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COMPARISON THEOREMS FOR THE THIRD ORDER TRINOMIAL DIFFERENTIAL EQUATIONS WITH DELAY ARGUMENT

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Abstract. In this paper we study asymptotic properties of the third order trinomial delay differential equation

$$y'''(t) - p(t)y'(t) + g(t)y(\tau(t)) = 0 \tag{$*$}$$

by transforming this equation to the binomial canonical equation. The results obtained essentially improve known results in the literature. On the other hand, the set of comparison principles obtained permits to extend immediately asymptotic criteria from ordinary to delay equations.

Keywords: comparison theorem, property (A), canonical operator

MSC 2010: 34C10

We consider the third order delay differential equation

$$y'''(t) - p(t)y'(t) + g(t)y(\tau(t)) = 0 \tag{1}$$

and the corresponding second order differential equation

$$v''(t) = p(t)v(t). \tag{2}$$

We always assume that

(i) $p(t)$ and $g(t) \in C([t_0,\infty))$, $p(t) \geqslant 0$, $g(t) > 0$, $\sup\{p(s)\colon\ s \geqslant t\} > 0$ for any $t \geqslant t_0$,

(ii) $\tau(t) \in C([t_0,\infty))$, $\tau(t) \leqslant t$ and $\tau(t) \to \infty$ as $t \to \infty$.

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We consider only nontrivial solutions of (1). Such a solution of (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

In general, most of the asymptotic results which have appeared in literature for delay differential equations are generalizations of known results for ordinary differential equations. Very often the method of proof of a generalized result is the same as that of the original result, sometimes requiring a severe restrictions on the delay. But when studying properties of the delay trinomial equations we cannot apply the technique used for equations without delay, since the presence of the term $-p(t)y'(t)$ and the delay argument does not allow it. On the other hand, the presence of the term $-p(t)y'(t)$ disables direct applications of Mahfoud's type of comparison theorems [15] for obtaining immediate results for the delay equation. Therefore not much is known about the asymptotic properties of solutions of delay trinomial equations.

In this chapter we propose two new methods to get over these difficulties. The first method employs Trench's theory of canonical operators, essentially utilizes the positive decreasing solution of (2) and is based on Mahfoud's comparison theorem to reduce the study of the asymptotic properties of solutions of (1) to that of an ordinary differential equations of the form (3), so that desirable generalizations of some asymptotic criteria from ordinary to delay equations of the same types become immediate. We give two illustrative applications of our technique by deriving some results from the corresponding ones in ordinary differential equations in [14] and [17].

The letter method also uses the positive decreasing solution of (2) and employs oscillation of a suitable second order equation for deducing desirable properties of (1).

In earlier papers [6], [7], [8], [10], [14] and [17] the authors have investigated a particular case of (1), namely the ordinary differential equation (without delay)

$$y'''(t) - p(t)y'(t) + g(t)y(t) = 0. \tag{3}$$

Lazer in [14] has shown that (3) has the following structure of nonoscillatory solutions:

Lemma 1. *Let* (i) *hold. Let $y(t)$ be a nonoscillatory solution of* (3). *Then there exists a $t_1 \geqslant t_0$ such that either*

$$y(t)y'(t) < 0 \quad \text{or} \tag{4}$$

$$y(t)y'(t) \geqslant 0 \tag{5}$$

for $t \geqslant t_1$ and moreover, if $y(t)$ satisfies (4) *then also*

$$(-1)^i y(t)y^{(i)}(t) > 0, \quad 0 \leqslant i \leqslant 3, \quad t \geqslant t_1. \tag{6}$$

It is known (see [14]) that (3) always has a solution satisfying (6).

Below we show that Eq. (1) has the same structure of nonoscillatory solutions as Eq. (3). We say that (1) has property (P_0) if every nonoscillatory solution $y(t)$ of (1) satisfies (6).

The prototype of results we wish to establish in the first part of the paper is the following theorem which is due to Lazer [14].

Theorem A. *Let* (i) *hold. If*

$$\int^{\infty}\Big[g(s)-\frac{2}{3\sqrt{3}}\big(p(s)\big)^{3/2}\Big]\,\mathrm{d}s=\infty,$$

then (3) *has property* (P_0).

This result has been improved and modified by several authors (see e.g. [8], [10], [17]). The following analogue of Theorem A is due to Škerlík [17].

Theorem B. *Let* (i) *hold. If*

$$\int^{\infty}\Big[s^2g(s)-sp(s)-\frac{2}{3\sqrt{3}s}\big(1+s^2p(s)\big)^{3/2}\Big]\,\mathrm{d}s=\infty,$$

then (3) *has property* (P_0).

We present a set of comparison theorems which enable us to deduce property (P_0) of the delay equation (1) from that of the ordinary equation (3) so that we will easily extend Theorem A and B to (1).

Preliminary results

All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all t large enough.

The following result shows the importance of Eq. (2).

