COMPATIBLE CONTACT STRUCTURES OF FIBERED SEIFERT LINKS IN HOMOLOGY 3-SPHERES

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Abstract. We study compatible contact structures of fibered Seifert multilinks in homology 3-spheres and especially give a necessary and sufficient condition for the contact structure to be tight in the case where the Seifert fibration is positively twisted. As a corollary we determine the strongly quasipositivity of fibered Seifert links in S^3. We also study the compatible contact structures of cablings along links in any 3-manifolds.

1. Introduction. A contact structure on a closed, oriented, smooth 3-manifold M is the kernel of a 1-form α on M satisfying $\alpha \wedge d\alpha \neq 0$ everywhere. In this paper, we only consider a positive contact form, i.e., a contact form α with $\alpha \wedge d\alpha > 0$. In [31], Thurston and Winkelnkemper used open book decompositions to show the existence of contact structures on any 3-manifolds. In [12], Giroux then focused on their idea, introduced the notion of contact structures supported by open book decompositions, and studied the correspondence between contact structures up to contactomorphisms and open book decompositions up to plumbings of positive Hopf bands (cf. [10]). Instead of the terminology "supported", we will say that the contact structure is "compatible" with an open book decomposition and vice versa.

In the study of open book decompositions of 3-manifolds, it is important to determine if the compatible contact structure is tight or overtwisted since it gives a rough classification of open book decompositions by Giroux's correspondence. An explicit construction sometimes helps us to determine the tightness. For example, in [9, 21], Etgü and Ozbagci gave explicit descriptions of contact structures transverse to the fibers of circle bundles and certain Seifert fibered manifolds and proved that such contact structures are Stein fillable. Stein fillable contact structures are known to be tight by Eliashberg and Gromov [7, 13].

The purpose of this paper is to give an explicit construction of contact structures compatible with fibered Seifert links in homology 3-spheres. We hereafter use the terminology "fibered link" instead of "open book decomposition". Following the book of Eisenbud and Neumann [5], we denote a Seifert fibered homology 3-sphere as $\Sigma(a_1, a_2, \ldots, a_k)$, where a_i's are the denominators of the Seifert invariants. The Seifert fibration has different properties depending on the sign of the product $a_1 a_2 \cdots a_k$; if $a_1 a_2 \cdots a_k > 0$ then the fibers

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of the Seifert fibration are twisted positively, as those of the positive Hopf fibration, and if $a_1 a_2 \cdots a_k < 0$ then they are negatively twisted.

A *Seifert link L* in $\Sigma(a_1, \ldots, a_k)$ is an oriented link whose exterior admits a Seifert fibration. A multilink $L(\underline{m})$ is a link each of whose link components is equipped with a non-zero integer, called the *multiplicity*, where $\underline{m} = (m_1, \ldots, m_n)$ represents the set of multiplicities. Note that we assign an orientation to each link component previously, called a *working orientation*, and the multiplicity is read with respect to this orientation. A multilink $L(\underline{m})$ is said to be *fibered* if its complement admits a fibration over S^1 such that the homology class of the union of oriented simple closed curves on the boundary of a small tubular neighborhood of each link component induced from a fiber surface is $p_i[\mathfrak{m}_i] + m_i[\mathfrak{l}_i]$, where $(\mathfrak{m}_i, \mathfrak{l}_i)$ is a meridian-longitude pair of that component chosen such that the orientation of the longitude agrees with its working orientation, m_i is the multiplicity and p_i is some integer. Note that a multilink is a usual link if all the multiplicities are in $\{-1, \ 1\}$. The criterion in [5, Theorem 11.1] determines the fiberedness of a Seifert multilink in $\Sigma(a_1, \ldots, a_k)$, from which we can see that Seifert multilinks are fibered in most cases (see Theorem 2.1 below).

Now we assign an orientation to the fibers of the Seifert fibration under the assumption $a_1 a_2 \cdots a_k \neq 0$, which we call the *orientation of the Seifert fibration*. If the orientations of all the components of $L(\underline{m})$ coincide with, or are opposite to, the orientation of the Seifert fibration then we say that the orientation of $L(\underline{m})$ is *canonical*.

In this paper we prove the following results.

THEOREM 1.1. *Let $L(\underline{m})$ be a fibered Seifert multilink in $\Sigma(a_1, a_2, \ldots, a_k)$ with $a_1 \cdots a_k > 0$. If the orientation of $L(\underline{m})$ is canonical then the compatible contact structure is Stein fillable. Otherwise it is overtwisted.*

The case $a_1 a_2 \cdots a_k < 0$ will also be discussed in this paper. As a consequence of our constructions in both cases, we determine the tightness of fibered Seifert links in S^3.

THEOREM 1.2. *Let L be a fibered Seifert link in $S^3 = \Sigma(a_1, a_2)$. Then the compatible contact structure of L is tight if and only if L is one of the following cases*:

(1) $a_1 a_2 > 0$ *and the orientation of L is canonical.*
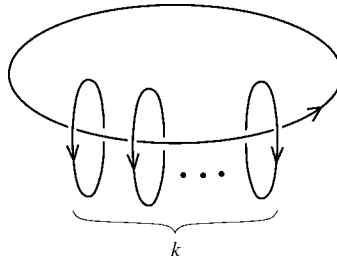(2) *L is an oriented link described in Figure 1 with $k \geq 1$.*

FIGURE 1. Fibered Seifert links in Case (2).

With a small additional effort, we can remove the fiberedness assumption by replacing 'tightness' by 'strongly quasipositivity' (see Section 7 for the definition of strongly quasipositive links).

COROLLARY 1.3. *Let L be a non-splittable Seifert link in S^3. Then, L is strongly quasipositive if and only if it is in Case* (1) *or* (2) *above, or in Case* (3) *stated below*:

(3) *L is a negative torus link consisting of even number of link components half of which have reversed orientations.*

Here a link L in S^3 is called *splittable* if $S^3 \setminus L$ contains an incompressible 2-sphere. The only splittable Seifert links are trivial links with several components.

The technique of cabling with contact structure can be used for studying cablings along fibered links in arbitrary 3-manifolds. Let $L(\underline{m})$ be a fibered multilink in an oriented, closed, smooth 3-manifold M with cabling in a solid torus N in M, and $L'(\underline{m}')$ a fibered multilink obtained from $L(\underline{m})$ by retracting N to its core curve. Note that $L'(\underline{m}')$ is always fibered. We say that a cabling is *positive* if $L(\underline{m}) \cap N$ intersects the fiber surface of $L'(\underline{m}')$ positively transversely, and otherwise it is called *negative*.

THEOREM 1.4. *Let $L(\underline{m})$ be a fibered multilink in an oriented, closed, smooth 3-manifold M with cabling in a solid torus N in M, and $L'(\underline{m}')$ the fibered multilink obtained from $L(\underline{m})$ by retracting N to its core curve. Let ξ and ξ' denote the contact structures on M compatible with $L(\underline{m})$ and $L'(\underline{m}')$, respectively.*

(1) *If ξ' is tight and the cabling is positive, then ξ is tight.*

(2) *If ξ' is tight, the cabling is negative and $L(\underline{m}) \cap N$ has at least two components, then ξ is overtwisted.*

(3) *If ξ' is tight, the cabling is negative, $L(\underline{m}) \cap N$ is connected, $p \geq 2$ and $q \leq -2$, then ξ is overtwisted.*

(4) *If ξ' is overtwisted then ξ is also overtwisted.*

Here p and q are the coefficients of the slope $q\mathfrak{m} + p\mathfrak{l}$ of the cabling with respect to the meridian-longitude pair $(\mathfrak{m}, \mathfrak{l})$ on ∂N which will be fixed in Subsection 8.1.

The compatible contact structures of cablings of multilinks are studied independently by Baker, Etnyre and van Horn-Morris [2]. In their paper, a fibered multilink is called a rational open book decomposition. The case $M = S^3$ had been studied by Hedden in [16] using a different method.

This paper is organized as follows. In Section 2, we fix the notation of Seifert fibered homology 3-spheres and Seifert multilinks following the book [5]. We introduce the notion of compatible contact structures for multilinks in Section 3. The case $a_1 \cdots a_k > 0$ is studied in Section 4, including the proof of Theorem 1.1, and the case $a_1 \cdots a_k < 0$ is in Section 5, where we give an explicit construction of contact structures and some criterion for detecting overtwisted disks. We then prove Theorem 1.2 in Section 6 and Corollary 1.3 in Section 7. In Section 8, we give the definitions of positive and negative cablings and the proof of Theorem 1.4. A conjecture about strongly quasipositive orientation is posed in the end of Section 7.

2. Preliminaries. In the following, $\mathrm{int}X$ and ∂X represent the interior and the boundary of a topological space X, respectively.

