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COMPATIBILITY CONDITIONS AT CORNERS BETWEEN WALLS AND INFLOW BOUNDARIES FOR FLUIDS OF MAXWELL TYPE

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Abstract. We prove an existence and uniqueness result for steady flows of fluids with differential constitutive laws of Maxwell type. The flows under consideration are small perturbations of plane Poiseuille flow, and the shear rate in the basic Poiseuille flow is assumed to be sufficiently small. The problem is posed on a rectangular domain, which has an inflow boundary and an outflow boundary. The main issue of interest is the nature of compatibility conditions which the data must satisfy in order to avoid singularities of the solution at the corners between the inflow boundary and the walls.

Wir zeigen einen Existenz- und Eindeutigkeitssatz für stationäre Strömungen von Flüssigkeiten mit differentiellen konstitutiven Gleichungen vom Maxwell'schen Typ. Die hier untersuchten Strömungen sind kleine Störungen einer ebenen Poiseuilleströmung, und es wird angenommen, daß die Scherrate in der ungestörten Poiseuilleströmung hinreichend klein ist. Das Problem ist gestellt in einem rechteckigen Gebiet, wobei die Flüssigkeit durch eine Seite hinein- und durch die gegenüberliegende Seite hinausfließt. Das Interesse konzentriert sich in erster Linie auf Kompatibilitätsbedingungen, welche die Daten des Problems erfüllen müssen, um Singularitäten an den Ecken zwischen der Seite, durch die die Flüssigkeit hineinströmt, und den Wänden zu vermeiden.

Доказываются существование и единственность стационарных течений жидкостей, выполняющих дифференциальное конститутивное уравнение Максвелловского типа. Изучаемые течения возмущают основное течение Пуазейля, и скорость этого течения Пуазейля предполагается малой. Проблема ставится на прямоугольной области; жидкость входит через одну из сторон и уходит через сторону, находящуюся напротив. Интерес сосредоточивается на условия совместимости, которые краевые условия должны выполнять, чтобы решение было несингулярное в углах между стороной входа и стенками.

Key words. Maxwell fluids, steady flow, behavior at corners, compatibility conditions

AMS(MOS) subject classifications. 35L80, 35M05, 76A10

1. Introduction. In [4], the author established an existence and uniqueness result for small perturbations of uniform flow of a Maxwell fluid transverse to a strip. For two space dimensions, it was shown that a well-posed problem is obtained if one prescribes the velocities at the inflow and outflow boundaries, and the diagonal components of the extra stress at the inflow boundary. The inflow value of the off-diagonal component of the extra stress can be determined as part of the solution.

In simulations of practically relevant flows, inflow boundaries typically join onto walls at a 90 degree angle. This raises the new issue which singularities might arise at such corners or which compatibility conditions should be imposed in order to avoid singularities. In contrast to reentrant corners, the corners between inflow boundaries and walls are generally

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regarded as "harmless" in numerical simulations, and the results of this paper provide some mathematical justification for this belief.

The construction of solutions in [4] is based on an iteration method, which alternates between solving a Stokes-type problem and determining the extra stresses by integration along streamlines. For the function spaces, one can use H^3 for velocities and H^2 for stresses. In the present paper, we reexamine the iteration procedure in the case where the inflow boundary joins onto a wall. It is found that if the prescribed data satisfy appropriate compatibility conditions, then it is still possible to carry out the iteration in the same function spaces. Hence singularities at the corner are very weak. In particular, stresses and velocity gradients remain bounded.

2. Formulation of the problem. The equations under study are the balance of momentum

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \operatorname{div}\,\mathbf{T} - \nabla p + \mathbf{f}, \tag{1}$$

the incompressibility condition

$$\operatorname{div}\,\mathbf{v} = 0, \tag{2}$$

and a constitutive relation of Maxwell type,

$$(\mathbf{v}\cdot\nabla)\mathbf{T} + \lambda\mathbf{T} = \eta\lambda(\nabla\mathbf{v} + (\nabla\mathbf{v})^T) + \mathbf{g}(\nabla\mathbf{v}, \mathbf{T}). \tag{3}$$

The tensor-valued function $\mathbf{g}$ is assumed to be smooth and $\mathbf{g}$ and its first derivatives are assumed to vanish when the arguments are zero.

Remark. For simplicity, we shall assume that a constitutive relation of the form (3) holds in two dimensions, with $\mathbf{T}$ being a 2×2-matrix. In general, T_{13} and T_{23} can be shown to vanish in purely two-dimensional motions, but even if the constitutive law in three dimensions has the form (3), the equation for T_{33} does not necessarily decouple from the rest. If it does not, then in the following we would also have to prescribe inflow data for T_{33} and require that they satisfy appropriate compatibility conditions. Otherwise, the analysis carries over without changes.

We consider two-dimensional flow in the domain $\Omega = (0,L)\times(0,1)$. The flow moves from left to right and the boundaries $y = 0, 1$ are walls where the no-slip boundary condition holds. We consider perturbations of a Poiseuille flow given by

$$p = P = -Gx,\ \mathbf{v} = \mathbf{V}(y) = (U(y), 0),\ \mathbf{T} = \Pi(y) = \begin{pmatrix} \Sigma(y) & \Upsilon(y) \\ \Upsilon(y) & \Gamma(y). \end{pmatrix} \tag{4}$$

If $G > 0$ is sufficiently small, the implicit function theorem can be used to determine $U(y)$ and $\Pi(y)$ as a solution of (1)-(3) (with $\mathbf{f} = 0$). To first order in G one obtains the Newtonian solution: $U(y) = \frac{G}{2\eta}y(1-y) + O(G^2)$, $\Sigma = O(G^2)$, $\Upsilon(y) = G(\frac{1}{2} - y) + O(G^2)$, $\Gamma = O(G^2)$. The Poiseuille flow given by (4) is perturbed by allowing a small body force $\mathbf{f}$ and by perturbing the boundary conditions. We use the following notation for the perturbations:

$$q = p - P, \ \mathbf{u} = \mathbf{v} - \mathbf{V}(y), \ \mathbf{S} = \mathbf{T} - \Pi(y). \tag{5}$$

In components, we shall write

$$\mathbf{u} = (u, v), \ \mathbf{S} = \begin{pmatrix} \sigma & \tau \\ \tau & \gamma. \end{pmatrix} \tag{6}$$