Lemma 2. *The operator* $Ly\equiv y'''(t)-p(t)y'(t)$ *can be written as*

$$Ly\equiv\frac{1}{v}\Big(v^2\Big(\frac{1}{v}\,y'\Big)'\Big)',$$

where $v(t)$ *is a solution of* (2).

P r o o f. Straightforward computation shows that

$$Ly=\frac{1}{v}\Big(v^2\Big(-\frac{v'}{v^2}\,y'+\frac{1}{v}\,y''\Big)\Big)'=y'''-\frac{v''}{v}\,y'=y'''(t)-p(t)y'(t).$$

$\square$

For our next considerations it is useful for the operator Ly to be in canonical form that is $\int^\infty v^{-2}(t)\,\mathrm{d}t = \int^\infty v(t)\,\mathrm{d}t = \infty$. Now we present some useful properties of positive solutions of (2). In the sequel we shall work only with positive solutions of (2).

Lemma 3. *Eq.* (2) *possesses the couple of solutions*

$$v(t) > 0, \quad v'(t) < 0 \quad \text{and} \quad v''(t) \geqslant 0 \tag{7}$$

and

$$v(t) > 0, \quad v'(t) > 0 \quad \text{and} \quad v''(t) \geqslant 0 \tag{8}$$

for all t large enough.

Following Kiguradze we say that a positive function $v(t)$ is of degree 0 $\{2\}$ if (7) $\{(8)\}$ holds. A solution of degree 0 is also called *principal solution* [9].

The following lemma complements the classical results dealing with (2) presented in [1] and [9] and permits to obtain a solution of degree 0 if the corresponding solution of degree 2 is known.

Lemma 4. *If $v_2(t)$ is a solution of* (2) *of degree* 2 *then*

$$v_1(t) = v_2(t) \int_t^\infty v_2^{-2}(s)\,\mathrm{d}s$$

is also a solution of (2) *and moreover, $v_1(t)$ is of degree* 0.

P r o o f. It is easy to see that if $v_2(t)$ is of degree 2 then $\int^\infty v_2^{-2}(s)\,\mathrm{d}s < \infty$ so $v_1(t)$ is well defined. It is easy to see that v_1 is a solution of (2) since

$$v_1'' = v_2'' \int_t^\infty v_2^{-2}(s)\,\mathrm{d}s = p(t)v_2 \int_t^\infty v_2^{-2}(s)\,\mathrm{d}s = p(t)v_1.$$

On the other hand,

$$v_1'(t) = v_2'(t) \int_t^\infty v_2^{-2}(s)\,\mathrm{d}s - \frac{1}{v_2(t)}.$$

Noting that

$$\frac{1}{v_2(t)} = \int_t^\infty v_2'(s)v_2^{-2}(s)\,\mathrm{d}s > v_2'(t) \int_t^\infty v_2^{-2}(s)\,\mathrm{d}s,$$

so we see that $v_1'(t) < 0$. So $v_1(t)$ is of degree 0. $\square$

The next result is obvious.

Lemma 5. *If $v_1(t)$ is a solution of degree 0 of (2) then $\int^\infty v_1^{-2}(t)\,\mathrm{d}t = \infty$.*

We integrate our previous results to:

Lemma 6. *If $v_1(t)$ is a solution of degree 0 of (2) and*

$$\int^\infty v_1(t)\,\mathrm{d}t = \infty \tag{9}$$

then the operator $Ly \equiv y'''(t) - p(t)y'(t)$ can be written in the canonical form as

$$Ly \equiv \frac{1}{v_1}\Big(v_1^2\Big(\frac{1}{v_1}\,y'\Big)'\Big)'.$$

For our further considerations a solution of degree 0 is the key solution for us because if (9) is satisfied then Eq. (1) can be represented in canonical form as

$$\Big(v_1^2\Big(\frac{1}{v_1}\,y'\Big)'\Big)' + v_1(t)g(t)y(\tau(t)) = 0 \tag{10}$$

and it is preferable to study properties of (10) than those of (1). So it is desirable to have a criterion guaranteeing (9). Let us denote $\tilde{P}(t) = \int_t^\infty p(s)\,\mathrm{d}s$ (it is supposed that $\int^\infty p(s)\,\mathrm{d}s < \infty$).

Lemma 7. *Assume that*

$$\int_{t_0}^\infty \mathrm{e}^{-\int_{t_0}^t \tilde{P}(s)\,\mathrm{d}s}\,\mathrm{d}t = \infty. \tag{11}$$

Then every solution of degree 0 of (2) satisfies (9).

