the form

$$y'' + p(x)y^{2n+1} = 0, \tag{1}$$

where n is a positive integer and p is positive and continuous on a compact interval of reals $[a, b]$ with $a < b$. Along with (1), Moore and Nehari consider the Rayleigh quotient

$$J(y) = \left(\int_a^b y'^2\,dx\right)^{n+1} \Big/ \left(\int_a^b py^{2n+2}\,dx\right) \tag{2}$$

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where the domain of J is

$$D[J] = \{y \in D'[a, b]: y(a) = 0 \quad \text{and} \quad y \not\equiv 0 \text{ on } [a, b]\}$$

where $D'[a, b]$ is the set of all continuous real-valued functions having sectionally continuous derivatives on $[a, b]$.

The following theorem is a portion of Theorem V of [8, p. 44] which deals with the boundary conditions $y(a) = y'(b) = 0$:

THEOREM (MOORE AND NEHARI). *For each pair of finite reals $a < b$, Eq.* (1) *has at least one solution $y \in C^2[a, b]$ which is positive in (a, b) and satisfies the boundary conditions*

$$y(a) = y'(b) = 0. \tag{3}$$

Furthermore this solution satisfies the property that

$$J(y) \leq J(u) \quad \text{for each } u \in D[J]. \tag{4}$$

Thus we see that (2) is minimized in $D[J]$ by a nontrivial $y \in C^2[a, b]$ which is a solution of (1) and (3). Such a solution will be called a *Moore–Nehari minimizing function* of (2) for the problem of (1) and (3).

The question of uniqueness of such a minimizing function is not in general answered. In case the boundary value problem (1) and (3) has a unique nontrivial solution which is positive in (a, b) we have, of course, uniqueness of the Moore–Nehari minimizing function.

If for our case we would assume in addition that p is monotone increasing on $[a, b]$, then a result of Moroney [10] assures us of the desired uniqueness. It would be highly desirable to have this result under less restriction on p. For our equation (1) together with (3), Moroney's proof can be modified to provide uniqueness if we assume in addition that $[(x - a)p(x)]' \geq 0$ on $[a, b]$. On the other hand, Moore and Nehari give an example of a p where the boundary value problem of (1) together with the boundary conditions $y(a) = y(b) = 0$ has two distinct solutions on (a, b).

It may first appear to the reader that the concept of a "Moore–Nehari minimizing function" is somewhat artificial. Due to the above remarks concerning uniqueness and certain relations to be established later, it will be shown that some of the most important results of the paper are independent of the concept.

We are now ready for our first result, which is basically a Lyapunov inequality.

THEOREM 1. *Let $\lambda_1(p)$ be the least positive value of $J(y)$ in* (2) *for $y \in D[J]$. Then it follows that*

$$(b - a)^{n+1}\lambda_1(p) \int_a^b p \, dx > 1. \tag{5}$$

Furthermore the inequality is sharp.

Proof. First, if y is any nontrivial solution of the problem (1) and (3), by integrating (1) after multiplying by y it follows from (3) that

$$\int_a^b y'^2 \, dx = \int_a^b p y^{2n+2} \, dx. \tag{6}$$

Consequently we have

$$\lambda_1(p) \leq \left(\int_a^b y'^2\,dx\right)^n = \left(\int_a^b py^{2n+2}\,dx\right)^n \tag{7}$$

where equality holds if y is a Moore–Nehari minimizing function of (2) for the problem of (1) and (3). Also, since $p(x) > 0$ on $[a, b]$, if $y(x) \neq 0$ on $(a, b]$ then we have

$$|y(x)| < |y(b)| \quad \text{for } x \in [a, b). \tag{8}$$

Now assume y is a Moore–Nehari minimizing function of (2) for the problem of (1) and (3). We have

$$\begin{aligned}
[y(b)]^{2n+2} &= \left(\int_a^b y'\,dx\right)^{2n+2} \\
&< (b-a)^{n+1}\left(\int_a^b y'^2\,dx\right)^{n+1} \\
&= (b-a)^{n+1}\lambda_1(p)\int_a^b py^{2n+2}\,dx \\
&\leq (b-a)^{n+1}\lambda_1(p)[y(b)]^{2n+2}\int_a^b p\,dx
\end{aligned}$$

from which (5) holds.