The boundary conditions which we prescribe, in addition to the no-slip condition at the walls, are as follows:

$$u = u_0(y), \ v = v_0(y), \ \sigma = \sigma_0(y), \ \gamma = \gamma_0(y), \ \text{at } x = 0,$$

$$u = u_1(y), \ v = v_1(y), \ \text{at } x = L. \tag{7}$$

Incompressibility requires that

$$\int_0^1 u_0(y)\, dy = \int_0^1 u_1(y)\, dy. \tag{8}$$

We are concerned with restrictions on the data that are needed to avoid corner singularities. Some restrictions on the velocities are obvious. In order to have continuity of velocity, we need

$$u_0(0) = u_0(1) = v_0(0) = v_0(1) = u_1(0) = u_1(1) = v_1(0) = v_1(1) = 0. \tag{9}$$

Moreover, continuity of velocity gradients and the incompressibility constraint require that

$$v_0'(0) = v_0'(1) = v_1'(0) = v_1'(1) = 0. \tag{10}$$

For further use, we note that formally we can compute all second derivatives of velocities at the corners in terms of the data. We obtain, for example

$$v_{xx}(0,0) = 0, \ u_{xy}(0,0) = -v_{yy}(0,0) = -v_0''(0), \ v_{xy}(0,0) = -u_{xx}(0,0) = 0,$$

$$u_{yy}(0,0) = u_0''(0). \tag{11}$$

Since the velocity vanishes at the corner, the term $(\mathbf{v}\cdot\nabla)\mathbf{T}$ disappears in (3). Hence we can compute the stresses in terms of the velocity gradients. Moreover, after differentiating (3) with respect to x and y, we can compute the first derivatives of the stresses. In order to obtain regular solutions, we shall therefore have to assume that σ_0, γ_0 and their first derivatives take on the right values at the corners.

To formulate a precise result, we need to introduce some function spaces. Let H^s denote the usual Sobolev spaces. In addition, for any non-negative integer n, we define $H_*^{n+1/2}(0,1)$ to be the set of all those functions $u \in H^{n+1/2}(0,1)$ such that the nth derivative $u^{(n)}$ has limiting values α and β at the endpoints in the sense that

$$\int_0^{1/2} y^{-1}|u^{(n)}(y)-\alpha|^2\,dy < \infty,\ \int_{1/2}^1 (1-y)^{-1}|u^{(n)}(y)-\beta|^2\,dy < \infty. \tag{12}$$

This space is closely related to the space $H_{00}^{n+1/2}$ defined in [3], which is obtained if the boundary values of u and its first n derivatives are restricted to vanish. In addition, we shall need the following weighted Sobolev spaces. Let $H_w^n(0,1)$ be the space of all functions which are of class H^n on any compact subinterval of $(0,1)$ and such that there exist constants α_0, ..., α_{n-1} and β_0, ..., β_{n-1} for which

$$\int_0^{1/2} y|u^{(n)}(y)|^2\,dy < \infty,\ \int_{1/2}^1 (1-y)|u^{(n)}(y)|^2\,dy < \infty,$$

$$\int_0^{1/2} y^{2(k-n)+1}\Big|u^{(k)}(y) - \frac{d^k}{dy^k}\Big[\sum_{i=0}^{n-1}\alpha_i y^i\Big]\Big|^2\,dy < \infty,$$

$$\int_{1/2}^1 (1-y)^{2(k-n)+1}\Big|u^{(k)}(y) - \frac{d^k}{dy^k}\Big[\sum_{i=0}^{n-1}\beta_i (1-y)^i\Big]\Big|^2\,dy < \infty,\ k=0,1,\ldots,n-1. \tag{13}$$

Moreover, let $H_\$^{n-1}(0,1)$ be the space of all functions in $H^{n-1}(0,1)$, which satisfy all the conditions in (13) except the first two, and let $H_\&^{n-2}(0,1)$ be the space of all functions in $H^{n-2}(0,1)$, which satisfy (13) for $k \le n-2$. We note that H_w^n and $H_*^{n-1/2}$ are both continuously embedded in $H_\$^{n-1}$. The former is clear from the definition; for the latter, see Remark 11.8 on page 70 of [3]. We denote the norm in H^s by $\|\cdot\|_s$ (it will be clear from context whether we are dealing with functions defined on $(0,1)$ or $(0,L)\times(0,1)$). The norms in $H_*^{n+1/2}$, H_w^n, $H_\$^n$ and $H_\&^n$ are denoted by $\|\cdot\|_{n+1/2,*}$, $\|\cdot\|_{n,w}$, $\|\cdot\|_{n,\$}$ and $\|\cdot\|_{n,\&}$.

The main result is as follows:

THEOREM. *Let G be chosen sufficiently small and let ϵ be sufficiently small relative to $G^{3/2}$. Assume that the data satisfy the following smallness hypothesis:*

$$\|\mathbf{f}\|_2 + \|u_0\|_{5/2,*} + \|v_0\|_{5/2,*} + \|u_1\|_{5/2} + \|v_1\|_{5/2} + \|\sigma_0\|_{2,w} + \|\gamma_0\|_{2,w} < \epsilon,$$

$$\|\sigma_0 + 2\eta v_0'\|_{1,\$} + \|\gamma_0 - 2\eta v_0'\|_{1,\$} < G\epsilon. \tag{14}$$

Moreover, assume that the prescribed velocities satisfy the compatibility conditions (8)-(10) and that σ_0 and γ_0 and their first derivatives assume compatible values at the end-points in the sense described above. Then there exists a solution of (1)-(3) which assumes the given boundary data and has the regularity $\mathbf{u} \in H^3$, $\mathbf{S} \in H^2$, $q \in H^2$. Except for an arbitrary constant in the pressure, this solution is the only one which has small norm.

We could of course make the second part of (14) superfluous by adjusting the size of ϵ. However, we obtain a somewhat sharper result the way the theorem is stated. The terms occuring in the second part of (14) are the differences between the prescribed stresses and the values which the stresses would have if the fluid were Newtonian. Since G is proportional to the Weissenberg number, it is physically reasonable to prescribe stresses close to the Newtonian ones when G is small.