P r o o f. Let $v_1(t)$ satisfy (7). Integrating (2) from t to ∞, one gets

$$v_1'(\infty) - v_1'(t) = \int_t^\infty p(s)v_1(s)\,\mathrm{d}s,$$

where $v_1'(\infty) = \lim\limits_{t\to\infty} v_1'(t)$. We claim $v_1'(\infty) = 0$. If not, then $\lim\limits_{t\to\infty} v_1'(t) = -l$, $l > 0$. Then $v_1'(t) \leqslant -l$. Integrating from t_1 to t, we have $v_1(t) \leqslant v_1(t_1) - l(t - t_1) \to -\infty$ as $t \to \infty$. This is a contradiction and we conclude that

$$-v_1'(t) = \int_t^\infty p(s)v_1(s)\,\mathrm{d}s \leqslant v_1(t)\int_t^\infty p(s)\,\mathrm{d}s = v_1(t)\tilde{P}(t).$$

Then integrating from t_1 to t, we have

$$v_1(t) \geqslant v_1(t_1)\mathrm{e}^{-\int_{t_1}^t \tilde{P}(s)\,\mathrm{d}s}.$$

Now it is easy to see that the last inequality together with (11) implies (9). $\square$

Example 1. For the second order equation

$$v'' = \frac{\alpha(\alpha-1)}{t^2}\, v$$

condition (11) reduces to $\alpha \in ([1-\sqrt{5}]/2, 0)$ and t^α is a solution of degree 0.

Let us denote

$$L_0 y = y, \quad L_1 y = \frac{1}{v_1}(L_0 y)', \quad L_2 y = v_1^2 (L_1 y)', \quad L_3 y = (L_2 y)'.$$

We recall that a nonoscillatory solution of (10) is of degree 0 if

$$(12) \qquad (-1)^i y L_i y > 0, \quad i = 0, 1, 2, 3$$

and a nonoscillatory solution of (10) is of degree 2 if

$$(13) \qquad y L_i y > 0, \quad i = 0, 1, 2, \quad y L_3 y < 0,$$

eventually.

The following result is a modification of a well-known lemma of Kiguradze [11].

Lemma 8. *Assume that $v_1(t)$ is a solution of degree 0 of (2) satisfying (9). Then every positive nonoscillatory solution of canonical representation of (1), namely Eq. (10), is of degree 0 or 2.*

Following Kiguradze we say that (10) has property (A) if each of its nonoscillatory solutions is of degree 0 (i.e. it satisfies (12)).

Lemma 9. *Assume that $v_1(t)$ is a solution of degree 0 of (2) satisfying (9). Then a positive solution of (10) satisfies (12) if and only if it satisfies (6).*

P r o o f. It is clear that $y(t)$ is a solution of (1) iff it is a solution of (10). Assume that $y(t)$ is positive.

$\rightarrow$ Assume that $y(t)$ satisfies (12). From $L_1 y(t) < 0$, we have $y'(t) < 0$. Then (1) implies $y'''(t) < 0$. Therefore we must have $y''(t) > 0$ eventually, because in the opposite case integrating the inequality $y'(t) < y'(t_1)$ we would have $y(t) \to -\infty$ as $t \to \infty$.

$\leftarrow$ Let (6) hold. Then $L_0 y(t) > 0$ and $L_1 y(t) < 0$. On the other hand, it follows form (10) that $L_3 y(t) < 0$. Thus $L_2 y(t)$ is decreasing. If we admit $L_2 y(t) < 0$, eventually, then $L_1 y(t) < -l < 0$ and integrating from t_1 to t, one gets $y(t) < y(t_1) - l \int_{t_1}^{t} v_1(s)\, ds \to -\infty$ as $t \to \infty$. Therefore $L_2 y(t) > 0$ and (6) holds. $\square$

Lemma 9 can be reformulated as

Theorem 1. *Assume that $v_1(t)$ is a solution of degree* 0 *of* (2) *satisfying* (9). *Then* (10) *has property* (A) *if and only if* (1) *has property* (P_0).

Now we can deal with property (P_0) of the trinomial equation (1) with help of property (A) of the binomial equation (10).

Remark. If the assumptions of Theorem 1 hold and $\int^\infty v_1(s)g(s)\,\mathrm{d}s = \infty$ then (see e.g. [4], [5], [13]) Eq. (10) has property (A), which means that Eq. (1) has property (P_0). Consequently, in the sequel we may assume that $\int^\infty v_1(s)g(s)\,\mathrm{d}s < \infty$.

Main results I

Our goal in this part is to present a comparison principle that permits to deduce property (P_0) of the delay equation (1) from that of the equation without deviating argument, so that desirable generalizations of criteria for property (P_0) from ordinary to delay equations of the same types become immediate.

The following comparison result is a modification of that of Kusano & Naito [12] or Dzurina [5].

Theorem 2. *Assume that $v_1(t)$ is a solution of degree* 0 *of* (2) *satisfying* (9). *Let*

$$\tau \in C^1, \quad \tau'(t) > 0. \tag{14}$$

If

$$\Big(v_1^2(t)\Big(\frac{1}{v_1(t)}\,y'(t)\Big)'\Big)' + \frac{v_1(\tau^{-1}(t))g(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))}\,y(t) = 0 \tag{15}$$

has property (A) *then so does* (10).

Combining Theorem 1 and Theorem 2, we get

Theorem 3. *Assume that $v_1(t)$ is a solution of degree* 0 *of* (2) *satisfying* (9). *Let* (14) *hold. If* (15) *has property* (A) *then* (1) *has property* (P_0).