The sharpness can be established by a slight modification of an example found in [5] which in turn is a modification of an example used to show that the Lyapunov inequality for linear equations is sharp.

COROLLARY 1. *For all $y \in D[J]$ it follows that*

$$(b-a)^{n+1}\left(\int_a^b y'^2\,dx\right)^{n+1}\int_a^b p\,dx > \int_a^b py^{2n+2}\,dx. \tag{9}$$

COROLLARY 2. *If $y \in C^2[a, b]$ is any solution of* (1) *and* (3) *which is positive on (a, b) then we have*

$$(b-a)^{n+1}Q_i^n\int_a^b p\,dx > 1 \quad \textit{for} \quad 1 \leq i \leq 4, \tag{10}$$

where

$$\begin{aligned}
Q_1 &= \int_a^b py^{2n+2}\,dx, \qquad Q_2 = [y(b)]^{2n+2}\int_a^b p\,dx \\
Q_3 &= \int_a^b y'^2\,dx \quad \textit{and} \quad Q_4 = (b-a)[y'(a)]^2.
\end{aligned} \tag{11}$$

Proof. As in the case of (9), $(10)_1$ and $(10)_3$ are immediate from (8). For $(10)_2$ and $(10)_4$, under our assumptions, $|y|$ is increasing and concave on $[a, b]$ and satisfies $y(a) = 0$. Also y satisfies (6), (7) and (8) as well as

$$|y'(x)| < |y'(a)| \quad \text{for } x \in (a, b].$$

Thus $(10)_2$ and $(10)_4$ follow from $(10)_1$ and $(10)_3$ respectively.

We next wish to establish comparison theorems for eigenvalues. When p is assumed to be positive and continuous on the whole real line, Nehari [12] has already established various results. In [12], however, the problem is not that of minimizing (2), but one of minimizing

$$K(y) = \int_a^b (y'^2 - (n+1)^{-1} p y^{2n+2})\, dx, \tag{12}$$

subject to the condition that the domain of K be $D[J]$ and

$$\int_a^b y'^2\, dx = \int_a^b p y^{2n+2}\, dx. \tag{6}$$

(Nehari actually considers a more general equation than (1), but we shall state his results as applied to our problem.) Nehari establishes the existence of a minimizing function for (12) which is a solution of (1) and (3) and is positive on $(a, b]$. By direct computation, any minimizing function for (12) is also one for (2) and vice versa. Also if $\nu_1(p)$ is the least positive value of (12) and $\lambda_1(p)$ is the one for (2), then

$$\lambda_1(p) = [n^{-1}(n+1)\nu_1(p)]^n. \tag{13}$$

We state two theorems of [12, p. 113–115] (again as applied to our problem), the second being one we shall generalize. For notation we let $\nu_1(p;\, a, b)$ denote the least positive value of (12) on $[a, b]$ when the interval itself is of importance. For simplicity we assume p satisfies previously mentioned conditions.

THEOREM I (NEHARI). *For reals $a' \leq a < b \leq b'$ it follows that*

$$\nu_1(p;\, a', b') < \nu_1(p;\, a, b) \tag{14}$$

unless $a' = a$ and $b' = b$.

THEOREM II (NEHARI). *For $0 < p_1(x) \leq p_2(x)$ on $[a, b]$ it follows that*

$$\nu_1(p_1) \geq \nu_1(p_2). \tag{15}$$

This second result above has a Sturmian flavor and can be generalized by replacing the condition $0 < p_1(x) \leq p_2(x)$ by an integral condition, as is given for the linear case of $n = 0$ by Nehari in [11]. St. Mary [14] also considers this problem in the linear case. The proof of the following theorem, and in fact a fair amount of the development of parts of the whole paper, are adapted from a paper of Fink [6], who uses a Rayleigh quotient together with a result of Banks [2] to study the behavior of $\lambda_1(p)$ for linear differential equations.

THEOREM 2. *Let p_1 and p_2 be positive and continuous on $[a, b]$ and suppose*

$$\int_s^b p_1\, dx \leq \int_s^b p_2\, dx \quad \textit{for each} \quad s \in [a, b]. \tag{16}$$

Then if $\lambda_1(p_1)$ and $\lambda_1(p_2)$ are the two corresponding minimum values of (2) for p_1 and p_2 respectively, it follows that

$$\lambda_1(p_1) \geq \lambda_1(p_2) \tag{17}$$

where equality holds if and only if $p_1(x) \equiv p_2(x)$ on $[a, b]$.