2.1. Notation of Seifert fibered homology 3-spheres. Let Σ be a homology 3-sphere. We use the topological description of Seifert links in [5, p. 60]. Let $\mathcal{S} = S^2 \setminus \mathrm{int}(D_1^2 \cup \cdots \cup D_k^2)$ be a 2-sphere with k holes and make an oriented, closed, smooth 3-manifold Σ from $\mathcal{S} \times S^1$ by gluing solid tori $(D^2 \times S^1)_1, \ldots, (D^2 \times S^1)_k$ along the boundary $\partial(\mathcal{S} \times S^1)$. To fix the notation, we first choose a section $\mathcal{S}^{\mathrm{sec}}$ of $\pi : \mathcal{S} \times S^1 \to \mathcal{S}$ and set

$$Q_i = (-\partial \mathcal{S}^{\mathrm{sec}}) \cap (D^2 \times S^1)_i \quad \text{and}$$

$$H = \text{typical oriented fiber of } \pi \text{ in } \partial(D^2 \times S^1)_i \,.$$

Suppose that the gluing map of $(D^2 \times S^1)_i$ to $\mathcal{S} \times S^1$ is given so that $a_i Q_i + b_i H$ is null-homologous in $(D^2 \times S^1)_i$, where $(a_i, b_i) \in \mathbf{Z}^2 \setminus \{(0, 0)\}$ and $\gcd(|a_i|, |b_i|) = 1$. To make the obtained 3-manifold Σ to be a homology 3-sphere, the integers a_i's and b_i's should satisfy the equality $\sum_{i=1}^{k} b_i a_1 \cdots a_{i-1} a_{i+1} \cdots a_k = \pm 1$. Following [5], in this paper, we always choose the coefficients a_i's and b_i's so that $\sum_{i=1}^{k} b_i a_1 \cdots a_{i-1} a_{i+1} \cdots a_k = 1$ by replacing (a_i, b_i) by $(-a_i, -b_i)$ for some i if necessary. Note that this equality ensures that if one of a_i's is zero then all the other a_i's satisfy $|a_i| = 1$, and if $a_i \neq 0$ for all $i = 1, \ldots, k$ then each pair (i, j) with $i \neq j$ satisfies $\gcd(|a_i|, |a_j|) = 1$. Since the 3-manifold Σ does not depend on the ambiguity of the choice of b_i's, we may denote it as $\Sigma = \Sigma(a_1, \ldots, a_k)$.

The core curve S_i of each solid torus $(D^2 \times S^1)_i$ is a fiber of the Seifert fibration after the gluings. We assign to S_i an orientation in such a way that the linking number of S_i and $a_i Q_i + b_i H$ equals 1. This orientation is called the *working orientation*.

Let $(\mathfrak{m}_i, \mathfrak{l}_i)$ be the preferred meridian-longitude pair of the link complement $\Sigma \setminus S_i$ chosen such that the orientation of the longitude $\mathfrak{l}_i$ agrees with the working orientation of S_i. Then $(\mathfrak{m}_i, \mathfrak{l}_i)$ and (Q_i, H) are related by the following equations (see [5, Lemma 7.5]):

$$(2.1) \qquad \begin{pmatrix} \mathfrak{m}_i \\ \mathfrak{l}_i \end{pmatrix} = \begin{pmatrix} a_i & b_i \\ -\sigma_i & \delta_i \end{pmatrix} \begin{pmatrix} Q_i \\ H \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q_i \\ H \end{pmatrix} = \begin{pmatrix} \delta_i & -b_i \\ \sigma_i & a_i \end{pmatrix} \begin{pmatrix} \mathfrak{m}_i \\ \mathfrak{l}_i \end{pmatrix},$$

where $\sigma_i = a_1 \cdots \hat{a}_i \cdots a_k$ and $\delta_i = \sum_{j \neq i} b_j a_1 \cdots \hat{a}_i \cdots \hat{a}_j \cdots a_k$. Note that they satisfy $a_i \delta_i + b_i \sigma_i = 1$.

Set $A = a_1 \cdots a_k$. In the case $A \neq 0$, the orientation of the Seifert fibration in $\mathcal{S} \times S^1 \to \mathcal{S}$ canonically extends into the fibers in $(D^2 \times S^1)_i$ for each $i = 1, \ldots, k$, namely the orientation of the Seifert fibration of $\Sigma(a_1, \ldots, a_k)$ becomes well-defined. Note that the working orientation on S_i coincides with the orientation of the Seifert fibration if and only if $a_i > 0$.

2.2. Fibered multilinks. We give the definition of fibered multilinks in 3-manifolds. The same notion appears in [2], where the fibration is called a *rational open book decomposition*.

Let M be an oriented, closed, smooth 3-manifold and L an unoriented link in M with n link components. We first assign an orientation to each link component of L, which we also call a *working orientation*. A *multilink* $L(\underline{m})$ in M is a link each of whose components is equipped with a non-zero integer, called the *multiplicity*, where $\underline{m} = (m_1, \ldots, m_n)$ represents the set of multiplicities. A multilink $L(\underline{m})$ is called *fibered* if there is a fibration $M \setminus L \to S^1$ such that the homology class of the union of oriented simple closed curves on the boundary of a small tubular neighborhood of each link component induced from a fiber surface is $p_i[\mathfrak{m}_i] + m_i[\mathfrak{l}_i]$, where $(\mathfrak{m}_i, \mathfrak{l}_i)$ is a meridian-longitude pair of that component chosen such that the orientation of the longitude agrees with its working orientation, m_i is the multiplicity and p_i is some integer (cf. [5, p. 28–29]).

2.3. Fibered Seifert multilinks. In the case $A \neq 0$, a Seifert link L in $\Sigma(a_1, \ldots, a_k)$ is a union of finite number of fibers of the Seifert fibration. We had introduced the working orientation for each link component S_i of L in Subsection 2.1. Using this working orientation, we assign a multiplicity to each S_i and make L a Seifert multilink. We denote this multilink as

$$L(\underline{m}) = (\Sigma(a_1, \ldots, a_k), m_1 S_1 \cup \cdots \cup m_n S_n),$$

where $1 \leq n \leq k$. In this paper, we call a trivial knot with multiplicity a *trivial multilink* and a Hopf link with multiplicities a *Hopf multilink*. The Milnor fibrations of singularities of complex polynomials $f(z_1, z_2) = z_1^{|m_1|}$ and $f(z_1, z_2) = z_1^{|m_1|} z_2^{|m_2|}$ at the origin and their mirror images show that trivial multilinks and Hopf multilinks are fibered. The fiberedness of other Seifert multilinks are determined by the following theorem.

THEOREM 2.1 (Eisenbud-Neumann, [5]). *Suppose that $L(\underline{m})$ is a Seifert multilink other than a trivial multilink and a Hopf multilink. Then $L(\underline{m})$ is fibered if and only if it satisfies $\sum_{i=1}^{n} m_i \sigma_i \neq 0$.*

The proof can be found in [5, p. 89–90]. From this theorem, we can see that Seifert multilinks are fibered in most cases.

Suppose that $L(\underline{m})$ is fibered. The interiors of the fiber surfaces of $L(\underline{m})$ intersect the fibers of the Seifert fibration transversely except for the case where $L(\underline{m})$ is a Hopf multilink (see [5, Theorem 11.2] and the proof therein). As mentioned in [5, Proposition 7.3], a Seifert multilink is invertible and this involution changes $L(\underline{m})$ into $L(-\underline{m})$. In particular, this reverses the sign of the intersection of the interiors of the fiber surfaces and the fibers of the Seifert fibration. In this paper, by choosing one of $L(\underline{m})$ and $L(-\underline{m})$ suitably, we often assume that the intersection is positive. We name it the *positive transverse property* and write it (PTP) for short.

DEFINITION 2.2. Suppose $A \neq 0$. A link component $m_i S_i$ of a fibered Seifert multilink $L(\underline{m})$ with (PTP) is called *positive* (resp. *negative*) if its orientation is consistent with (resp. opposite to) the orientation of the Seifert fibration. If the orientations of the link components of $L(\underline{m})$ are either all positive or all negative then we say that the orientation of $L(\underline{m})$ is *canonical*.

3. Fibered multilinks and contact structures.

3.1. A Lutz tube. We first introduce terminologies in 3-dimensional contact topology briefly, see for instance [22, 11] for general references.