3. Iterative construction of solution. We use essentially the same iterative procedure as in [4]. We apply the operation $(\mathbf{v} \cdot \nabla) + \lambda + (\nabla \mathbf{v})^T$ to the equation of motion (1) and we use (3) to substitute for $(\mathbf{v} \cdot \nabla)\mathbf{T}$. With ϕ denoting $(\mathbf{v} \cdot \nabla)p + \lambda p$, we find an equation of the form

$$\eta\lambda\Delta\mathbf{v} - \nabla\phi + \mathbf{h}(\mathbf{v}, \nabla\mathbf{v}, \nabla^2\mathbf{v}, \mathbf{T}, \nabla\mathbf{T}, \mathbf{f}, \nabla\mathbf{f}) = 0. \tag{15}$$

Here $\mathbf{h}$ is a complicated nonlinearity which we do not write out explicitly. For the perturbed quantities defined in (5), equation (15) takes on the form

$$\eta\lambda\Delta\mathbf{u} - \nabla\psi + \mathbf{k}(G, y, \mathbf{u}, \nabla\mathbf{u}, \nabla^2\mathbf{u}, \mathbf{S}, \nabla\mathbf{S}, \mathbf{f}, \nabla\mathbf{f}) = 0, \tag{16}$$

and equation (3) takes the form

$$((\mathbf{V} + \mathbf{u}) \cdot \nabla)\mathbf{S} + \lambda\mathbf{S} = \eta\lambda(\nabla\mathbf{u} + (\nabla\mathbf{u})^T) + \mathbf{r}(G, y, \mathbf{S}, \mathbf{u}, \nabla\mathbf{u}),$$

$$\mathbf{r}(G, y, \mathbf{S}, \mathbf{u}, \nabla\mathbf{u}) = -(\mathbf{u} \cdot \nabla)\Pi + \mathbf{g}(\nabla\mathbf{u} + \nabla\mathbf{V}, \mathbf{S} + \Pi) - \mathbf{g}(\nabla\mathbf{V}, \Pi). \tag{17}$$

Here $\psi = \phi + \lambda G x + GU(y)$ denotes the perturbation to ϕ.

The iterative construction of the solution alternates between solving the Stokes problem (16) and integrating (17) for the stresses. At each step of the iteration, we first determine a new velocity field from

$$\eta\lambda\Delta\mathbf{u}^{n+1} - \nabla\psi^{n+1} + \mathbf{k}(G, y, \mathbf{u}^n, \nabla\mathbf{u}^n, \nabla^2\mathbf{u}^n, \mathbf{S}^n, \nabla\mathbf{S}^n, \mathbf{f}, \nabla\mathbf{f}) = 0,$$

$$\text{div } \mathbf{u}^{n+1} = 0,$$

$$u^{n+1} = u_0(y),\ v^{n+1} = v_0(y),\ \text{at } x = 0,\ u^{n+1} = u_1(y),\ v^{n+1} = v_1(y) \text{ at } x = L,$$

$$u^{n+1} = v^{n+1} = 0 \text{ at } y = 0, 1. \tag{18}$$

We then determine a new stress from solving

$$((\mathbf{V} + \mathbf{u}^{n+1}) \cdot \nabla)\mathbf{S}^{n+1} + \lambda\mathbf{S}^{n+1} = \eta\lambda(\nabla\mathbf{u}^{n+1} + (\nabla\mathbf{u}^{n+1})^T)$$

$$+\mathbf{r}(G, y, \mathbf{S}^{n+1}, \mathbf{u}^{n+1}, \nabla\mathbf{u}^{n+1}), \tag{19}$$

subject to the initial conditions

$$\sigma^{n+1} = \sigma_0(y),\ \gamma^{n+1} = \gamma_0(y),\ \text{at } x = 0, \tag{20}$$

and an initial condition for τ^{n+1} which is determined by taking the curl of (1)

$$\rho \text{ curl } (((\mathbf{V} + \mathbf{u}^{n+1}) \cdot \nabla)\mathbf{u}^{n+1} + (\mathbf{u}^{n+1} \cdot \nabla)\mathbf{V}) = \text{curl div } \mathbf{S}^{n+1} + \text{curl } \mathbf{f},\ \text{at } x = 0. \tag{21}$$

It will be explained later how (21) is used to find an initial condition for τ^{n+1}.

As in [4], the proof is based on establishing that, for an appropriate choice of δ, the mapping $(\mathbf{u}^n, \mathbf{S}^n) \to (\mathbf{u}^{n+1}, \mathbf{S}^{n+1})$ as defined above is a contraction in a complete metric space. We choose δ sufficiently large relative to $\epsilon/\sqrt{G}$, but small relative to G. The complete metric space in which we show convergence of the iteration is the set $Z = \{(\mathbf{u}, \mathbf{S}) \mid \|\mathbf{u}\|_3 + \|\mathbf{S}\|_2 \leq \delta\}$, equipped with the metric $\|\mathbf{u}\|_2 + \|\mathbf{S}\|_1$. We first show that this space is mapped into itself by the iteration. In the following, we particularly emphasize those steps where the argument differs from [4]. The main issues are the presence of corners in the boundary (for the Stokes part of the iteration) and the vanishing velocity at the walls (leading to a degeneracy of the hyperbolic equation (19)).

4. Solution of Stokes problem. Let us assume that $(\mathbf{u}^n, \mathbf{S}^n)$ lies in Z. We first consider the solution of the Stokes problem (18). The following estimate holds:

$$\|\mathbf{u}^{n+1}\|_3 \leq C(\|\mathbf{k}\|_1 + \|u_0\|_{5/2} + \|v_0\|_{5/2} + \|u_1\|_{5/2} + \|v_1\|_{5/2}). \tag{22}$$

We note that $\|\mathbf{k}\|_1$ can be estimated by a constant times $\epsilon + \delta(\epsilon + G + \delta)$, and hence the right hand side of (22) is small relative to δ. The estimate (22) follows from the results in Chapter 7 of [2]. Since we have assumed the compatibility conditions (9) and (10) at the corners, we need only check that the equation

$$\sinh\,(\frac{\pi}{2}\lambda) = \pm\lambda \tag{23}$$

has no roots in the strip $-2 \leq \text{ Im } \lambda \leq 2$, other than the obvious roots 0 and $\pm i$. We set $\lambda = s + it$ in (23) and compare real and imaginary parts. Since both sides of (23) are odd functions of λ, we need only consider $t > 0$. We obtain

$$\cos(\frac{\pi}{2}t)\sinh(\frac{\pi}{2}s) = \pm s, \ \sin(\frac{\pi}{2}t)\cosh(\frac{\pi}{2}s) = \pm t. \tag{24}$$

One easily sees that $t = 2$ is impossible and $t = 0$ or $t = 1$ yields $s = 0$. Moreover, it follows from the second equation in (24) that for $0 < t < 2$ only the plus sign is possible on the right hand sides. For $1 < t < 2$, this implies that the terms in the first equation of (24) have opposite signs and hence cannot be equal. If $0 < t < 1$, then $\sin(\frac{\pi}{2}t) > t$, and hence the left hand side in the second equation of (24) is always bigger than the right hand side.