Proof. Let y_1 be a Moore–Nehari minimizing function of (2) for the problem of (1) and (3) relative to p_1. We recall that y_1 is positive and concave on $(a, b]$. Then we have

$$\begin{aligned}\lambda_1(p_1) &= \left(\int_a^b y'^2\,dx\right)^{n+1} \Big/ \left(\int_a^b p_1 y_1^{2n+2}\,dx\right)\\ &= \left(\int_a^b y_1'^2\,dx\right)^{n+1} \Big/ \left(\int_0^M \left(\int_{\sigma(y)}^b p_1\,dx\right) dy\right)\\ &\geq \left(\int_a^b y_1'^2\,dx\right)^{n+1} \Big/ \left(\int_0^M \left(\int_{\sigma(y)}^b p_2\,dx\right) dy\right)\\ &= \left(\int_a^b y_1'^2\,dx\right)^{n+1} \Big/ \left(\int_a^b p_2 y_1^{2n+2}\,dx\right)\\ &= \lambda_1(p_2).\end{aligned}$$

The second equation above is due to y_1 increasing to $y(a) = 0$ and to an integral equation of Banks [2] where $M = [y_1(b)]^{2n+2}$ and $\sigma:[0, M] \to [a, b]$ is defined by

$$\sigma(y) = \inf\{x \in [a, b]:[y_1(x)]^{2n+2} \geq y\}.$$

By the continuity of p_1 and p_2 it is clear that the first inequality above becomes strict if the inequality in (16) is strict for some value of s. This is true whenever (16) holds and $p_1 \not\equiv p_2$ on $[a, b]$ and consequently this implies $\lambda_1(p_1) > \lambda_1(p_2)$. On the other hand, if $p_1 \equiv p_2$ we obviously have equality.

We have various corollaries of the theorem. Our first corollary is analogous to Theorem 5 of [14], which in turn goes back to Beesack and Schwarz [4].

Two continuous functions p and q are said to be *equimeasurable* on a compact interval $[a, b]$ if for each real y

$$m\{x \in [a, b] : p(x) \geq y\} = m\{x \in [a, b] : q(x) \geq y\},$$

where m denotes Lebesgue measure. Now for any such continuous function p there are two continuous functions, p^+ and p^-, which are equimeasurable with p and which are monotone decreasing and monotone increasing respectively on $[a, b]$. With p^+ and p^- so defined it can be established that

$$\int_s^b p^+\,dx \leq \int_s^b p\,dx \leq \int_s^b p^- \quad \text{for each} \quad s \in [a, b] \tag{18}$$

and thus we have

COROLLARY 1. *Let p be positive and continuous on $[a, b]$ and let p^+ and p^- be defined as above. Then we have*

$$\lambda_1(p^-) \leq \lambda_1(p) \leq \lambda_1(p^+). \tag{19}$$

As a preliminary to the next corollary we go back to (7) found in the proof of Theorem 1. From this, for two positive continuous functions p_1 and p_2 on an interval $[a, b]$ we have

$$[\lambda_1(p_1)]^{1/n} - [\lambda_1(p_2)]^{1/n} + \int_a^b (p_2 - p_1) y_2^{2n+2}\,dx \leq \int_a^b p_1(y_1^{2n+2} - y_2^{2n+2})\,dx, \tag{20}$$

where $y_1 \in C^2[a, b]$ is any nontrivial solution of (1) and (3) relative to p_1 and where y_2 is a Moore–Nehari minimizing function of (2) for the problem of (1) and (3) relative to p_2. Now the integral equation of Banks [2] can again be used to allow us to write

$$\int_a^b (p_2 - p_1) y_2^{2n+2} \, dx = \int_0^{M_2} \left(\int_{\sigma_2(y)}^b (p_2 - p_1) \, dx \right) dx, \tag{21}$$

where $M_2 = [y_2(b)]^{2n+2}$ and $\sigma_2 : [0, M_2] \to [a, b]$ is defined by

$$\sigma_2(y) = \inf \{ x \in [a, b] : [y_2(x)]^{2n+2} \geq y \}.$$

COROLLARY 2. *Let p_1 and p_2 be as in the theorem with y_1 and y_2 being as described above. Then it follows that*

$$|y_1(x_0)| > |y_2(x_0)| \quad \text{for some } x_0 \in (a, b] \tag{22}$$

unless $y_1 \equiv y_2$ on $[a, b]$ which in turn implies $p_1 \equiv p_2$.