A *contact structure* on M is the 2-plane field given by the kernel of a 1-form α satisfying $\alpha \wedge d\alpha \neq 0$ everywhere on M. In this paper, we only consider a contact structure given by the kernel of a 1-form α satisfying $\alpha \wedge d\alpha > 0$, called a *positive contact form*. A vector field R_α on M determined by the conditions $d\alpha(R_\alpha, \cdot) \equiv 0$ and $\alpha(R_\alpha) \equiv 1$ is called the *Reeb vector field* of α. The 3-manifold M equipped with a contact structure ξ is called a *contact manifold* and denoted as (M, ξ). Two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are said to be *contactomorphic* if there exists a diffeomorphism $\varphi : M_1 \to M_2$ such that $d\varphi : TM_1 \to TM_2$ satisfies $d\varphi(\xi_1) = \xi_2$. A disk D in (M, ξ) is called *overtwisted* if D is tangent to ξ at each point on ∂D. If (M, ξ) has an overtwisted disk then we say that ξ is *overtwisted* and otherwise that ξ is *tight*. A typical example of overtwisted contact structures is given as follows: Let α be the contact form on $\mathbf{R}^3$ given by

$$\alpha = \cos r dz + r \sin r d\theta,$$

where (r, θ, z) are coordinates of $\mathbf{R}^3$ with polar coordinates (r, θ). The contact structure $\ker \alpha$ is as shown in Figure 2. We can find an overtwisted disk in the tube $\{(r, \theta, z); |r| \le \pi + \varepsilon\}$, where $\varepsilon > 0$ is a sufficiently small real number. Hence, this contact structure is overtwisted.
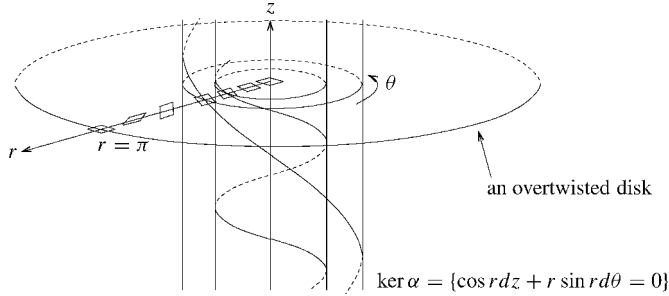

FIGURE 2. A typical example of overtwisted contact structures.

Now we introduce an effective way to describe a contact structure on $D^2 \times S^1$. Let α be a 1-form on $D^2 \times S^1$ given by $\alpha = h_2 d\mu - h_1 d\lambda$, where (r, μ, λ) are coordinates of $D^2 \times S^1$ with polar coordinates (r, μ) of D^2, and h_1 and h_2 are real-valued smooth functions with parameter r. We have

$$d\alpha = h_2' dr \wedge d\mu - h_1' dr \wedge d\lambda,$$
$$\alpha \wedge d\alpha = (h_1' h_2 - h_1 h_2') dr \wedge d\mu \wedge d\lambda,$$

where h_1' and h_2' are the derivatives of h_1 and h_2 with parameter r, respectively. Hence, α is a positive contact form if and only if $h_1' h_2 - h_1 h_2' > 0$. We now plot (h_1, h_2) on the xy-plane. Since $(h_2, -h_1)$ represents a vector normal to the 2-plane of the contact structure $\ker \alpha$, we

can regard the line connecting $(0, 0)$ and (h_1, h_2) as the slope of $\ker \alpha$. The Reeb vector field R_α of α is given as

$$R_\alpha = \frac{1}{h_1' h_2 - h_1 h_2'} \left(h_1' \frac{\partial}{\partial \mu} + h_2' \frac{\partial}{\partial \lambda} \right).$$

The parameter r varies from 0 to 1, namely from $\{(0, 0)\} \times S^1$ to the boundary of $D^2 \times S^1$, and the pair of functions $(h_1(r), h_2(r))$ defines a curve γ on the xy-plane. In summary, the curve γ has the following properties:

- Since $h_1' h_2 - h_1 h_2' > 0$, $(0, 0) \notin \gamma([0, 1])$ and γ moves in the clockwise orientation.
- The line connecting $(0, 0)$ and (h_1, h_2) represents the slope of $\ker \alpha$ and the vector $(h_2, -h_1)$ represents the positive side of $\ker \alpha$.
- The speed vector (h_1', h_2') is parallel to R_α and points in the same direction.

See Figure 3. To make α a well-defined contact form in a neighborhood of $r = 0$, we choose γ near $r = 0$ such that $(h_1, h_2) = (-c, r^2)$ or $(h_1, h_2) = (c, -r^2)$ with some positive constant c, so that α has the form $\alpha = r^2 d\mu + c d\lambda$ or $\alpha = -(r^2 d\mu + c d\lambda)$ near $r = 0$, respectively.
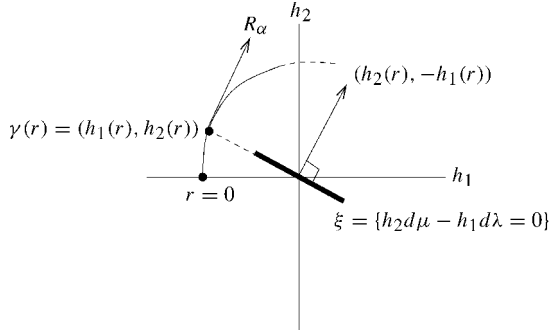

FIGURE 3.   How to read $\xi = \ker \alpha$ and R_α from the curve $\gamma(r) = (h_1(r), h_2(r))$.

If the curve γ intersects the positive x-axis, then the contact structure $\ker \alpha$ on $D^2 \times S^1$ has an overtwisted disk, similar to Figure 2, whose boundary corresponds to the intersection point of γ and the positive x-axis. In this paper, we call the tube $(D^2 \times S^1, \ker \alpha)$ a *Lutz tube* and use it frequently to show the existence of an overtwisted disk.

3.2. Contact structures compatible with multilinks.   The notion of compatible contact structures of fibered links can be generalized to fibered multilinks canonically. This idea also appears in [2]. Let M be a closed, oriented, smooth 3-manifold.

DEFINITION 3.1.   A fibered multilink $L(\underline{m})$ in M is said to be *compatible* with a contact structure $\xi = \ker \alpha$ on M if $L(\underline{m})$ is positively transverse to ξ and $d\alpha$ is a volume form on the interiors of the fiber surfaces of $L(\underline{m})$.

The next lemma gives a useful interpretation of the notion of compatible contact structures in terms of Reeb vector fields. In this paper we mainly use this characterization.

LEMMA 3.2.   *A fibered multilink $L(\underline{m})$ in M is compatible with a contact structure ξ on M if and only if there exists a contact form α on M with $\xi = \ker \alpha$ such that the Reeb vector field R_α is tangent to $L(\underline{m})$ and positively transverse to the interiors of the fiber surfaces of $L(\underline{m})$, and its orientation is consistent with that of $L(\underline{m})$ induced from the fiber surfaces.*

PROOF.   The proof for a fibered link in [10, Lemma 3.5] works in this case also.   □

Now we introduce two fundamental facts concerning compatible contact structures of fibered multilinks, following the fibered link case.

PROPOSITION 3.3.   *Any fibered multilink in M admits a compatible contact structure.*

Although the proof is analogous to the one in [31], since an explicit contact form of the compatible contact structure will be needed in the proof of Lemma 8.4 later, we prove the assertion here with presenting the contact form. A similar proof can be found in [2].

PROOF.   Let $L(\underline{m})$ be a fibered multilink in M with n link components $m_1 S_1, \ldots, m_n S_n$ and $N(S_i)$ a small compact tubular neighborhood of S_i in M for $i = 1, \ldots, n$. We denote by F_t the fiber surface of $L(\underline{m})$ over $t \in S^1 = [0, 1]/0 \sim 1$ and choose a diffeomorphism $\phi_t : F_0 \to F_t$ of the fibration of $L(\underline{m})$ in such a way that

$$\phi_t(r_i, \mu_i, \lambda_i) = \left(r_i, \mu_i + \frac{t}{|m_i|}, \lambda_i \right)$$

in $N(S_i)$, where (r_i, μ_i, λ_i) are coordinates of $N(S_i) = D^2 \times S^1$ chosen such that (r_i, μ_i) are the polar coordinates of D^2 and the orientation of λ agrees with that of the corresponding link component of $L(\underline{m})$. For convenience, we set the coordinates (r_i, μ_i) such that the radius of D^2 is 1.

Let θ_i be the coordinate function on the curve $-(F_0 \setminus \operatorname{int} N(S_i))$ given as $\theta_i = -\lambda_i$. Then, as in [31], we can find a 1-form β on $F_0 \cap (\mathcal{S} \times S^1)$ such that $d\beta$ is a volume form on $F_0 \cap (\mathcal{S} \times S^1)$ and $\beta = -(1/r_i)d\theta_i$ near $\partial N(S_i)$. The manifold M is constructed from $F_0 \times [0, 1]$ by identifying $(x, 1) \sim (\phi_1(x), 0)$ for each $x \in F_0$ and then filling the boundary components by the solid tori $N(S_i)$'s. According to this construction, we define a 1-form α_0 on $\mathcal{S} \times S^1$ as

$$\alpha_0 = (1 - t)\beta + t\phi_1^*(\beta) + Rdt \,,$$

with $R > 0$, which is given near $\partial N(S_i)$ as

(3.1) $$\alpha_0 = -\frac{1}{r_i}d\theta_i + Rdt = \frac{1}{r_i}d\lambda_i + R(v_i d\mu_i - u_i d\lambda_i) \,,$$

where (u_i, v_i) is a vector representing the oriented boundary of $F_0 \setminus \operatorname{int} N(S_i)$ on $\partial N(S_i)$ with coordinates (μ_i, λ_i); in other words, $(v_i, -u_i)$ is a vector positively normal to F_0 on $\partial N(S_i)$. Note that $v_i > 0$. We choose R sufficiently large so that α_0 becomes a positive contact form on $\mathcal{S} \times S^1$.