Applying Theorem 1 to (15), one gets

Theorem 4. *Assume that $v_1(t)$ is a solution of degree 0 of (2) satisfying (9). Let (14) hold. Then (15) has property (A) if and only*

$$y'''(t) - p(t)y'(t) + \frac{v_1(\tau^{-1}(t))g(\tau^{-1}(t))}{v_1(t)\tau'(\tau^{-1}(t))}\, y(t) = 0 \tag{16}$$

has property (P_0).

Finally, Theorems 3 and 4 provide

Theorem 5. *Assume that $v_1(t)$ is a solution of degree 0 of (2) satisfying (9). Let (14) hold. If (16) has property (P_0) then so does (1).*

This comparison theorem enables us to extend immediately Theorem A and B to delay equations.

Theorem 6. *Assume that $v_1(t)$ is a solution of degree 0 of (2) satisfying (9). Let (14) hold. If either*

$$\int^{\infty} \Big[\frac{g(t)v_1(t)}{v_1(\tau(t))} - \frac{2}{3\sqrt{3}} p^{3/2}(\tau(t))\tau'(t)\Big]\,\mathrm{d}t = \infty \quad \text{or}$$
$$\int^{\infty} \Big[\frac{\tau^2(t)g(t)v_1(t)}{v_1(\tau(t))} - p(\tau(t))\tau(t)\tau'(t) - \frac{2}{3\sqrt{3}\tau(t)}[1+\tau^2(t)p(\tau(t))]^{3/2}\tau'(t)\Big]\,\mathrm{d}t = \infty,$$

then (1) has property (P_0).

Proof. Applying Theorem A and B to (16) we get in view of Theorem 5 the assertion of the theorem. $\square$

Main results II

Now we present another comparison method for deducing property (P_0) of (1) from the absence of positive solutions of a suitable second order differential inequality. To simplify notation we set

$$\tilde{g}(t) = \frac{v_1(\tau^{-1}(t))g(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))} \quad \text{and} \quad \tilde{G}(t) = v_1(t)\int_{\tau^{-1}(t)}^{\infty} v_1(s)g(s)\,\mathrm{d}s.$$

Theorem 7. *Assume that $v_1(t)$ is a solution of degree* 0 *of* (2) *satisfying* (9). *Let* (14) *hold. If the second order differential inequality*

$$(17)\qquad \big(v_1^2(t)z'(t)\big)' + \left[v_1(t)\int_{\tau^{-1}(t)}^{\infty} v_1(s)g(s)\,\mathrm{d}s\right]z(t) \leqslant 0$$

has no positive solution then (1) *has property* (P_0).

P r o o f. Taking Theorem 3 into account it is sufficient to show that (17) implies property (A) of (15). Assume the contrary, that is, (15) has a positive solution $y(t)$ satisfying (13). Integrating (15) from t to ∞, we obtain

$$L_2y(t) = c + \int_t^{\infty} \tilde{g}(s)y(s)\,\mathrm{d}s, \quad c = \lim_{t\to\infty} L_2y(t).$$

Since $y(t) = y(t_1) + \int_{t_1}^t v_1(x)L_1y(x)\,\mathrm{d}x$ we have

$$L_2y(t) \geqslant \int_t^{\infty} \tilde{g}(s)\int_{t_1}^{s} v_1(x)L_1y(x)\,\mathrm{d}x\,\mathrm{d}s \geqslant \int_t^{\infty} \tilde{g}(s)\int_{t}^{s} v_1(x)L_1y(x)\,\mathrm{d}x\,\mathrm{d}s.$$

Changing order of integration leads to

$$L_2y(t) \geqslant \int_t^{\infty} L_1y(x)v_1(x)\int_x^{\infty} \tilde{g}(s)\,\mathrm{d}x\,\mathrm{d}s = \int_t^{\infty} L_1y(x)\tilde{G}(x)\,\mathrm{d}x.$$

Integrating from t_1 to t, we get

$$(18)\qquad L_1y(t) \geqslant L_1y(t_1) + \int_{t_1}^{t} \frac{1}{v_1^2(s)}\int_s^{\infty} L_1y(x)\tilde{G}(x)\,\mathrm{d}x\,\mathrm{d}s.$$

Let us denote the right hand side of (18) by $z(t)$, then $z(t) > 0$ and

$$\big(v_1^2(t)z'(t)\big)' + \tilde{G}(t)L_1y(t) = 0.$$

Since $L_1y(t) \geqslant z(t)$, we see that $z(t)$ is a positive solution of (17). This is a contradiction and the proof is complete. $\square$

Theorem 8. *Assume that $v_1(t)$ is a solution of degree 0 of (2) satisfying (9). Let (14) hold. If*

$$\liminf_{t\to\infty}\left(\int_{t_1}^{t}\frac{1}{v_1^2(s)}\,\mathrm{d}s\right)\left(\int_{t}^{\infty}v_1(s)\int_{\tau^{-1}(s)}^{\infty}v_1(x)g(x)\,\mathrm{d}x\,\mathrm{d}s\right)>\frac{1}{4} \tag{19}$$

then (1) has property (P_0).