Proof. The conditions on p_1 and p_2 allow us to conclude from (17) and (21) that the left-hand side of (20) is nonnegative and indeed positive if $p_1 \not\equiv p_2$. Since p_1 is positive, (22) being false would yield a contradiction to (20) unless $p_1 \equiv p_2$ and $y_1 \equiv y_2$.

Finally we have a corollary which relates to Theorem 2 as well as the preceding corollary. This result can be used to place an upper bound on the functional in $(10)_2$ depending on the "behavior" of p in $[a, b]$.

COROLLARY 3. *Suppose p_1 and p_2 are positive and continuous on $[a, b]$ with*

$$P_1(s) \equiv \int_s^b p_1 \, dx \quad \textit{and} \quad P_2(s) \equiv \int_s^b p_2 \, dx$$

satisfying

$$P_1(s) \geq P_2(s) \quad \textit{on } (a, s_0) \quad \textit{and} \quad P_1(s) \leq P_2(s) \quad \text{on } (s_0, b) \tag{23}$$

for some $s_0 \in [a, b]$. Also suppose y_1 and y_2 correspond to p_1 and p_2 as described following (20) *and satisfy*

$$|y_1(x)| \leq |y_2(x)| \quad \textit{for all } x \in [a, b]. \tag{24}$$

Then it follows that

$$\int_a^b p_1 \, dx \geq \int_a^b p_2 \, dx. \tag{25}$$

Furthermore, strict inequality holds unless $p_1 \equiv p_2$ on $[a, b]$.

Proof. The result obviously holds if $p_1 \equiv p_2$, so assume otherwise. By our assumptions, if (25) does not hold in the strict sense we may assume from (23) that $s_0 = a$ and consequently (16) is satisfied. Thus from (17) we have $\lambda_1(p_1) > \lambda_1(p_2)$. Now (20), (24) and (21) lead us to a contradiction.

We remark here that the conditions (23) are weaker than the corresponding "one-crossing" conditions on p_1 and p_2, namely

$$p_1(x) \geq p_2(x) \quad \text{on } (a, x_0) \quad \text{and} \quad p_1(x) \leq p_2(x) \quad \text{on } (x_0, b) \tag{23'}$$

for some $x_0 \in [a, b]$.

In order to place other bounds on the functionals in (5) and (10) which are sharper than previous results for certain classes of functions we shall make comparisons with solutions of constant coefficient equations.

If y_n is the real solution of the initial-value problem $y'' + y^{2n+1} = 0$, $y(0) = 0$, $y'(0) = 1$ then we let z_n denote the first positive zero of y_n'. It follows from [1] that

$$\begin{aligned} z_n &= (n+1)^{1/2(n+1)} \int_0^1 (1 - v^{2n+2})^{-1/2}\, dv \\ &= 2^{-1}(n+1)^{-(2n+1)/(2n+2)} B((2n+2)^{-1}, 2^{-1}) \\ &= (n+1)^{1/(2n+2)} \Gamma(1 + (2n+2)^{-1}) \Gamma(2^{-1}) / \Gamma(2^{-1} + (2n+2)^{-1}), \end{aligned} \tag{26}$$

where B and Γ are the complete beta and gamma functions respectively.

When y is a real solution of the boundary-value problem

$$\begin{gathered} y'' + Cy^{2n+2} = 0 \\ y(a) = 0, \qquad y'(b) = 0, \qquad y(x) \neq 0 \quad \text{on} \quad (a, b] \end{gathered} \tag{27}$$

where $a < b$ and $C > 0$ are reals then

$$\begin{aligned} (b-a)^2 [y(b)]^{2n} C &= z_n^2 (n+1)^{n/(n+1)}, \\ (b-a)^{2n+2} [y'(a)]^{2n} C &= z_n^{2n+2}, \\ (b-a)^{n+2} \lambda_1(C) C &= ((n+1)/(n+2))^n z_n^{2n+2}. \end{aligned} \tag{28}$$

Elementary computation yields

$$\begin{aligned} (n+1) &< z_n^2 (n+1)^{n/(n+1)} < (n+1)\pi^2/4, \\ (n+1) &< z_n^{2n+2} < (\pi/2)^{2n+2}, \end{aligned} \tag{29}$$

while fairly tedious computations with the aid of [1] yields

$$z_n^2 (n+1)^{n/(n+1)} < \{(n+1)/(n+2)\}^n z_n^{2n+2}. \tag{30}$$

Also from (26) we see that z_n is asymptotic to $(n+1)^{1/(2n+2)}$ as $n \to \infty$.