By the trace theorem, it is clear that the right hand side of (22) is an upper bound for the $H^{1/2}$-norm of the second derivatives of $\mathbf{u}^{n+1}$ on the inflow boundary $x = 0$. For future use, we actually need to estimate the norm of the second derivatives in $H^{1/2}_*$. We have $u_{yy} = u_0''$, $v_{yy} = v_0''$, and these are in $H^{1/2}_*$ by assumption. Moreover, because of the boundary condition at the walls, we have u_{xx}, $v_{xx} \in H^1((0,L);L^2(0,1)) \cap L^2((0,L);H^1_0(0,1))$, and hence their traces lie in the interpolation space $H^{1/2}_{00}(0,1)$ (see [2]). Finally, the incompressibility condition implies that $u_{xy} = -v_{yy}$, $v_{xy} = -u_{xx}$.

Remark. It is of interest to what extent the analysis here might be generalizable to other flow problems; in particular, what angles at the corner might be accommodated. For a general angle ω, the analogue of (23) is $\sinh(\omega\lambda) = \pm\lambda\sin\omega$. One obtains H^3 velocities as long as this equation has no roots (other than 0 and $\pm i$) in the strip $-2 \leq \text{Im } \lambda \leq 2$. Numerical solutions show that this is the case if the angle is less than approximately 126 degrees (see [1]). It is probable that H^3-velocities are not really needed; one would have to work in fractional order spaces to avoid this. The minimum one would expect to need is $H^{2+\theta}$ for velocities and $H^{1+\theta}$ for stresses ($\theta > 0$); in that case any convex angle can be allowed.

5. Inflow boundary condition for the shear stress. We next consider the determination of an inflow boundary condition for τ^{n+1}. A rearrangement of (21) yields

$$\rho \text{ curl } (((\mathbf{V}+\mathbf{u}^{n+1})\cdot\nabla)\mathbf{u}^{n+1}+(\mathbf{u}^{n+1}\cdot\nabla)\mathbf{V}) - \text{curl } \mathbf{f}$$

$$= \frac{\partial^2}{\partial x \partial y}(\gamma^{n+1}-\sigma^{n+1}) + \frac{\partial^2 \tau^{n+1}}{\partial x^2} - \frac{\partial^2 \tau^{n+1}}{\partial y^2}. \tag{25}$$

Moreover, from (19) we find

$$\frac{\partial \sigma^{n+1}}{\partial x} = \frac{1}{U+u^{n+1}}\Big(-v^{n+1}\frac{\partial \sigma^{n+1}}{\partial y} - \lambda\sigma^{n+1} + 2\eta\lambda\frac{\partial u^{n+1}}{\partial x}$$

$$+r_{11}(G,y,\mathbf{S}^{n+1},\mathbf{u}^{n+1},\nabla\mathbf{u}^{n+1})\Big),$$

$$\frac{\partial \gamma^{n+1}}{\partial x} = \frac{1}{U+u^{n+1}}\Big(-v^{n+1}\frac{\partial \gamma^{n+1}}{\partial y} - \lambda\gamma^{n+1} + 2\eta\lambda\frac{\partial v^{n+1}}{\partial y}$$

$$+r_{22}(G,y,\mathbf{S}^{n+1},\mathbf{u}^{n+1},\nabla\mathbf{u}^{n+1})\Big),$$

$$\frac{\partial \tau^{n+1}}{\partial x} = \frac{1}{U+u^{n+1}}\Big(-v^{n+1}\frac{\partial \tau^{n+1}}{\partial y} - \lambda\tau^{n+1} + \eta\lambda(\frac{\partial u^{n+1}}{\partial y} + \frac{\partial v^{n+1}}{\partial x})$$

$$+r_{12}(G,y,\mathbf{S}^{n+1},\mathbf{u}^{n+1},\nabla\mathbf{u}^{n+1})\Big), \tag{26}$$

We differentiate the last equation once more with respect to x and obtain

$$\frac{\partial^2 \tau^{n+1}}{\partial x^2} = -\frac{1}{U+u^{n+1}}\frac{\partial u^{n+1}}{\partial x}\frac{\partial \tau^{n+1}}{\partial x}$$

$$+\frac{1}{U+u^{n+1}}\Big(-\frac{\partial v^{n+1}}{\partial x}\frac{\partial \tau^{n+1}}{\partial y} - \lambda\frac{\partial \tau^{n+1}}{\partial x} + \eta\lambda(\frac{\partial^2 u^{n+1}}{\partial x\partial y} + \frac{\partial^2 v^{n+1}}{\partial x^2})$$

$$+\frac{\partial}{\partial x}r_{12}(G,y,\mathbf{S}^{n+1},\mathbf{u}^{n+1},\nabla\mathbf{u}^{n+1})\Big) - \frac{v^{n+1}}{U+u^{n+1}}\frac{\partial^2\tau^{n+1}}{\partial y\partial x}. \tag{27}$$

We substitute (26) for $\frac{\partial\tau^{n+1}}{\partial x}$, $\frac{\partial\sigma^{n+1}}{\partial x}$ and $\frac{\partial\gamma^{n+1}}{\partial x}$ in (27) and then substitute (26) and (27) into (25). In this way, we obtain a second order ordinary differential equation for τ^{n+1}, which needs to be solved. We first determine the values of τ^{n+1} and its derivatives at the corners. Recall that the velocities vanish at the corners, and hence (19) implies that

$$-\lambda\mathbf{S}^{n+1} + \eta\lambda(\nabla\mathbf{u}^{n+1}+(\nabla\mathbf{u}^{n+1})^T) + \mathbf{r}(G,y,\mathbf{S}^{n+1},\mathbf{u}^{n+1},\nabla\mathbf{u}^{n+1}) = 0. \tag{28}$$