For each $N(S_i)$, we extend α_0 into $N(S_i)$ by describing a curve $\gamma(r_i)$ on the xy-plane explained in Subsection 3.1. The endpoint $(h_1(1), h_2(1))$ of $\gamma(r_i)$ is given as $(h_1(1), h_2(1)) =$

$(Ru_i - 1, Rv_i)$ and the speed vector $\gamma'(r_i)$ at $r_i = 1$ is $(h'_1(r_i), h'_2(r_i)) = (1, 0)$. Hence, we can describe a curve $\gamma(r_i)$ representing a positive contact form on $N(S_i)$ such that

- $(h_1, h_2) = (-c, r^2)$ near $r = 0$ with $c > 0$,
- $\gamma(1)$ and $\gamma'(1)$ satisfy the above conditions, and
- $\gamma'(r_i)$ rotates monotonously.

Thus the contact form α_0 is extended into $N(S_i)$. We denote the obtained contact form on M as α.

Since the fibers of the Seifert fibration intersect $F_t \cap (S \times S^1)$ positively transversely, $\ker \alpha$ is compatible with $L(\underline{m})$ on $S \times S^1$. In each $N(S_i)$, we can isotope F_t into the position shown in Figure 4 such that $\ker \alpha$ is compatible with $L(\underline{m})$. This completes the proof. $\qquad\square$
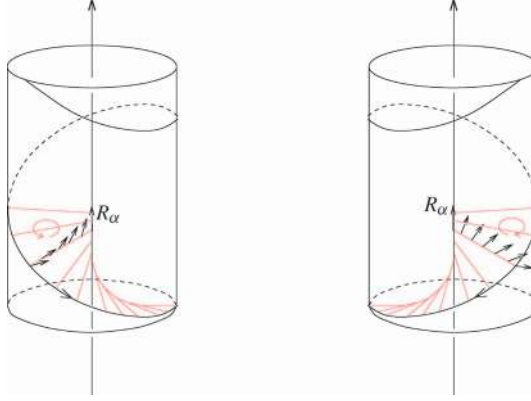

FIGURE 4.   The compatibility in the neighborhood $N(S_i)$.

PROPOSITION 3.4.   *If two contact structures on M are compatible with the same fibered multilink in M then they are contactomorphic.*

PROOF.   The proof for a fibered link in [12] works in this case also (cf. [22, Proposition 9.2.7]). $\qquad\square$

4.   Case $a_1 a_2 \cdots a_k > 0$.

4.1.   Explicit construction of the contact structure.

Throughout this section, we always assume $A = a_1 \cdots a_k > 0$. Theorem 1.1 follows from the explicit construction of compatible contact structures described below.

PROPOSITION 4.1.   *Let $L(\underline{m}) = (\Sigma, m_1 S_1 \cup \cdots \cup m_n S_n)$ be a fibered Seifert multilink in a homology 3-sphere $\Sigma = \Sigma(a_1, \ldots, a_k)$ with $A > 0$. Assume* (PTP). *Then there exists a positive contact form α on Σ with the following properties*:

(1)   *$L(\underline{m})$ is compatible with the contact structure $\xi = \ker \alpha$.*

(2)   *The Reeb vector field R_α of α is tangent to the fibers of the Seifert fibration on $S \times S^1$.*

(3)   *The neighborhood $(D^2 \times S^1)_i$ of each negative component $m_i S_i$ of $L(\underline{m})$ contains a Lutz tube. In particular, it contains an overtwisted disk.*

(4)   *On the other $(D^2 \times S^1)_i$'s, $\ker \alpha$ is transverse to the fibers of the Seifert fibration.*

REMARK 4.2.   The most canonical way to construct a contact structure compatible with a given fibered link is to use the fiber surface as done in [31]. However this is difficult in our situation because there is no systematic way to describe the fiber surface. The idea of the proof of Theorem 1.1 is that we choose the contact form such that its Reeb vector field is tangent to the fibers of the Seifert fibration everywhere except in small neighborhoods of the negative components. This makes sure that the contact structure is compatible with the fibered multilink in the most part. The rest is done by describing possible local positions of the fiber surfaces along the exceptional components.

REMARK 4.3.   The existence of S^1-invariant contact forms on orientable Seifert fibered 3-manifolds is known in [18]. The existence of a contact structure transverse to the fibers of a Seifert fibration had been studied in [29] for circle bundles over closed surfaces and in [19] for Seifert fibered 3-manifolds. The transverse contact structures are always Stein fillable as mentioned in [4, Theorem 4.2] (cf. 9, 21). This fact will be used in the proof of Theorem 1.1.

To prove Proposition 4.1, we apply the argument in the proof in [31] to the Seifert fibration. We denote the boundary component $(-\partial \mathcal{S}) \cap D_i^2$ of $\mathcal{S}$ by C_i.

LEMMA 4.4.   *Suppose $A > 0$ and let U_i be a collar neighborhood of C_i in $\mathcal{S}$ with coordinates $(r_i, \theta_i) \in [1, 2) \times S^1$ satisfying $\{(r_i, \theta_i) \, ; \, r_i = 1\} = C_i$. Then there exists a 1-form β on $\mathcal{S}$ which satisfies the following properties*:

(1) $d\beta > 0$ on $\mathcal{S}$.

(2)   *If $b_i/a_i \leq 0$ then $\beta = R_i r_i d\theta_i$ with $-b_i/a_i < R_i$ near C_i on U_i.*

(3)   *If $b_i/a_i > 0$ then $\beta = (R_i/r_i)d\theta_i$ with $-b_i/a_i < R_i < 0$ near C_i on U_i.*

PROOF.   Since $\sum_{i=1}^{k}(-b_i/a_i) = -1/A < 0$, we can choose $R_1, \ldots, R_k$ such that they satisfy the inequalities in (2) and (3) and the inequality $\sum_{i=1}^{k} R_i < 0$. Let Ω be a volume form on $\mathcal{S}$ which satisfies

- $\int_{\mathcal{S}} \Omega = -\sum_{i=1}^{k} R_i > 0$,
- $\Omega = R_i dr_i \wedge d\theta_i$ near C_i with $b_i/a_i \leq 0$, and
- $\Omega = -(R_i/r_i^2)dr_i \wedge d\theta_i$ near C_i with $b_i/a_i > 0$.

Let η be any 1-form on $\mathcal{S}$ which equals $R_i r_i d\theta_i$ if $b_i/a_i \leq 0$ and $(R_i/r_i)d\theta_i$ if $b_i/a_i > 0$ near C_i. By Stokes' theorem, we have

$$\int_{\mathcal{S}} (\Omega - d\eta) = \int_{\mathcal{S}} \Omega - \int_{\partial \mathcal{S}} \eta = \int_{\mathcal{S}} \Omega + \sum_{i=1}^{k} \int_{C_i} R_i d\theta_i$$

$$= \int_{\mathcal{S}} \Omega + \sum_{i=1}^{k} R_i = 0 \, .$$

Here C_i is oriented as $-\partial \mathcal{S}$. The closed 2-form $\Omega - d\eta$ represents the trivial class in cohomology vanishing near $\partial \mathcal{S}$. By de Rham's theorem, there is a 1-form γ on $\mathcal{S}$ vanishing near $\partial \mathcal{S}$ and satisfying $d\gamma = \Omega - d\eta$. Define $\beta = \eta + \gamma$, then $d\beta = \Omega$ is a volume form on $\mathcal{S}$ and β satisfies properties (2) and (3) near $\partial \mathcal{S}$ as required.   □

We prepare two further lemmas which will be used for constructing the contact form on $(D^2 \times S^1)_i$. Let $B = [1, 2) \times S^1 \times S^1 \subset \mathcal{S} \times S^1$ be a neighborhood of a boundary component of $\mathcal{S} \times S^1$ with coordinates (r, θ, t). We glue $D^2 \times S^1$ to B as

$$\mu\mathfrak{m} + \lambda\mathfrak{l} = (a\mu - \sigma\lambda)Q + (b\mu + \delta\lambda)H,$$

where $(\mathfrak{m}, \mathfrak{l})$ is a standard meridian-longitude pair of $\partial D^2 \times S^1 \subset D^2 \times S^1$, Q is the oriented curve given by $\{1\} \times S^1 \times \{\text{a point}\} \subset \partial B$, H is a typical fiber of the projection $[1, 2) \times S^1 \times S^1 \to [2, 1) \times S^1$ which omits the third entry, and $a, b, \sigma, \delta \in \mathbf{Z}$ are given according to the relations (2.1). The fibers $H = \sigma\mathfrak{m} + a\mathfrak{l}$ of the Seifert fibration on $\partial D^2 \times S^1$ are canonically extended to the interior of $D^2 \times S^1$.