Barnes [3] uses the concepts of "increasing and decreasing on the average" relative to the density function when placing various bounds on the frequencies of oscillation of a vibrating string.

We shall say that a continuous function $p : [a, b] \to R$ is *monotone increasing (decreasing) on the average from the right on* $[a, b]$ if

$$P(s) \equiv \frac{1}{b-s} \int_s^b p\, dx \tag{31}$$

is monotone increasing (decreasing) on $[a, b]$. We change the phrase "from the right" to "from the left" when (31) is changed to

$$P(s) \equiv \frac{1}{s-a} \int_a^s p\, dx. \tag{31$'$}$$

It follows that if p is continuous and monotone increasing on $[a, b]$ then it is also true on the average from the right and the left. As an example we mention that $p(x) = (x-1)^2$ is monotone increasing on the average from the right, but not from the left on $[0, 3]$.

The reason for using (31) goes back to conditions (16) in Theorem 2. If we were to change the boundary conditions (3) to

$$y'(a) = y(b) = 0 \tag{3)'$$

we would make changes in (16) and also use (31)′ in the following theorem.

THEOREM 3. *Let p be positive and continuous on $[a, b]$. Then we have*

$$(b - a)^{n+1}\lambda_1(p) \int_a^b p\, dx \leq ((n + 1)/(n + 2))^n z_n^{2n+2} \tag{32}$$

if p is monotone increasing on the average from the right on $[a, b]$, and

$$(b - a)^{n+1}\lambda_1(p) \int_a^b p\, dx \geq ((n + 1)/(n + 2))^n z_n^{2n+2} \tag{32)'$$

if p is monotone decreasing on the average from the right on $[a, b]$.

Furthermore, equality holds if and only if p is identically a positive constant on $[a, b]$.

Proof. We provide a proof for (32) only.

Let p satisfy the hypothesis and let

$$C = (b - a)^{-1} \int_a^b p\, dx.$$

Then in Theorem 2, $p_2 \equiv p$ and $p_1 \equiv C$ satisfy (16) and consequently $\lambda_1(C) \geq \lambda_1(p)$ with equality if and only if $p \equiv C$. Now by using (28) for computing $\lambda_1(C)$ we obtain (32) as an easy consequence.

We are now able to sharpen Corollary 2 of Theorem 1 in a similar fashion.

COROLLARY 1. *If $y \in C^2[a, b]$ is any solution of (1) and (3) which is positive on $(a, b]$ and if p is monotone decreasing on the average from the right on $[a, b]$ then*

$$(b - a)^{n+1} Q_i^n \int_a^b p\, dx \geq ((n + 1)/(n + 2))^n z_n^{2n+2}, \tag{33}$$

for $i = 1, 2, 3, 4$ where Q_i is defined by (11).

$$(b - a)^{n+1} Q_i^n \int_a^b p\, dx \leq ((n + 1)/(n + 2))^n z_n^{n+2}, \tag{34}$$

If y is a Moore–Nehari minimizing function of (2) for the problem of (1) and (3) then for $i = 1, 3$ when p is monotone increasing on the average from the right on $[a, b]$.

The proof of Corollary 2 of Theorem 1 can be easily modified to provide a proof here. In the case of (34) we need equality in (7) which will be the case for a Moore–Nehari minimizing function. An interesting question is whether p being monotone increasing on the average from the right on $[a, b]$ is sufficient to assure uniqueness of the solution of (1) and (3) which is positive on (a, b).

We have not yet obtained upper bounds for the functionals in $(10)_2$ and $(10)_4$. Our next theorem will provide one for $(10)_2$. First we have a lemma.