This equation determines the value of $\mathbf{S}^{n+1}$ at the corners. By assumption, the values of σ_0 and γ_0 are chosen consistent with (28) and we must seek τ^{n+1} such that its values at the corners are also given by (28). By differentiating (19) with respect to x and y, we formally obtain the following identities at the corners:

$$-\lambda\frac{\partial \mathbf{S}^{n+1}}{\partial x}+\eta\lambda\frac{\partial}{\partial x}(\nabla\mathbf{u}^{n+1}+(\nabla\mathbf{u}^{n+1})^T)+\frac{\partial}{\partial x}\mathbf{r}(G,y,\mathbf{S}^{n+1},\mathbf{u}^{n+1},\nabla\mathbf{u}^{n+1})=0, \tag{29}$$

$$(\frac{\partial U}{\partial y}+\frac{\partial u^{n+1}}{\partial y})\frac{\partial \mathbf{S}^{n+1}}{\partial x}=-\lambda\frac{\partial \mathbf{S}^{n+1}}{\partial y}$$

$$+\eta\lambda\frac{\partial}{\partial y}(\nabla\mathbf{u}^{n+1}+(\nabla\mathbf{u}^{n+1})^T)+\frac{\partial}{\partial y}\mathbf{r}(G,y,\mathbf{S}^{n+1},\mathbf{u}^{n+1},\nabla\mathbf{u}^{n+1}). \tag{30}$$

We can now compute $\partial\mathbf{S}^{n+1}/\partial x$ from (29) and then $\partial\mathbf{S}^{n+1}/\partial y$ from (30). It was assumed that the values of σ_0' and γ_0' at the corner are compatible with this procedure, and we shall seek τ^{n+1} such that $\partial\tau^{n+1}/\partial y$ is also compatible. We note that by applying l'Hospital's rule to the right hand side of (26), we obtain equation (30). Hence the determination of the derivatives of $\mathbf{S}^{n+1}$ from (29) and (30) is consistent with (26). We now seek an inflow value for τ^{n+1} in the form $\tau^{n+1}=\tilde{\tau}^{n+1}+\tau_0$, where τ_0 is a given function (e.g. a third degree polynomial) such that τ_0 and its derivative assume compatible boundary values. Since τ_0 is given entirely in terms of the data, its size is of order ϵ. We seek $\tilde{\tau}^{n+1}$ in the space $H^2_{w,0}=\{\tau\in H^2_w \mid \tau(0)=\tau'(0)=\tau(1)=\tau'(1)=0\}$.

Before we proceed further, we need some simple properties of the weighted Sobolev spaces defined in the previous section. We state these properties as a lemma.

LEMMA.

a) *$H^1_\$(0,1)$ is a Banach algebra.*

b) *$H^1_w(0,1)$ is a Banach algebra.*

c) *The operator of differentiation is continuous from $H^n_w(0,1)$ into $H^{n-1}_w(0,1)$.*

d) *Let U be a continuously differentiable function on $[0,1]$ which has simple zeros at the endpoints and is positive in the interior. Let $d(y)=\min(y,1-y)$. Assume that*

$$a=\int_0^{1/2} y^{-1}(\frac{y}{U}-\frac{1}{U'(0)})^2\,dy+\int_{1/2}^1(1-y)^{-1}(\frac{1-y}{U}+\frac{1}{U'(1)})^2\,dy<\infty,$$

$$b=\int_0^{1/2} y\Big(\frac{U-yU'}{U^2}\Big)^2\,dy+\int_{1/2}^1(1-y)\Big(\frac{U+(1-y)U'}{U^2}\Big)^2\,dy<\infty. \tag{31}$$

Let

$$\kappa(U)=\sqrt{a}+\sqrt{b}+c,\ c=\max_{[0,1]}\Big(\frac{d(y)}{U}+\Big|\frac{d(y)^2U'}{U^2}\Big|\Big). \tag{32}$$

Let $H^1_{\$,\#}$ be the subspace of all functions in $H^1_{\$}$ which vanish at the endpoints. Then the mapping $\phi \to \phi/U$ is continuous from $H^1_{\$,\#}$ into H^1_w, with norm bounded by a constant times $\kappa(U)$.

e) *There is a constant C such that for any function $u \in H^{n+1/2}_*(0,1)$ and any $G > 0$ there exists $\tilde{u} \in H^{n+1}_w$ with*

$$\|u - \tilde{u}\|_{n,\$} \le CG\|u\|_{n+1/2,*} \text{ and } \|\tilde{u}\|_{n+1,w} \le \frac{C}{G}\|u\|_{n+1/2,*}. \tag{33}$$

We remark that for (31) to hold it is sufficient that U' is Hölder continuous. We sketch the proof of the lemma. First we show that $H^1_w(0,1)$ is continuously imbedded in $C(0,1)$. From the definition it is clear that H^1_w is the direct sum of the space of linear functions and the space $\{\phi \in H^1_{\text{loc}}(0,1) \mid d(y)^{-1/2}\phi \in L^2(0,1),\ d(y)^{1/2}\phi' \in L^2(0,1)\}$. If ϕ is in this latter space, then, by the Cauchy-Schwarz inequality, $\phi\phi' \in L^1(0,1)$ and hence $\phi^2 \in C(0,1)$. A fortiori, $H^1_{\$}$ is also imbedded in $C(0,1)$. Parts a and b of the lemma follow easily. Part c of the lemma is obvious from the definitions. For part d, we first decompose $H^1_{\$,\#}$ as $X+Y$, where X is the space of all third degree polynomials which vanish at the endpoints 0 and 1, and $Y = \{\phi \in H^1(0,1) \mid d(y)^{-3/2}\phi \in L^2(0,1),\ d(y)^{-1/2}\phi' \in L^2(0,1)\}$. We note that $(\phi/U)' = \phi'/U - \phi U'/U^2$. If ϕ lies in Y, it follows that ϕ/U lies in H^1_w, with norm bounded by $c\|\phi\|_Y$. If ϕ lies in X, we can write it in the form $\phi(y) = \alpha y + \beta y^2 - (\alpha+\beta)y^3 = (2\alpha+\beta)(1-y) - (3\alpha+2\beta)(1-y)^2 + (\alpha+\beta)(1-y)^3$. We then obtain

$$\int_0^{1/2} y^{-1}\Big(\frac{\phi}{U} - \frac{\alpha}{U'(0)}\Big)^2 dy + \int_0^{1/2} y\Big(\frac{\phi'}{U} - \frac{\phi U'}{U^2}\Big)^2 dy$$

$$= \alpha^2 \int_0^{1/2} y^{-1}\Big(\frac{y}{U} - \frac{1}{U'(0)}\Big)^2 dy + \alpha^2 \int_0^{1/2} y\Big(\frac{1}{U} - \frac{yU'}{U^2}\Big)^2 dy + \ldots \tag{34}$$