Proof. It is known (see e.g. [4], [13]) that (19) is sufficient for (17) to have no positive solution. The assertion of the theorem follows from Theorem 7. □

Example 2. Let us consider the equation

$$y''' - \frac{2}{t^2}\,y' + \frac{a}{t^3}\,y(0.5t) = 0. \tag{E_1}$$

Clearly $v_1(t) = 1/t$ is a solution of degree 0 of the corresponding Eq. (2). Since (9) and (14) hold, applying Theorem 8 to (E_1) we ensure that (E_1) has property (P_0) if $a > 54$.

Generalization I

There is a natural question what to do if we are not able to solve (2). In this case, as stated below, we can replace the needed solution $v_1(t)$ by its asymptotic expression as $t \to \infty$ and Theorems 5, 7 and 8 still work. Although we can solve (2) only in some particular cases we have various kinds of necessary and sufficient conditions (see e.g. [1], [2], [9] and [19]) for the asymptotic expression of a solution of (2) as $t \to \infty$.

We note that $\tilde{v}_1(t)$ is an asymptotic expression as $t \to \infty$ of a function $v_1(t)$ if $\lim\limits_{t\to\infty} v_1(t)/\tilde{v}_1(t) = 1$. We will denote it by $v_1 \sim \tilde{v}_1$. Obviously, for any $\lambda \in (0,1)$ we have

$$\lambda\tilde{v}_1(t) \leqslant v_1(t) \leqslant \frac{1}{\lambda}\,\tilde{v}_1(t), \tag{20}$$

eventually.

The following result is a simple modification of a comparison principle which is due to Kusano & Naito [12] and Dzurina [5].

Theorem 9. *Assume that $v(t)$ is a solution of degree* 0 *of* (2), $v(t) \sim \tilde{v}_1(t)$ *and*

$$\int^{\infty} \tilde{v}_1(t)\,\mathrm{d}t = \infty. \tag{21}$$

Let (14) *hold and* $0 < \lambda < 1$. *If*

$$\Bigl(\frac{1}{\lambda^2}\,{\tilde{v}_1}^2(t)\Bigl(\frac{1}{\lambda\tilde{v}_1(t)}\,y'(t)\Bigr)'\Bigr)' + \frac{\lambda\tilde{v}_1(\tau^{-1}(t))g(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))}\,y(t) = 0 \tag{22}$$

has property (A) *then so does* (15).

The next comparison principle is a simple combination of Theorem 9 and Theorem 3.

Theorem 10. *Assume that $v(t)$ is a solution of degree* 0 *of* (2) *and* $v(t) \sim \tilde{v}_1(t)$. *Let* (14) *and* (21) *hold. If*

$$\liminf_{t\to\infty}\Bigl(\int_{t_1}^{t}\frac{1}{{\tilde{v}_1}^2(s)}\,\mathrm{d}s\Bigr)\Bigl(\int_t^{\infty}\tilde{v}_1(s)\int_{\tau^{-1}(s)}^{\infty}\tilde{v}_1(x)g(x)\,\mathrm{d}x\,\mathrm{d}s\Bigr) > \frac{1}{4} \tag{23}$$

then (1) *has property* (P_0).

P r o o f. It is clear from (23) that there exists a $\lambda \in (0,1)$ such that

$$\liminf_{t\to\infty}\Bigl(\int_{t_1}^{t}\frac{\lambda^2}{{\tilde{v}_1}^2(s)}\,\mathrm{d}s\Bigr)\Bigl(\int_t^{\infty}\lambda\tilde{v}_1(s)\int_{\tau^{-1}(s)}^{\infty}\lambda\tilde{v}_1(x)g(x)\,\mathrm{d}x\,\mathrm{d}s\Bigr) > \frac{1}{4}. \tag{24}$$

On the other hand (24) guarantees that

$$\Bigl(\frac{{\tilde{v}_1}^2(t)}{\lambda^2}\,z'(t)\Bigr)' + \Bigl[\lambda\tilde{v}_1(t)\int_{\tau^{-1}(t)}^{\infty}\lambda\tilde{v}_1(s)g(s)\,\mathrm{d}s\Bigr]z(t) \leqslant 0$$

has no positive solution. Thus taking the proof of Theorem 7 into account we see that this suffices for property (A) of (22). Then by Theorem 9 Eq. (15) enjoys property (A) too. Applying Theorem 3, we get property (P_0) of (1). $\square$

Roughly speaking, the previous theorem says that we can replace v_1 by its asymptotic representation $\tilde{v}_1$ in (19) and Theorem 8 holds true.

We present several results for the asymptotic expression of $v_1(t)$.

Lemma 10. *Assume that* $\int^{\infty} sp(s)\,\mathrm{d}s < \infty$. *Then* $v_1(t) \sim \tilde{v}_1(t) = 1$.