LEMMA. *Let y be a solution of (1) and (3) which is positive on $(a, b]$. Then there are positive constants C_1 and C_2 satisfying*

$$z_n^2(n + 1)^{n/(n+1)}[y'(a)]^{-2n}(b - a)^{-2n-2} \leq C_1 \leq C_2 \leq z_n^{2n+2}[y(b)]^{-2n}(b - a)^{-2}, \tag{35}$$

and such that when $0 < C \leq C_1$. Then the solution u of (27) satisfies

$$0 < y(x) \leq u(x) \quad on\ [a, b] \tag{36}$$

and when $C_2 \leq C$ *it satisfies*

$$0 < u(x) \leq y(x) \quad on\ [a, b]. \tag{37}$$

Discussion of a proof. It can be shown by a straightforward argument that the constants C_1 and C_2 exist. This basically is due to the fact that when $x \in (a, b]$ is fixed and u is a positive solution of (27) then $u(x)$ is a decreasing function of C with $u(x) \to 0$ as $C \to \infty$ and $u(x) \to \infty$ as $C \to 0$.

The constant C_2 is thus the infimum, and in fact the minimum, of the values of C for which (37) holds, and C_1 is the maximum of the values of C for which (36) holds.

It can also be shown that when C is such that the solution u of (27) satisfies the initial condition

$$u'(a) = (b - a)^{-1}y(b)$$

then u is dominated by y on $[a, b]$. This value of C is from (28) the largest constant in (35).

Similarly when C is such that the solution u of (27) satisfies the terminal condition $u(b) = (b - a)y'(a)$ then u dominates y on $[a, b]$. Again (28) provides the value of the least constant in (35).

We now state our final result on bounds.

THEOREM 4. *Let p be positive, continuous and monotone increasing on the average from the right on $[a, b]$. Let y be a corresponding Moore–Nehari minimizing function of* (2) *for the problem of* (1) *and* (3). *Then it follows that*

$$(b - a)[y(b)]^{2n} \int_a^b p\, dx \leq z_n^{2n+2}. \tag{38}$$

Proof. We are in position to apply the lemma and Corollary 3 of Theorem 2. We let $p_2 = p$, $y_2 = y$, $p_1 = C_2$ (given by the lemma) and y_1 be the corresponding solution of (27) on $[a, b]$, which is denoted by u in (37). Now (37) yields (24), and since p is monotone increasing on the average from the right on $[a, b]$ we have in Corollary 3 of Theorem 2 that

$$(b - s)^{-1}P_1(s) - (b - s)^{-1}P_2(s)$$

is monotone decreasing on $[a, b]$. This is sufficient to establish (23). For on the one hand, if

$$(b - a)^{-1}P_1(a) - (b - a)^{-1}P_2(a) \leq 0,$$

then it certainly follows that

$$(b - s)^{-1}P_1(s) - (b - s)^{-1}P_2(s) \leq 0 \quad \text{on } [a, b),$$

which implies

$$P_1(s) \leq P_2(s) \quad \text{on } [a, b]$$

and $s_0 = a$ applies in (23). On the other hand, if

$$(b - a)^{-1}P_1(a) - (b - a)^{-1}P_2(a) > 0,$$

then

$$s_0 = \sup \{s \in [a, b) : (b - s)^{-1}P_1(s) - (b - s)^{-1}P_2(s) > 0\}$$

will apply in (23) to establish $P_1(s) \geq P_2(s)$ on (a, s_0) and $P_1(s) \leq P_2(s)$ on (s_0, b). Consequently we have

$$\int_a^b p\,dx \leq (b - a)C_2 \leq z_n^{2n+2}[y(b)]^{-2n}(b - a)^{-1},$$

from which (38) follows.

A similar upper bound for $(10)_4$ does not appear to be as easy to obtain.

By using the lemma again when p is monotone decreasing on the average from the right on $[a, b]$ we can place a lower bound of $z_n^2(n + 1)^{n/(n+1)}$ on the functional in $(10)_4$. By (30), however, the bounds in (33) are better.

Other authors have results which we may compare with ours. Hooker [7] considers a more general equation than (1), but his Theorem 3.1 as applied to (1) is given as

THEOREM (HOOKER). *Suppose* $P^* = \max \{p(x) : x \in [a, b]\}$ *and* $m > 0$ *satisfies the condition*

$$P^*m^{2n}(b - a)^{2n+2} = \pi^2/4. \tag{37}$$

Then for all $\mu \in (0, m)$ *the solution* y_μ *of the initial value problem*

$$y'' + p(x)y^{2n+1} = 0, \tag{1}$$

$$y(a) = 0, \qquad y'(a) = \mu, \tag{38}$$

exists on $[a, b]$ *and* $y_\mu'(x) > 0$ *on* (a, b); *in particular,* $y_\mu(x) \neq 0$ *on* $(a, b]$ *for all such values of* μ.