A similar calculation holds for the interval $(1/2,1)$. The integrals on the right hand side of (34) are bounded by a and b, respectively, and the terms indicated by dots involve only integrals of bounded functions. This completes the proof of d. For part e, we use the fact that, in the notation of [5],

$$H^{n+1}_w(0,1) = W^{n+1}_2((0,1),\rho^{-1},\rho^{2n+1}) + P_{2n+1}(0,1),$$

$$H^{n+1/2}_*(0,1) = W^{n+1/2}_2((0,1),1,\rho^{2n+1}) + P_{2n+1}(0,1),$$

$$H^n_{\$}(0,1) = W^n_2((0,1),\rho,\rho^{2n+1}) + P_{2n+1}(0,1), \tag{35}$$

where $P_{2n+1}(0,1)$ denotes the space of polynomials of degree $2n+1$. Using the results of Chapter 3 in [5], we therefore find that $H^{n+1/2}_*$ is equal to the interpolation space $[H^{n+1}_w, H^n_{\$}]_{1/2}$, and part e follows immediately.

We are now in a position to discuss the various terms occuring in (25). The left hand side does not depend on τ^{n+1} and its norm in H^0_w can be estimated by a constant times $\epsilon+(G+\nu)\nu$, where we set $\nu=\|\mathbf{u}^{n+1}\|_{5/2,*}+\|\nabla\mathbf{u}^{n+1}\|_{3/2,*}+\|\nabla^2\mathbf{u}^{n+1}\|_{1/2,*}$. Using parts a,c and d of the lemma, we find that the mapping

$$\tilde{\tau}^{n+1}\to\frac{\partial^2}{\partial x\partial y}(\gamma^{n+1}-\sigma^{n+1}), \tag{36}$$

where the right hand side is given by (26), is smooth from $H^2_{w,0}$ into H^0_w. Moreover, we obtain the estimate

$$\begin{aligned}\|\frac{\partial^2}{\partial x\partial y}(\gamma^{n+1}-\sigma^{n+1})\|_{0,w}\leq C\kappa(U+u^{n+1})\Big((G+\epsilon)\epsilon+G(\nu+\|\tilde{\tau}^{n+1}\|_{1,\$}+\epsilon)\\ +(\nu+\|\tilde{\tau}^{n+1}\|_{1,\$}+\epsilon)^2\Big).\end{aligned} \tag{37}$$

Here we have taken into account the second part of (14) and the explicit form of $\mathbf{r}$ as given by (17). We note that $\kappa(U+u^{n+1})$ is of order $1/G$.

We next use part e of the lemma to construct τ_1^{n+1} such that

$$\begin{aligned}\|\eta(\frac{\partial u^{n+1}}{\partial y}+\frac{\partial v^{n+1}}{\partial x})-\tau_0-\tau_1^{n+1}\|_{1,\$}\leq CG\nu,\\ \|\tau_1^{n+1}\|_{2,w}\leq C(\frac{\nu}{G}+\epsilon).\end{aligned} \tag{38}$$

From (28)-(30), it follows that the corner values of $\tau_0-\eta(\frac{\partial u^{n+1}}{\partial y}+\frac{\partial v^{n+1}}{\partial x})$ and its first derivative are bounded by a constant times $G\epsilon$, and hence we can choose τ_1^{n+1} in such a way that τ_1^{n+1} and its derivatives vanish on the boundaries and (38) still holds. Let $\tau_2^{n+1}=\tilde{\tau}^{n+1}-\tau_1^{n+1}$. Using (26), we obtain the estimate

$$\begin{aligned}\|\frac{\partial\tau^{n+1}}{\partial x}\|_{1,w}\leq C\kappa(U+u^{n+1})\Big(\epsilon\|\tau^{n+1}\|_{2,w}+\|\tau^{n+1}-\eta(\frac{\partial u^{n+1}}{\partial y}+\frac{\partial v^{n+1}}{\partial x})\|_{1,\$}\\ +G(\nu+\|\tilde{\tau}^{n+1}\|_{1,\$}+\epsilon)+(\nu+\|\tilde{\tau}^{n+1}\|_{1,\$}+\epsilon)^2\Big).\end{aligned} \tag{39}$$

We note that

$$\|\tau^{n+1}-\eta(\frac{\partial u^{n+1}}{\partial y}+\frac{\partial v^{n+1}}{\partial x})\|_{1,\$}\leq CG\nu+\|\tau_2^{n+1}\|_{1,\$},$$

$$\|\tau^{n+1}\|_{2,w} \leq C(\epsilon + \frac{\nu}{G}) + \|\tau_2^{n+1}\|_{2,w}. \tag{40}$$

Moreover, we can split $\partial\tau^{n+1}/\partial x$ in the form

$$\frac{\partial\tau^{n+1}}{\partial x} = -\frac{\lambda}{U+u^{n+1}}\tau_2^{n+1} + \chi^{n+1}, \tag{41}$$

and the remainder term χ^{n+1} satisfies the estimates

$$\|\chi^{n+1}\|_{1,w} \leq C\kappa(U+u^{n+1})\Big(\epsilon\|\tau^{n+1}\|_{2,w} + G(\nu + \|\tilde{\tau}^{n+1}\|_{1,\$} + \epsilon) + (\nu + \|\tilde{\tau}^{n+1}\|_{1,\$} + \epsilon)^2\Big),$$

$$\|\chi^{n+1}\|_{0,\$} \leq C\kappa(U+u^{n+1})\Big(\epsilon\|\tau^{n+1}\|_{1,\$} + (G+\nu+\|\tilde{\tau}^{n+1}\|_{1,\$}+\epsilon)(\nu+\|\tilde{\tau}^{n+1}\|_{0,\&}+\epsilon)\Big). \tag{42}$$