Proof. Let $t_1 \geqslant t_0$ be such that $\int_{t_1}^{\infty} sp(s)\,\mathrm{d}s < 1$. Let $v_1(t)$ be a solution of degree 0. In the proof of Lemma 7 we have shown that $\lim_{t\to\infty} v_1'(t) = 0$. Moreover, $\lim_{t\to\infty} v_1(t) = l \geqslant 0$. We need to show that $l > 0$. We assume the contrary. Then integrating (2) twice from $t \geqslant t_1$ to ∞, one gets

$$\begin{aligned} v(t) = \int_t^{\infty}\int_x^{\infty} p(s)v(s)\,\mathrm{d}s\,\mathrm{d}x &= \int_t^{\infty} (s-t)p(s)v(s)\,\mathrm{d}s \leqslant \int_t^{\infty} sp(s)v(s)\,\mathrm{d}s \\ &\leqslant v(t)\int_t^{\infty} sp(s)\,\mathrm{d}s < v(t), \end{aligned}$$

a contradiction. Then $v_1 \sim l \neq 0$ and so $l^{-1}v_1$ is the required asymptotic expression. □

Combining Theorem 10 with Lemma 10 we get the following criterion:

Corollary 1. *Let* (14) *hold. Assume that* $\int^{\infty} sp(s)\,\mathrm{d}s < \infty$. *If*

$$\liminf_{t\to\infty} t\left(\int_t^{\infty}\int_{\tau^{-1}(s)}^{\infty} g(x)\,\mathrm{d}x\,\mathrm{d}s\right) > \frac{1}{4}$$

then (1) *has property* (P_0).

The following result is recalled from [9, Corollary 9.2].

Lemma 11. *Assume that* $p(t) = \lambda^2 + \varphi(t)$, $\lambda > 0$. *If* $\int^{\infty} |\varphi(t)|\,\mathrm{d}t < \infty$. *Then* $v_1(t) \sim \tilde{v}_1(t) = \mathrm{e}^{-\lambda t}$.

The following criterion can be found in [1].

Lemma 12. *Assume that* $p(t) = 1 + \varphi(t)$. *If* $\lim_{t\to\infty} \varphi(t) = 0$ *and* $\int^{\infty} \varphi^2(t)\,\mathrm{d}t < \infty$ *then* $v_1(t) \sim \tilde{v}_1(t) = \exp[-t - \frac{1}{2}\int_{t_0}^{t} \varphi(s)\,\mathrm{d}s]$.

Since condition (21) fails for $\tilde{v}_1(t) = \mathrm{e}^{-t}$ and $\tilde{v}_1(t) = \exp[-t - \frac{1}{2}\int_{t_0}^{t} \varphi(s)\,\mathrm{d}s]$, Theorem 10 cannot be applied at present. We will recall Lemmas 11 and 12 later after we expand our theory.

In our above results, it has been supposed that (9) holds. Our next considerations are intended to make it possible to deduce property (P_0) of (1) even if a solution of degree 0 of (2) satisfies the opposite condition, namely

$$\int^{\infty} v_1(t)\,\mathrm{d}t < \infty. \tag{25}$$

Let us denote

$$\begin{aligned} r_0(t) &= \int_t^{\infty} v_1(s)\,\mathrm{d}s, \\ r_1(t) &= v_1(t)\Big(\int_t^{\infty} v_1(s)\,\mathrm{d}s\Big)^{-2}, \\ r_2(t) &= v_1^{-2}(t)\int_t^{\infty} v_1(s)\,\mathrm{d}s, \\ r_3(t) &= v_1(t). \end{aligned} \tag{26}$$

Theorem 11. *Assume that $v_1(t)$ is a solution of degree 0 of (2) satisfying (25). Then the operator $Ly \equiv y'''(t) - p(t)y'(t)$ can be represented in canonical form as*

$$Ly \equiv \frac{1}{r_3}\Big(\frac{1}{r_2}\Big(\frac{1}{r_1}\Big(\frac{y}{r_0}\Big)'\Big)'\Big)'.$$

Proof. Direct computation shows that

$$Ly = \frac{1}{r_3}\left(\frac{1}{r_2}\left[y''\,\frac{\int_t^{\infty} v_1(s)\,\mathrm{d}s}{v_1} - y'\,\frac{v_1'\int_t^{\infty} v_1(s)\,\mathrm{d}s}{v_1^2}\right]\right)' = y''' - p(t)y'.$$

Now we verify that Ly is in canonical form, i.e. $\int^{\infty} r_i(t)\,\mathrm{d}t = \infty$ for $i = 1, 2$. Let us denote $V(t) = \int_t^{\infty} v_1(s)\,\mathrm{d}s$. It is easy to see that (25) implies $\lim_{t\to\infty} V(t) = \lim_{t\to\infty} v_1(t) = 0$. Moreover, it follows from the proof of Lemma 7 that $\lim_{t\to\infty} v_1'(t) = 0$. Therefore

$$\int_{t_0}^{\infty} r_1(t)\,\mathrm{d}t = \int_{t_0}^{\infty} \frac{-V'(t)}{V^2(t)}\,\mathrm{d}t = \lim_{t\to\infty}\Big(\frac{1}{V(t)} - \frac{1}{V(t_0)}\Big) = \infty.$$