We may use $(10)_4$ to obtain a similar result as indicated below. Whenever

$$P^* < \frac{\pi^2}{4(b - a)} \int_a^b p\,dx,$$

Hooker's result is better, and our result is better when the inequality is reversed. Our result may be stated by simply changing the first sentence of Hooker's theorem to read: "Suppose $m_1 > 0$ satisfies the property that

$$(b - a)^{2n+1}m_1^{2n} \int_a^b p\,dx = 1\text{''}, \tag{37$'$}$$

where m is changed to m_1 in the second sentence of Hooker's theorem.

Now if the conclusion does not hold that $y_\mu(x) > 0$ or $y_\mu'(x) > 0$ on (a, b) then there is a least value of $c \in (a, b)$ such that $y_\mu'(c) = 0$. We may now apply $(10)_4$ on $[a, c]$ to reach the contradiction

$$1 = (b - a)^{2n+1}m_1^{2n} \int_a^b p\,dx \geq (c - a)^{2n+1}[y_\mu'(a)]^{2n} \int_a^c p\,dx > 1.$$

When p is monotone decreasing from the right on $[a, b]$ and n is such that

$$\{(n + 1)/(n + 2)\}^n z_n^{2n+2} > \pi^2/4,$$

from (33), our results are always better since

$$P^* < (b - a)^{-1} \int_a^b p\,dx$$

is impossible.

Moore and Nehari [8] also place some lower bounds on the functional $J(y)$ when p is quite specific, but they do not discuss our general situation.

In closing we wish to point out that by using differential inequalities some of the bounds we have obtained will work for other nonlinear differential equations. For example, if

$$h(y) \equiv y'' + py^{2n+1},$$

we may in Corollary 2 of Theorem 1 assume only that $y \in C^2[a, b]$ satisfies

$$yh(y) \geq 0 \tag{1)*$$

and (3) as well as being positive on $(a, b]$.

The same remark is true for our inequality (2) where p is assumed to be p_1 and y to be y_1 ; which, of course, has application in Corollaries 2 and 3 of Theorem 2.

Again, rather than Eq. (1) we may, by change of independent variable, consider the equation

$$\{r(t)u'\}' + q(t)u^{2n+1} = 0, \tag{1)'$$

where n is a positive integer, $q \in C[a, b]$, $r \in C'[a, b]$ and r and q are positive on $[a, b]$.

By letting

$$x = \int_a^t [r(s)]^{-1}\, ds \equiv g(t),$$

it follows that (1)′ can be written in the form of (1) on an interval $[0, d]$, where $d = g(b)$, and where

$$p(x) \equiv r(g^{-1}(x))q(g^{-1}(x)) = r(t)q(t),$$

$$y(x) \equiv u(g^{-1}(x)) = u(t).$$

In this case we let $\lambda_1(q, r)$ denote the minimum of the functional

$$J^*(u) = \left(\int_a^b ru'^2\, dt\right)^{n+1} \Big/ \left(\int_a^b qu^{2n+2}\, dt\right) \tag{2)'$$

where $D[J^*] = D[J]$. Then by change of variable we have

$$d^{n+1}\lambda_1(p) \int_0^d p(x)\, dx = \left(\int_a^b [r(t)]^{-1}\, dt\right)^{n+1} \lambda_1(q, r) \int_a^b q(t)\, dt, \tag{39}$$

as well as identities for other functionals in (10). We shall not present the details except to say that in Theorem 3, (32) is valid with the new functional in (39) replacing the old if $r(t)q(t)$ is monotone increasing on $[a, b]$. A corresponding situation develops in (33).

Finally, St. Mary [14] considers a linear equation such as (1)′ above where $n = 0$ and with the assumption that $q(t)$ be nonnegative on the average from the right, i.e.

$$Q(s) \equiv \frac{1}{b - s}\int_s^b q\, dt \geq 0 \quad \text{on} \quad [a, b],$$

rather than assuming $q(t) > 0$ on $[a, b]$. Intuitively we feel our results should be true as well under these relaxed conditions, but due to the dependence of our results on

Theorem V of Moore and Nehari where they assume a positive condition, we have not attempted to generalize our results.

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