We can now discuss the terms on the right hand side of (27). We obtain

$$\|\frac{1}{U+u^{n+1}}\frac{\partial u^{n+1}}{\partial x}\frac{\partial\tau^{n+1}}{\partial x}\|_{0,w} + \|\frac{v^{n+1}}{U+u^{n+1}}\frac{\partial^2\tau^{n+1}}{\partial x\partial y}\|_{0,w}$$

$$\leq C\frac{\epsilon}{G}\|\frac{\partial\tau^{n+1}}{\partial x}\|_{1,w}. \tag{43}$$

Moreover, we find

$$\Big\|\frac{1}{U+u^{n+1}}\Big(-\frac{\partial v^{n+1}}{\partial x}\frac{\partial\tau^{n+1}}{\partial y} - \lambda\chi^{n+1} + \eta\lambda(\frac{\partial^2 u^{n+1}}{\partial x\partial y} + \frac{\partial^2 v^{n+1}}{\partial x^2})$$

$$+\frac{\partial}{\partial x}r_{12}(G, y, \mathbf{S}^{n+1}, \mathbf{u}^{n+1}, \nabla\mathbf{u}^{n+1})\Big)\Big\|_{0,w} \leq \frac{C}{G}\Big(\nu\|\tau^{n+1}\|_{1,\$} + \|\chi^{n+1}\|_{0,\$}$$

$$+\nu + (G+\nu+\epsilon+\|\tilde{\tau}^{n+1}\|_{1,w})(\nu+\epsilon+\|\tilde{\tau}^{n+1}\|_{1,w} + \|\frac{\partial\mathbf{S}^{n+1}}{\partial x}\|_{0,\$})\Big). \tag{44}$$

Equation (25) now assumes the form

$$\frac{\partial\tau_2^{n+1}}{\partial y^2} - \frac{\lambda^2}{(U+u^{n+1})^2}\tau_2^{n+1} = \omega^{n+1}, \tag{45}$$

and if we assume that $\epsilon << G$, $\nu << G$ and $\|\tau^{n+1}\|_{1,\$} << G$, then the above estimates yield, after a straightforward calculation,

$$\|\omega^{n+1}\|_{0,w} \leq C(\frac{\epsilon}{G} + \frac{\nu}{G} + \frac{\epsilon^2}{G^2}\|\tau_2^{n+1}\|_{2,w} + (1+\frac{\epsilon}{G^2})\|\tau_2^{n+1}\|_{1,\$} + \frac{1}{G}\|\tau_2^{n+1}\|_{0,\&}). \tag{46}$$

Let $A\tau^{n+1}$ denote the expression on the left hand side of (45). The following lemma holds.

LEMMA. *The operator A is bijective from $H^2_{w,0}$ onto H^0_w. Moreover, an estimate of the form*

$$\|\tau\|_{2,w} + \frac{1}{G}\|\tau\|_{1,\$} + \frac{1}{G^2}\|\tau\|_{0,\&} \le C\|A\tau\|_{0,w} \tag{47}$$

holds.

To prove the lemma, we first derive some energy estimates. We consider the equation $A\tau = \omega$ and, to simplify notation, we set $\tilde{U} = U + u^{n+1}$. We multiply the equation by $\tilde{U}\tau''/G$, and obtain after an integration by parts

$$\int_0^1 \frac{\tilde{U}}{G}|\tau''|^2 + \frac{\lambda^2}{G\tilde{U}}|\tau'|^2\, dy - \int_0^1 \frac{\lambda^2 \tilde{U}'}{G\tilde{U}^2}\tau\tau'\, dy = \int_0^1 \frac{\tilde{U}}{G}\omega\tau''\, dy. \tag{48}$$

Next we multiply the equation by $-\tau/G\tilde{U}$ and obtain

$$\int_0^1 \frac{1}{G\tilde{U}}|\tau'|^2 + \frac{\lambda^2}{G\tilde{U}^3}|\tau^2|\, dy - \int_0^1 \frac{\tilde{U}'}{G\tilde{U}^2}\tau\tau'\, dy = -\int_0^1 \frac{1}{G\tilde{U}}\omega\tau\, dy. \tag{49}$$

Using the Cauchy-Schwarz inequality, we find

$$\left|\int_0^1 \frac{\tilde{U}'}{G\tilde{U}^2}\tau\tau'\, dy\right| \le \max_{[0,1]}|\tilde{U}'|\Big(\int_0^1 \frac{1}{G\tilde{U}^3}|\tau^2|\, dy\Big)^{1/2}\Big(\int_0^1 \frac{1}{G\tilde{U}}|\tau'|^2\, dy\Big)^{1/2}. \tag{50}$$

The first term on the right hand side of (50) is of order G and hence small. The estimate (47) now follows immediately. Hence A is injective and has closed range. To show that the range is dense, we approximate $\tilde{U}$ by a sequence of analytic functions $\tilde{U}^m$. Let A^m be the corresponding operator. If ω has compact support in $(0,1)$, the behavior of solutions of $A^m\tau = \omega$ near the endpoints of the interval can be described by Frobenius' theory and it is easy to show that there is a unique solution in $H^2_{w,0}$. Hence A^m has dense range and since an energy estimate like the one above holds uniformly in m, it follows that A is onto.

The lemma in conjunction with the implicit function theorem and the estimate (46) now yields the existence of a unique small solution of (45) and hence of (25). In addition, we obtain the following estimate for the inflow value of τ^{n+1}:

$$\|\tau^{n+1}\|_{2,w} \le C\Big(\frac{\epsilon}{G} + \frac{\nu}{G}\Big) \le C\Big(\frac{\epsilon}{G} + \delta\Big). \tag{51}$$

For the last inequality we have taken into account the bound for ν obtained in Section 4.