LEMMA 4.5.   *Suppose $a \neq 0$ and either* (i) $0 \leq -b/a < R$ and $\alpha_0 = Rr d\theta + dt$ or (ii) $-b/a < R < 0$ and $\alpha_0 = (R/r)d\theta + dt$, *where α_0 is a contact form on B. Then there exists a contact form α on $B \cup (D^2 \times S^1)$ with the following properties*:

(1) $\alpha = \alpha_0$ *on B.*

(2) $\ker \alpha$ *is transverse to the fibers of the Seifert fibration in $D^2 \times S^1$.*

(3) R_α *is tangent to $\{(0, 0)\} \times S^1$ and the direction of R_α is consistent with the orientation of the Seifert fibration.*

(4) R_α *rotates monotonously with respect to the parameter $r \in [0, 1]$.*

PROOF.   We consider Case (i). Let σ and δ be integers satisfying the relations (2.1). Denote the gluing map of $D^2 \times S^1$ to B by φ, then we have

$$\varphi^*\alpha_0 = Rr d(a\mu - \sigma\lambda) + d(b\mu + \delta\lambda) = (b + aRr)d\mu + (\delta - \sigma Rr)d\lambda$$

(4.1)
$$= a\left(\frac{b}{a} + Rr\right)d\mu + \frac{1}{a}\left(1 - a\sigma\left(\frac{b}{a} + Rr\right)\right)d\lambda.$$

If $a > 0$ then $a(b/a + Rr) > 0$ near $r = 1$. Hence, on the xy-plane, the point $(h_1(1), h_2(1))$ lies in the region $y > 0$. Since R_{α_0} is positively transverse to $\ker \alpha_0$ at $r = 1$, we can describe a smooth curve $\gamma(r) = (h_1(r), h_2(r))$ on the xy-plane representing a positive contact form on $B \cup (D^2 \times S^1)$ such that

- $(h_1, h_2) = (-c, r^2)$ near $r = 0$ with $c > 0$,
- $h_2 d\mu - h_1 d\lambda = \varphi^*\alpha_0$ near $r = 1$, and
- $\gamma'(r)$ rotates monotonously,

as shown in Figure 5. This satisfies the required properties.

If $a < 0$ then $a(b/a + Rr) < 0$ near $r = 1$ and hence the point $(h_1(1), h_2(1))$ lies in the region $y < 0$. We choose a smooth curve $\gamma(r)$ such that

- $(h_1, h_2) = (c, -r^2)$ near $r = 0$ with $c > 0$,
- $h_2 d\mu - h_1 d\lambda = \varphi^*\alpha_0$ near $r = 1$, and
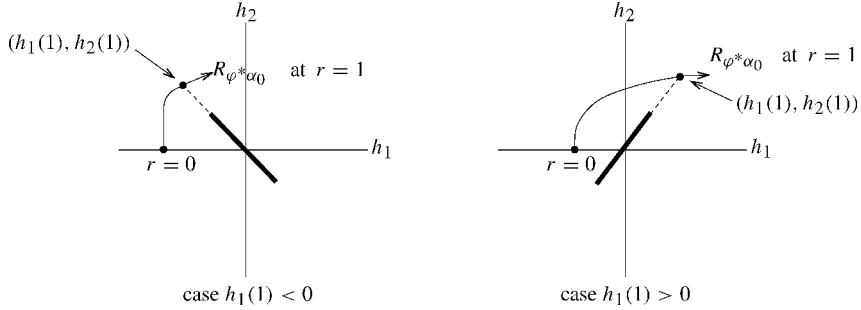
FIGURE 5.   Curves representing contact forms on $D^2 \times S^1$ in Lemma 4.5 (the figures are in the case $a > 0$).

- $\gamma'(r)$ rotates monotonously.

Note that such a curve $\gamma(r)$ is given by the π-rotation of the figures in Figure 5. The contact form α on $B \cup (D^2 \times S^1)$ defined by this curve satisfies the required properties as before.

The proof for Case (ii) is similar.   □

LEMMA 4.6.   *Let α_0 be a contact form on B given by either* (i) $\alpha_0 = Rrd\theta + dt$ *with $R > 0$ or* (ii) $\alpha_0 = (R/r)d\theta + dt$ *with $R < 0$. Then there exists a contact form α on $B \cup (D^2 \times S^1)$ with the following properties*:

(1) $\alpha = \alpha_0$ *on B.*

(2) $\ker\alpha$ *is transverse to the fibers of the Seifert fibration in $D^2 \times S^1$ except on a torus $\{r_1\} \times S^1 \times S^1$ embedded in $D^2 \times S^1$ for some $r_1 \in (0, 1)$.*

(3) R_α *is tangent to $\{(0, 0)\} \times S^1$ and the direction of R_α is opposite to the orientation of the Seifert fibration.*

(4) R_α *rotates monotonously with respect to the parameter $r \in [0, 1]$.*
Furthermore, if R satisfies $R > -b/a$ then $(D^2 \times S^1, \ker\alpha)$ contains a Lutz tube.
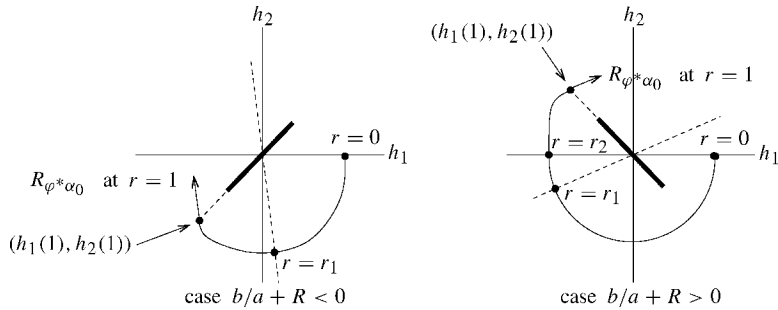

FIGURE 6.   Curves representing contact forms on $D^2 \times S^1$ in Lemma 4.6.

PROOF. The proof is analogous to the proof of Lemma 4.5. In Case (i) with $a > 0$, we choose a curve γ on the xy-plane such that $(h_1, h_2) = (c, -r^2)$ near $r = 0$ with $c > 0$ as shown in Figure 6. This satisfies the required properties. If $R > -b/a$ then a Lutz tube appears at $r = r_2$ as described on the right in the figure. The proofs in the case $a < 0$ and Case (ii) are similar.   □

PROOF OF PROPOSITION 4.1. Let α_0 be the 1-form on $\mathcal{S} \times S^1$ defined by $\alpha_0 = \beta + dt$, where β is a 1-form constructed in Lemma 4.4 and t is the coordinate of S^1, which is assumed to be consistent with the orientation of the Seifert fibration. Since $\beta \wedge d\beta$ is a 3-form on $\mathcal{S}$, we have $\beta \wedge d\beta = 0$ and

$$\alpha_0 \wedge d\alpha_0 = \beta \wedge d\beta + dt \wedge d\beta = d\beta \wedge dt > 0\,.$$

Thus α_0 is a positive contact form on $\mathcal{S} \times S^1$ and its Reeb vector field is given by $R_{\alpha_0} = \partial/\partial t$. Note that, since R_{α_0} is tangent to the fibers of $\pi : \mathcal{S} \times S^1 \to \mathcal{S}$ in the same direction, (PTP) implies that R_{α_0} is positively transverse to the fiber surfaces of $L(\underline{m})$ in $\mathcal{S} \times S^1$.

Now we extend α_0 into $(D^2 \times S^1)_i$ in the following way. If either $m_i S_i$ is a positive component or $i > n$ then we use the construction of a contact form in Lemma 4.5, otherwise we use the construction in Lemma 4.6. We denote the extended contact form on Σ by α.

From the construction, we only need to check property (1) in the assertion. Due to Lemma 3.2, it is enough to check if R_α is tangent to $L(\underline{m})$ in the same direction and positively transverse to the interiors of the fiber surfaces of $L(\underline{m})$. This positive transversality had already been established in $\mathcal{S} \times S^1$.

We first check the positive transversality in the neighborhood $(D^2 \times S^1)_i$ of a positive component $m_i S_i$. Figure 7 shows the mutual positions of the fiber surfaces F, the oriented fibers H of the Seifert fibration and the Reeb vector field R_α on $(D^2 \times S^1)_i$ in the case $a_i > 0$. The orientations of the link component $m_i S_i$ and the fibers H are as shown in the figure since $m_i S_i$ is a positive component, $a_i > 0$, $\sigma_i > 0$, and H is given as $H = \sigma_i m_i + a_i l_i$. The
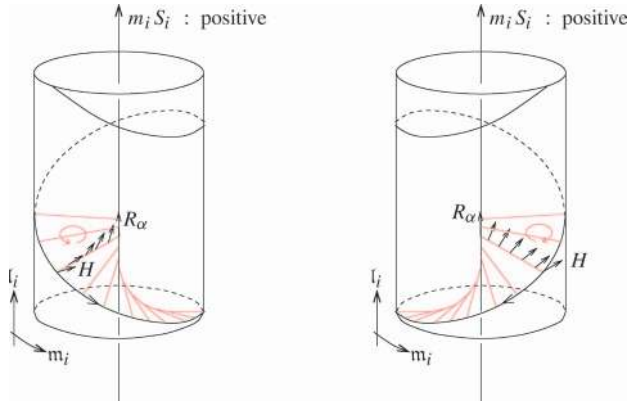

FIGURE 7. The compatibility in the neighborhood $(D^2 \times S^1)_i$ of a positive component $m_i S_i$ in the case $a_i > 0$.