To ensure that $\int^{\infty} r_2(t)\,\mathrm{d}t = \infty$ it is sufficient to show that $\lim_{t\to\infty} r_2(t) > 0$. Applying L'Hospital rule, we get

$$\lim_{t\to\infty} r_2(t) = \lim_{t\to\infty} \frac{1}{-2v_1'(t)} = \infty.$$

□

Under the conditions of Theorem 11, Eq. (1) can be represented as the canonical equation

$$\Big(\frac{1}{r_2(t)}\Big(\frac{1}{r_1(t)}\Big(\frac{y(t)}{r_0(t)}\Big)'\Big)'\Big)' + r_3(t)g(t)y(\tau(t)) = 0$$

or, setting $y(t) = r_0(t)z(t)$, as

$$\Big(\frac{1}{r_2(t)}\Big(\frac{1}{r_1(t)}z'(t)\Big)'\Big)' + r_0(\tau(t))r_3(t)g(t)z(\tau(t)) = 0. \tag{27}$$

For (27) we set

$$L_0z = z, \quad L_1z = \frac{1}{r_1}(L_0z)', \quad L_2z = \frac{1}{r_2}(L_1z)', \quad L_3z = (L_2z)'.$$

Theorem 12. *Assume that* $v_1(t)$ *is a solution of degree* 0 *of* (2) *satisfying* (25). *If Eq.* (27) *has property* (A) *then Eq.* (1) *has property* (P_0).

Proof. Assume that $y(t)$ is a positive solution of (1). Then $z(t) = y(t)/r_0(t)$ is a positive solution of (27). Consequently, $z(t)$ is of degree 0, i.e.

$$L_0z > 0, \quad L_1z < 0, \quad L_2z > 0, \quad L_3z < 0,$$

eventually. Then

$$0 > (L_0z)' = \frac{y'r_0(t) + yv_1(t)}{r_0^2(t)}.$$

Considering the sign properties of the terms on the right hand side, one gets $y'(t) < 0$. Eq. (1) then implies $y'''(t) < 0$. Now exactly as in the proof of Lemma 7, we get $y''(t) > 0$, eventually. The proof is complete now. $\square$

The following result is an analogue of Theorem 2 for Eq. (27).

Theorem 13. *Assume that* $v_1(t)$ *is a solution of degree* 0 *of* (2) *satisfying* (25). *Let* (14) *hold. If equation*

$$\Big(\frac{1}{r_2(t)}\Big(\frac{1}{r_1(t)}z'(t)\Big)'\Big)' + \frac{r_0(t)r_3(\tau^{-1}(t))g(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))}z(t) = 0 \tag{28}$$

has property (A) *then so does* (27).

Combining Theorems 12 and 13 we can deduce property (P_0) of the delay trinomial equation from property (A) of the binomial equation without delay.

Theorem 14. *Assume that $v_1(t)$ is a solution of degree* 0 *of* (2) *satisfying* (25). *Let* (14) *hold. If Eq.* (28) *has property* (A) *then Eq.* (1) *has property* (P_0).

Now we are prepared to extend Theorem 7 to the case when $v_1(t)$ satisfies (25).

Theorem 15. *Assume that $v_1(t)$ is a solution of degree* 0 *of* (2) *satisfying* (25). *Let* (14) *hold. If the second order differential inequality*

$$\Big(\frac{1}{r_2(t)}u'(t)\Big)' + \Big[r_1(t)\int_{\tau^{-1}(t)}^{\infty} r_0(\tau(s))r_3(s)\,g(s)\,\mathrm{d}s\Big]u(t) \leqslant 0 \tag{29}$$

has no positive solution then (1) *has property* (P_0).

The proof follows all steps of the proof of Theorem 7 and so it is omitted.

Adding a criterion for absence of a positive solution of (29) we get in view of Theorem 15 a criterion for property (P_0).

Theorem 16. *Assume that $v_1(t)$ is a solution of degree* 0 *of* (2) *satisfying* (25). *Let* (14) *hold. If*

$$\liminf_{t\to\infty}\Big(\int_{t_0}^{t} r_2(s)\,\mathrm{d}s\Big)\Big(\int_{t}^{\infty} r_1(s)\int_{\tau^{-1}(s)}^{\infty} r_0(\tau(x))r_3(x)\,g(x)\,\mathrm{d}x\,\mathrm{d}s\Big) > \frac{1}{4} \tag{30}$$

then (1) *has property* (P_0).