6. Determination of stresses. We now have inflow values for all stress components, which we can use to solve equation (19). Using (26), (27) and the analogue of (27) for the other stress components, we can also compute the inflow values of $\partial \mathbf{S}^{n+1}/\partial x$ and $\partial^2 \mathbf{S}^{n+1}/\partial x^2$ at the inflow boundary. It is not hard to verify that

$$\|\mathbf{S}^{n+1}\|_{2,w} + \|\frac{\partial \mathbf{S}^{n+1}}{\partial x}\|_{1,w} + \|\frac{\partial^2 \mathbf{S}^{n+1}}{\partial x^2}\|_{0,w} \leq C(\frac{\epsilon}{G} + \delta). \tag{52}$$

Since the velocity field is Lipschitz continuous, (19) can be solved uniquely by integration along characteristics. We need to derive estimates for the solution. The following discussion is limited to formal energy estimates; a rigorous justification is easily obtained by approximating the data and coefficients by smoother functions and passing to the limit. We multiply (19) by $\mathbf{S}^{n+1}$ and integrate. This yields

$$\lambda \int_0^1 \int_0^L |\mathbf{S}^{n+1}|^2 \, dx \, dy$$

$$+\frac{1}{2}\int_0^1 (U(y) + u^{n+1}(L,y))|\mathbf{S}^{n+1}(L,y)|^2 \, dy - \frac{1}{2}\int_0^1 (U(y) + u^{n+1}(0,y))|\mathbf{S}^{n+1}(0,y)|^2 \, dy$$

$$= \int_0^1 \int_0^L \eta\lambda(\nabla \mathbf{u}^{n+1} + (\nabla \mathbf{u}^{n+1})^T) : \mathbf{S}^{n+1} + \mathbf{r}(G, y, \mathbf{S}^{n+1}, \mathbf{u}^{n+1}, \nabla \mathbf{u}^{n+1}) : \mathbf{S}^{n+1} \, dx \, dy. \tag{53}$$

After differentiating (19) with respect to x and y, analogous energy estimates can be derived for derivatives of $\mathbf{S}^{n+1}$. We note that on the left hand side of (53) the second integral is positive and can be discarded in the estimates and the third integral can be estimated in terms of the inflow data. By combining (53) with the analogous equations for derivatives, one obtains

$$\|\mathbf{S}^{n+1}\|_2 \leq C(\sqrt{G}(\frac{\epsilon}{G} + \delta) + \nu + G(\nu + \|\mathbf{S}^{n+1}\|_2) + (\nu + \|\mathbf{S}^{n+1}\|_2)^2). \tag{54}$$

Here the first term on the right results from the inflow values of $\mathbf{S}^{n+1}$ and its derivatives and the remaining terms result from the terms on the right hand side of (19). After taking account of the relative sizes of ϵ, ν, δ and G, (54) simplifies to

$$\|\mathbf{S}^{n+1}\|_2 \leq C(\frac{\epsilon}{\sqrt{G}} + \delta\sqrt{G}). \tag{55}$$

The quantity on the right is small relative to δ. This concludes the proof that the iteration maps Z into itself.

7. Contraction estimates. To show that the mapping defined by the iteration is a contraction on Z, one derives estimates similar to those above for the differences between the iterates. Since no essentially new ideas are involved, we shall not give a complete derivation, but only show one step. By taking differences, we obtain from (19)

$$((\mathbf{V} + \mathbf{u}^{n+1}) \cdot \nabla)(\mathbf{S}^{n+1} - \mathbf{S}^n) + \lambda(\mathbf{S}^{n+1} - \mathbf{S}^n) = ((\mathbf{u}^n - \mathbf{u}^{n+1}) \cdot \nabla)\mathbf{S}^n$$
$$+\eta\lambda(\nabla(\mathbf{u}^{n+1} - \mathbf{u}^n) + (\nabla(\mathbf{u}^{n+1} - \mathbf{u}^n))^T)$$
$$+\mathbf{r}(G, y, \mathbf{S}^{n+1}, \mathbf{u}^{n+1}, \nabla\mathbf{u}^{n+1}) - \mathbf{r}(G, y, \mathbf{S}^n, \mathbf{u}^n, \nabla\mathbf{u}^n), \tag{56}$$

Using the same energy estimates as in Section 6 above, we find

$$\|\mathbf{S}^{n+1} - \mathbf{S}^n\|_1 \le C\Big(\sqrt{G}\|\mathbf{S}^{n+1}(0,\cdot) - \mathbf{S}^n(0,\cdot)\|_{1,w} + \sqrt{G}\|\frac{\partial \mathbf{S}^{n+1}}{\partial x}(0,\cdot) - \frac{\partial \mathbf{S}^n}{\partial x}(0,\cdot)\|_{0,w} + \|rhs\|_1\Big). \tag{57}$$

Here rhs stands for the right hand side of (56). We obtain

$$\|rhs\|_1 \le C\Big(\|\mathbf{S}^n\|_2(\|\mathbf{u}^{n+1} - \mathbf{u}^n\|_2 + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_2$$
$$+(G + \|\mathbf{S}^{n+1}\|_2 + \|\mathbf{S}^n\|_2 + \|\mathbf{u}^{n+1}\|_3 + \|\mathbf{u}^n\|_3)(\|\mathbf{S}^{n+1} - \mathbf{S}^n\|_1 + \|\mathbf{u}^{n+1} - \mathbf{u}^n\|_2)\Big). \tag{58}$$

Using the a priori bounds for $\|\mathbf{S}^n\|_2$ and $\|\mathbf{u}^n\|_3$ which have already been derived, we find

$$\|\mathbf{S}^{n+1} - \mathbf{S}^n\|_1 \le C\Big(\sqrt{G}\|\mathbf{S}^{n+1}(0,\cdot) - \mathbf{S}^n(0,\cdot)\|_{1,w} + \sqrt{G}\|\frac{\partial \mathbf{S}^{n+1}}{\partial x}(0,\cdot) - \frac{\partial \mathbf{S}^n}{\partial x}(0,\cdot)\|_{0,w}$$
$$+\|\mathbf{u}^{n+1} - \mathbf{u}^n\|_2 + (G + \delta)\|\mathbf{S}^{n+1} - \mathbf{S}^n\|_1\Big). \tag{59}$$

In a similar fashion, we obtain estimates analogous to those of Sections 4 and 5. The contraction property follows by combining those estimates.

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