Reeb vector field R_α had already been given in the above construction. Now there are three possibilities of the framing of the fiber surface F, namely it is either positive, negative, or parallel to $m_i S_i$. The case of positive framing is described on the left in the figure and the case of negative framing is on the right. The parallel case is omitted. In either case, we can isotope the fiber surfaces F in $(D^2 \times S^1)_i$ so that it satisfies property (1). Note that the vectors of R_α on the right figure in Figure 7 are directed under the fiber surface. The proof in the case $a_i < 0$ is similar, in which case the figures are those in Figure 7 with replacing $(\mathfrak{m}_i, \mathfrak{l}_i)$ by $(-\mathfrak{m}_i, -\mathfrak{l}_i)$.

Property (1) in $(D^2 \times S^1)_i$ with $i > n$ can also be checked from the figure because the fiber surfaces in $(D^2 \times S^1)_i$ consist of horizontal disks.

Suppose that $m_i S_i$ is a negative component. We assume that $a_i > 0$. Then the orientations of the link component $m_i S_i$ and the fibers H become as shown in Figure 8. There is only one possibility of the framing of the fiber surface F, which is shown in the figure, otherwise they do not satisfy (PTP) on the boundary of $(D^2 \times S^1)_i$. As shown in the figure, we can isotope the fiber surface F in $(D^2 \times S^1)_i$ so that it satisfies property (1). The proof in the case $a_i < 0$ is similar and the figure is as in Figure 8 with replacing $(\mathfrak{m}_i, \mathfrak{l}_i)$ by $(-\mathfrak{m}_i, -\mathfrak{l}_i)$.   □
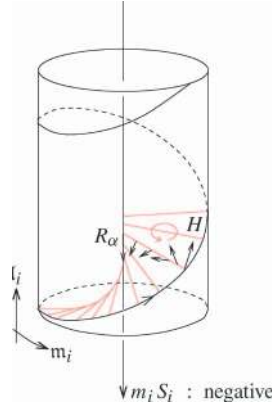

FIGURE 8.   The compatibility on the neighborhood $(D^2 \times S^1)_i$ of a negative component $m_i S_i$ in the case $a_i > 0$.

4.2.   Proof of Theorem 1.1.   The next lemma will be used in the proof of Theorem 1.1.

LEMMA 4.7.   *If $A > 0$ then every fibered Seifert multilink has at least one positive component.*

PROOF.   Let F be a fiber surface of a fibered Seifert multilink $L(\underline{m})$ and assume that $L(\underline{m})$ has no positive component. The fibers of the Seifert fibration are given as $H = \sigma_i \mathfrak{m}_i + a_i \mathfrak{l}_i$, where $\sigma_i a_i = A > 0$. Let $\gamma_i = u_i \mathfrak{m}_i + v_i \mathfrak{l}_i$ be the oriented boundary $\partial(F \cap (D^2 \times S^1)_i) \setminus m_i S_i$, where $u_i \in \mathbf{Z}$ and $v_i \in \mathbf{Z} \setminus \{0\}$ are chosen such that the number of connected components of $\partial(F \cap (D^2 \times S^1)_i) \setminus m_i S_i$ is equal to $\gcd(|u_i|, |v_i|)$ in the case $u_i \neq 0$ and $|v_i|$

otherwise. From (PTP), we have the inequality $I(\gamma_i, H) = u_i a_i - v_i \sigma_i > 0$, where $I(\gamma_i, H)$ is the algebraic intersection number of γ_i and H on $\partial(D^2 \times S^1)_i$. Furthermore, the fiber surface F along $m_i S_i$ is given as shown in Figure 9 and we can verify the inequality $a_i v_i > 0$ from these figures.
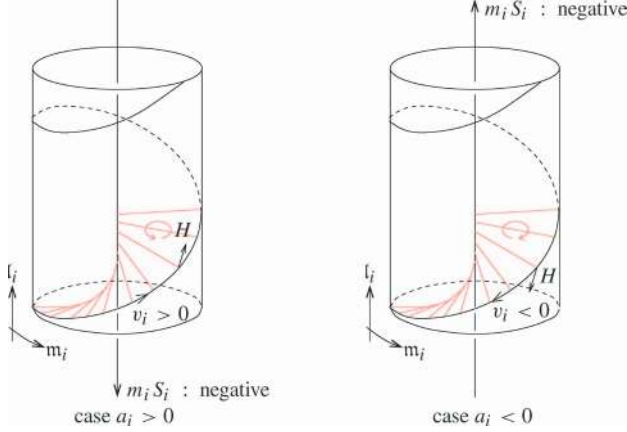

FIGURE 9.   The framing of F along $m_i S_i$.

For each $i = 1, \ldots, n$,

$$u_i \mathfrak{m}_i + v_i \mathfrak{l}_i = (a_i u_i - \sigma_i v_i) Q_i + (b_i u_i + \delta_i v_i) H \,.$$

The union of these curves is homologous to the boundary of the fiber surface because it is a Seifert surface, and hence the sum $\sum_{i=1}^{n} (u_i \mathfrak{m}_i + v_i \mathfrak{l}_i)$ is null-homologous in the complement $\Sigma \setminus L(\underline{m})$. This complement is obtained from $\mathcal{S} \times S^1$ by gluing $(D^2 \times S^1)_i$, for $i = n+1, \ldots, k$, in such a way that $a_i Q_i + b_i H$ corresponds to the meridian of $(D^2 \times S^1)_i$. Hence there exists a non-zero vector $(w_{n+1}, \ldots, w_k)$ which satisfies

$$\sum_{i=1}^{n} ((a_i u_i - \sigma_i v_i) Q_i + (b_i u_i + \delta_i v_i) H) + \sum_{i=n+1}^{k} w_i (a_i Q_i + b_i H) = 0 \,.$$

Since $\sum_{i=1}^{k} Q_i = 0$ in $H_1(\mathcal{S} \times S^1)$ is the unique relation which we can use for vanishing the coefficients of Q_i's, all coefficients of Q_i's must be the same value. Hence we have the equality

$$\sum_{i=1}^{n} \left(Q_i + \frac{b_i u_i + \delta_i v_i}{a_i u_i - \sigma_i v_i} H \right) + \sum_{i=n+1}^{k} \left(Q_i + \frac{b_i}{a_i} H \right) = 0 \,,$$

which implies

$$
(4.2) \quad
\begin{aligned}
0 &= \sum_{i=1}^{n} \frac{b_i u_i + \delta_i v_i}{a_i u_i - \sigma_i v_i} + \sum_{i=n+1}^{k} \frac{b_i}{a_i} = \sum_{i=1}^{n} \left(\frac{b_i}{a_i} + \frac{v_i}{a_i(a_i u_i - \sigma_i v_i)} \right) + \sum_{i=n+1}^{k} \frac{b_i}{a_i} \\
&= \frac{1}{A} + \sum_{i=1}^{n} \frac{v_i}{a_i(a_i u_i - \sigma_i v_i)} \, .
\end{aligned}
$$

However the right-hand side of this equation must be strictly positive since $a_i u_i - \sigma_i v_i > 0$ and $a_i v_i > 0$, which is a contradiction.   □

PROOF OF THEOREM 1.1. We first remark that it is enough to observe the tightness for a specific contact form whose contact structure is compatible with $L(\underline{m})$ by Proposition 3.4. Assume that $L(\underline{m})$ is not a Hopf multilink in S^3. If all components of $L(\underline{m})$ are negative then it does not satisfy (PTP) by Lemma 4.7. In this case, we reverse the orientation of $L(\underline{m})$ as $L(-\underline{m})$ so that all components become positive. If all components of $L(\underline{m})$ are positive, then the compatible contact structure constructed according to the recipe in Proposition 4.1 is positively transverse to the fibers of the Seifert fibration everywhere. In particular, it is known that such a contact structure is always tight (see [20] and [19, Corollary 2.2]). Moreover, since the monodromy of the fibration of $L(\underline{m})$ is periodic, according to [4, Theorem 4.2], we can conclude that the contact structure is Stein fillable.

Suppose that $L(\underline{m})$ has at least one positive component and one negative component. In this case, even if we reverse the orientation of $L(\underline{m})$ by involution, $L(\underline{m})$ still has a negative component. Therefore, in either case, the contact structure $\ker \alpha$ has an overtwisted disk by property (3) in Proposition 4.1.