Example 3. Let us consider the delay equation

$$y'''(t) - y'(t) + a\mathrm{e}^{(1-\lambda)t}y(\lambda t) = 0, \qquad a > 0, \quad 0 < \lambda < 1. \tag{E_2}$$

Obviously $v(t) = \mathrm{e}^{-t}$ is a solution of degree 0 of (2) and satisfies (25). Therefore $r_0(t) = r_3(t) = \mathrm{e}^{-t}$ and $r_1(t) = r_2(t) = \mathrm{e}^{t}$. Then condition (30) reduces to $a > \frac{1}{2}\lambda$, which by Theorem 16 guarantees property (P_0) of (E_2). Moreover, it is clear from our results here and from the classical comparison theorems (see e.g. [12], [4]) that the equation

$$y'''(t) - y'(t) + g(t)y(\lambda t) = 0, \qquad 0 < \lambda < 1$$

has property (P_0) provided that

$$g(t) > \frac{\lambda}{2}\,\mathrm{e}^{(1-\lambda)t}.$$

Remark. Arguing as in *Generalization I* it can be shown that Theorems 12–16 hold true even if we replace $v_1(t)$ by $\tilde{v}_1(t)$ in (26). Now we can return to Lemmas 11 and 12.

Example 4. Let us consider the delay equation

$$(E_3)\qquad y'''(t)-\frac{t^2+k}{t^2}\,y'(t)+g(t)y(\tau(t))=0,\quad k\in\mathbb{R},\quad \tau\in C^1,\quad \tau'(t)>0.$$

As $\varphi(t)=kt^{-2}$ satisfies the condition of Lemma 11, we have $v_1(t)\sim \mathrm{e}^{-t}$. By Theorem 16 Eq. (E_3) enjoys property (P_0) if

$$\liminf_{t\to\infty}\mathrm{e}^t\left(\int_t^\infty \mathrm{e}^s\int_{\tau^{-1}(s)}^\infty \mathrm{e}^{-x-\tau(x)}g(x)\,\mathrm{d}x\,\mathrm{d}s\right)>\frac{1}{4}.$$

Example 5. Let us consider the delay equation

$$(E_4)\qquad y'''(t)-\frac{t-2}{t}\,y'(t)+g(t)y(\tau(t))=0,\quad \tau\in C^1,\quad \tau'(t)>0.$$

Then $\varphi(t)=-2/t$ satisfies the conditions of Lemma 12. Thus $v_1(t)\sim t\mathrm{e}^{-t}$. It follows from Theorem 16 that Eq. (E_4) enjoys property (P_0) if

$$\liminf_{t\to\infty}\left(\int_3^t\frac{s+1}{s^2}\,\mathrm{e}^s\,\mathrm{d}s\right)\left(\int_t^\infty\frac{s\mathrm{e}^s}{(s+1)^2}\int_{\tau^{-1}(s)}^\infty x(1+\tau(x))\mathrm{e}^{-x-\tau(x)}g(x)\,\mathrm{d}x\,\mathrm{d}s\right)>\frac{1}{4}.$$

Remark. If a solution of degree 0 of (2) satisfies (9) then employing an additional condition, namely

$$\int_{t_0}^\infty v_1(s_3)\int_{s_3}^\infty v_1^{-2}(s_2)\int_{s_2}^\infty v_1(s_1)g(s_1)ds_1ds_2ds_3=\infty,$$

our results here concerning property (P_0) can be formulated in stronger form as *every nonoscillatory solution* $y(t)$ *of* (1) *satisfies* $\lim_{t\to\infty}y(t)=0$. Really, if $y(t)y'(t)<0$ and we assume $\lim_{t\to\infty}y(t)=l>0$ then $y(\tau(t))>l$. Integrating (10) twice from t to ∞ and then from t_1 to t, we have

$$\begin{aligned}y(t)&=y(t_1)-\int_{t_1}^t v_1(s_3)\int_{s_3}^\infty v_1^{-2}(s_2)\int_{s_2}^\infty v_1(s_1)g(s_1)y(\tau(s_1))ds_1ds_2ds_3\\&\leqslant y(t_1)-l\int_{t_1}^t v_1(s_3)\int_{s_3}^\infty v_1^{-2}(s_2)\int_{s_2}^\infty v_1(s_1)g(s_1)ds_1ds_2ds_3\to-\infty\end{aligned}$$

as $t\to\infty$. Consequently, $\lim_{t\to\infty}y(t)=0$.

Remark. We have presented a set of comparison theorems of higher order of generality. Theorem 5 enables us immediately to extend criteria for property (P_0) of trinomial third order differential equations to delay equations. So we were able to generalize results of Lazer, Skerlik and, as a matter of fact, we can generalize many more results.

On the other hand, Theorems 7 and 15 permit to deduce property (P_0) of (1) from the absence of positive solutions of the corresponding second order differential equation, covering both cases whether or not $\int^\infty v_1(s)\,\mathrm{d}s$ is convergent.

Theorems 8, 10, 16 present easily verifiable criteria for the desired property of (1) and Theorem 10 is applicable even if we are not able to find $v_1(t)$ but its asymptotic form is known.

Corollary 1 is applicable to a wide class of equations and its assumptions include only coefficients $\tau(t)$, $p(t)$ and $g(t)$. A result of this type is not known from earlier papers on property (P_0) of (1).

Presented comparison theorems generalize earlier ones od Dzurina [3], Parhi and Padhi [16].

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