Finally we consider the case where $L(\underline{m})$ is a Hopf multilink. Let $m_1 S_1$ and $m_2 S_2$ denote the multilink components of $L(\underline{m})$, i.e., $L(\underline{m}) = (\Sigma(1, 1), m_1 S_1 \cup m_2 S_2)$. If $m_1 + m_2 \neq 0$ then $L(\underline{m})$ satisfies (PTP) up to the reversal of the orientation of $L(\underline{m})$. Hence the above proof works in this case. Suppose that $m_1 + m_2 = 0$. Note that the fiber surface of this multilink is tangent to the fibers of the Seifert fibration of $\Sigma(1, 1)$, though we can choose another Seifert fibration such that (PTP) is satisfied. Since the orientation of $L(\underline{m})$ is not canonical, it is enough to check that the compatible contact structure is overtwisted. This follows immediately since the fiber surface of $L(\underline{m})$ is a disjoint union of the fiber surfaces of a negative Hopf link and the compatible contact structure is same as that of the negative Hopf link.   □

5. Case $a_1 a_2 \cdots a_k < 0$.

5.1. Explicit construction of the contact structure. Throughout this section, we assume $A = a_1 \cdots a_k < 0$. We start from the following lemma.

LEMMA 5.1. *If $A < 0$ then every fibered Seifert multilink has at least one negative component.*
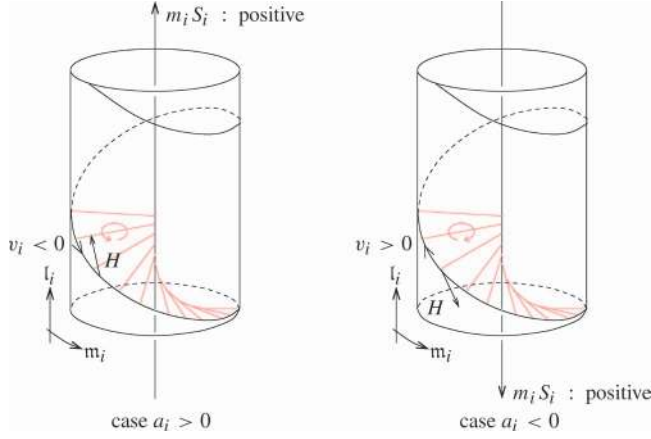
FIGURE 10.   The framing of F along $m_i S_i$.

PROOF.   The proof is analogous to that of Lemma 4.7. In the present case, the framing of the fiber surface F along $m_i S_i$ becomes as shown in Figure 10, from which we have $a_i v_i < 0$. Hence the right-hand side of equation (4.2) is strictly negative since $a_i u_i - \sigma_i v_i > 0$ and $a_i v_i < 0$. This is a contradiction.   □

The main assertion in this section is the following.

PROPOSITION 5.2.   *Let $L(\underline{m}) = (\Sigma, m_1 S_1 \cup \cdots \cup m_n S_n)$ be a fibered Seifert multilink $L(\underline{m})$ in a homology 3-sphere $\Sigma = \Sigma(a_1, \ldots, a_k)$ with $A < 0$. Assume* (PTP). *Fix an index i_0 of some negative component of $L(\underline{m})$. Then there exists a positive contact form α on Σ with the following properties*:

(1) $L(\underline{m})$ *is compatible with the contact structure $\xi = \ker \alpha$.*

(2)   *The Reeb vector field R_α of α is tangent to the fibers of the Seifert fibration on $\mathcal{S} \times S^1$.*

(3)   *The neighborhood $(D^2 \times S^1)_i$ of each negative component $m_i S_i$, except $m_{i_0} S_{i_0}$, contains a Lutz tube. In particular, it contains an overtwisted disk.*

(4)   *On the other $(D^2 \times S^1)_i$'s, except $i = i_0$, $\ker \alpha$ is transverse to the fibers of the Seifert fibration.*
In particular, if $L(\underline{m})$ has at least two negative components then the contact structure $\ker \alpha$ is overtwisted.

Before proving this proposition, we prepare a lemma similar to Lemma 4.4.

LEMMA 5.3.   *Suppose $A < 0$ and fix an index i_0. Let U_i be a collar neighborhood of C_i in $\mathcal{S}$ with coordinates $(r_i, \theta_i) \in [1, 2) \times S^1$ satisfying $\{(r_i, \theta_i)\,;\, r_i = 1\} = C_i$. Then there exists a 1-form β on $\mathcal{S}$ which satisfies the following properties*:

(1) $d\beta > 0$ *on $\mathcal{S}$.*

(2)   *If $b_i/a_i \leq 0$ and $i \neq i_0$ then $\beta = R_i r_i d\theta_i$ with $-b_i/a_i < R_i$ near C_i on U_i.*

(3) *If $b_i/a_i > 0$ and $i \neq i_0$ then $\beta = (R_i/r_i)d\theta_i$ with $-b_i/a_i < R_i < 0$ near C_i on U_i.*

(4) *If $b_{i_0}/a_{i_0} - 1/A < 0$ then $\beta = R_{i_0}r_{i_0}d\theta_{i_0}$ with $0 < R_{i_0} < -b_{i_0}/a_{i_0} + 1/A$ near C_{i_0} on U_{i_0}.*

(5) *If $b_{i_0}/a_{i_0} - 1/A \geq 0$ then $\beta = (R_{i_0}/r_{i_0})d\theta_{i_0}$ with $R_{i_0} < -b_{i_0}/a_{i_0} + 1/A$ near C_{i_0} on U_{i_0}.*

PROOF. Since $\sum_{i \neq i_0}(-b_i/a_i) + (-b_{i_0}/a_{i_0} + 1/A) = 0$, we can choose $R_1, \ldots, R_k$ such that they satisfy the above inequalities and the inequality $\sum_{i=1}^k R_i < 0$. The 1-form β required can be constructed from these R_i's in the same way as in the proof of Lemma 4.4. □

PROOF OF PROPOSITION 5.2. We make a contact form α_0 on $\mathcal{S} \times S^1$ from the 1-form β in Lemma 5.3 and extend it to $(D^2 \times S^1)_i$ as in the proof of Proposition 4.1. Properties (2), (3), (4) in the assertion follow from this construction. Let α denote the obtained contact form on M.

Suppose that $i \neq i_0$, $m_i S_i$ is a positive component and $a_i > 0$. From equation (4.1), we have $h_1(1) < 0$, $h_2(1) > 0$, $h_1'(1) < 0$ and $h_2'(1) > 0$. Hence the mutual positions of the fiber surface F, the oriented fibers H of the Seifert fibration and the Reeb vector field R_α on $(D^2 \times S^1)_i$ become as shown on the left in Figure 11. The contact structure α in this case is determined by the curve described on the right. From these figures, we can easily check that these satisfy property (1) in the assertion.
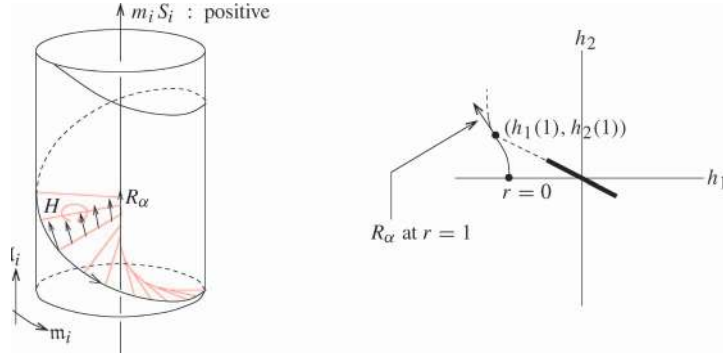

FIGURE 11. The mutual positions of F, H and R_α in the case where $m_i S_i$ is a positive component.

If $m_i S_i$ is negative and $a_i > 0$ then we have the same inequalities. Hence their mutual positions become as shown in Figure 12 and property (1) holds. If $i = i_0$ then $h_2(1) > 0$ may not hold, but this does not make any problem since $m_{i_0} S_{i_0}$ is a negative component. Thus property (1) holds.

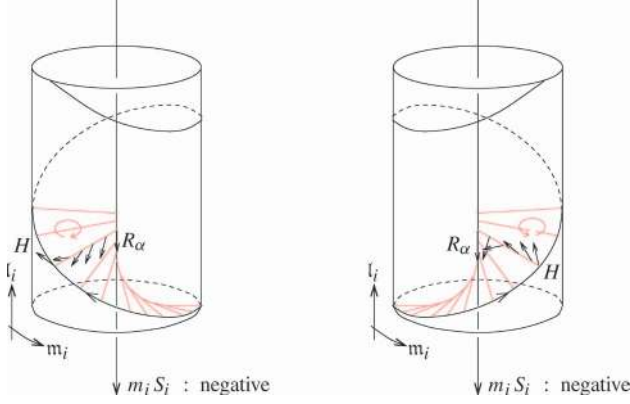The proof is analogous in the case $a_i < 0$.   □

FIGURE 12. The mutual positions of F, H and R_α in the case where $m_i S_i$ is a negative component.

5.2. Some criterion to detect overtwisted disks.

In this subsection, we show two lemmas which give sufficient conditions for the contact structure in Proposition 5.2 to be overtwisted.

LEMMA 5.4. *Suppose $A < 0$ and let $m_{i_0} S_{i_0}$ be a negative component of $L(\underline{m})$. Suppose further that there exists a_{i_1} among $a_1, \ldots, a_k$ which satisfies the inequality*

$$\frac{1}{|a_{i_1}|}\left(\frac{1}{|a_{i_0}|} - \frac{1}{|a_{i_1}|}\right) > -\frac{1}{A}\,.$$

Then the contact structure in Proposition 5.2 is overtwisted.

PROOF. From the inequality in the assumption, we have $|a_{i_1}| > |a_{i_0}|$. In particular, $i_0 \neq i_1$. We can assume that $m_{i_1} S_{i_1}$ is a positive component, since otherwise the contact structure is overtwisted by Proposition 5.2. We will find $R_1, \ldots, R_k$ in Lemma 5.3 which satisfy

$$|a_{i_0}|\left(R_{i_0} + \frac{b_{i_0}}{a_{i_0}}\right) = -|a_{i_1}|\left(R_{i_1} + \frac{b_{i_1}}{a_{i_1}}\right) < 0\,.$$

Set $X = R_{i_0} + b_{i_0}/a_{i_0}$ and $Y = R_{i_1} + b_{i_1}/a_{i_1}$. They should satisfy the conditions in Lemma 5.3, that is, $X - 1/A < 0$ and $Y > 0$.

For a sufficiently small $\varepsilon > 0$, we set R_i's for $i \neq i_0, i_1$ such that they satisfy the conditions in Lemma 5.3 and the equality

$$\sum_{i \neq i_0, i_1}\left(R_i + \frac{b_i}{a_i}\right) = \varepsilon\,.$$

In the case $k = 2$, we set $\varepsilon = 0$. We need the inequality $\sum_{i=1}^{k} R_i < 0$ and hence X and Y should satisfy

$$0 > \sum_{i \neq i_0, i_1} R_i + R_{i_0} + R_{i_1} = \varepsilon - \sum_{i \neq i_0, i_1} \frac{b_i}{a_i} + R_{i_0} + R_{i_1} = \varepsilon - \frac{1}{A} + X + Y \, .$$

Now we assume that the following inequality holds:

(5.1) $$|b_{i_0} + a_{i_0} R_{i_0}| = -|a_{i_0}|X < \frac{1}{|a_{i_0}|} \, .$$

The difference of the slopes of a meridional disk and a Legendrian curve on $\partial(D^2 \times S^1)_{i_0}$ is given as

$$\frac{(a_{i_0} Q_{i_0} + b_{i_0} H)}{a_{i_0}} - (Q_{i_0} - R_{i_0} H) = \left(\frac{b_{i_0}}{a_{i_0}} + R_{i_0} \right) H \, .$$

Since $b_{i_0}/a_{i_0} + R_{i_0} = X < 1/A < 0$, the slope of the Legendrian curve is a bit higher than that of the meridional disk as shown in Figure 13. Let γ be the boundary of the meridional disk. Since the distance of two neighboring intersection points of H and γ is $1/|a_{i_0}|$, inequality (5.1) ensures that we can isotope γ on $\partial(D^2 \times S^1)_{i_0}$ so that it is Legendrian except for a short vertical interval of length $|b_{i_0} + a_{i_0} R_{i_0}|$. We denote by Δ_{i_0} the meridional disk bounded by this isotoped γ.
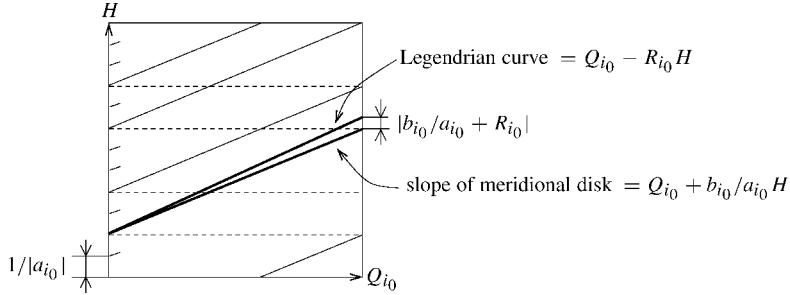

FIGURE 13. The slopes of a meridional disk and a Legendrian curve on the boundary of $(D^2 \times S^1)_{i_0}$.

We also obtain a similar disk Δ_{i_1} in $(D^2 \times S^1)_{i_1}$, assuming the inequality

$$|b_{i_1} + a_{i_1} R_{i_1}| = |a_{i_1}|Y < \frac{1}{|a_{i_1}|} \, .$$

In this case, the slope of the Legendrian curve is a bit lower than that of the meridional disk since $b_{i_1}/a_{i_1} + R_{i_1} = Y > 0$ (cf. Figure 15).

In summary, we have assumed for a point (X, Y) to satisfy the following conditions:

(5.2)
$$\begin{cases} |a_{i_0}|X + |a_{i_1}|Y = 0, \\[2mm] X + Y < -\varepsilon + \dfrac{1}{A}, \\[2mm] -\dfrac{1}{a_{i_0}^2} < X < \dfrac{1}{A}, \\[2mm] 0 < Y < \dfrac{1}{a_{i_1}^2}. \end{cases}$$

Note that we always have the inequality $-1/a_{i_0}^2 < 1/A$, because $1/|a_{i_1}|(1/|a_{i_0}| - 1/|a_{i_1}|) > -1/A$ implies $|a_{i_0}| < |a_{i_1}|$ and hence

$$-\frac{1}{a_{i_0}^2} < -\frac{1}{|a_{i_0}||a_{i_1}|} \le \frac{1}{A}.$$

Now we describe the region on the XY-plane where (X, Y) satisfies the inequalities in the above conditions, which is shown in Figure 14. Note that we used the inequality

$$\frac{1}{a_{i_0}^2} - \frac{1}{a_{i_1}^2} > \frac{1}{|a_{i_1}|}\left(\frac{1}{|a_{i_0}|} - \frac{1}{|a_{i_1}|}\right) > -\frac{1}{A}$$

when we described this region. The equality and inequalities in (5.2) have a solution if and only if the line $|a_{i_0}|X + |a_{i_1}|Y = 0$ intersects this region, i.e., the inequality

$$|a_{i_0}|\left(-\frac{1}{a_{i_1}^2} + \frac{1}{A}\right) + |a_{i_1}|\left(\frac{1}{a_{i_1}^2}\right) > 0$$

holds, and this follows from the assumption. Thus the embedded disks Δ_{i_0} and Δ_{i_1} exist.
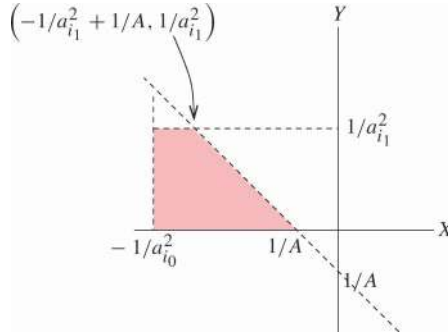

FIGURE 14. The region where (X, Y) satisfies the required inequalities.

Finally we connect these disks by a band B whose two sides are Legendrian as shown in Figure 15. We here explain this more precisely. We first remark that the lengths of the two short vertical intervals on the boundaries of Δ_{i_0} and Δ_{i_1} are the same since

$$|b_{i_0} + a_{i_0}R_{i_0}| = -|a_{i_0}|X = |a_{i_1}|Y = |b_{i_1} + a_{i_1}R_{i_1}|.$$

Let p_0, q_0 be the endpoints of the vertical interval of the boundary of Δ_{i_0} and let p_1 and q_1 be those of Δ_{i_1}. Choose a vertical annulus $W = H \times [0, 1]$ between $(D^2 \times S^1)_{i_0}$ and $(D^2 \times S^1)_{i_1}$ as shown in Figure 15 and let $\mathcal{F}_W$ denote the foliation on W determined by ξ. Note that $\mathcal{F}_W$ is non-singular and every leaf of $\mathcal{F}_W$ connects the connected components of ∂W because ξ is transverse to H. By shifting Δ_{i_0} if necessary, we can assume that p_0 and p_1 are the endpoints of the same leaf of $\mathcal{F}_W$. Since the lengths of the short vertical intervals are the same, by shifting both of Δ_{i_0} and Δ_{i_1} simultaneously, we can find positions of Δ_{i_0} and Δ_{i_1} such that p_0 and p_1 are the endpoints of a leaf of $\mathcal{F}_W$ and q_0 and q_1 are also the endpoints of another leaf of $\mathcal{F}_W$. Now we choose the band B to be a curved rectangle such that its boundary consists of these leaves and the short vertical intervals and it is tangent to the contact structure ξ along the leaves of $\mathcal{F}_W$ on the boundary as shown in Figure 15. The union $\Delta_{i_0} \cup B \cup \Delta_{i_1}$ is a disk embedded in Σ with a polygonal Legendrian boundary. We then isotope it in a neighborhood of the corners of the polygonal Legendrian boundary so that it becomes a smooth embedded disk with smooth Legendrian boundary. From the construction, the contact structure ξ is tangent to this disk along its boundary. Hence it is an overtwisted disk. $\qquad\square$
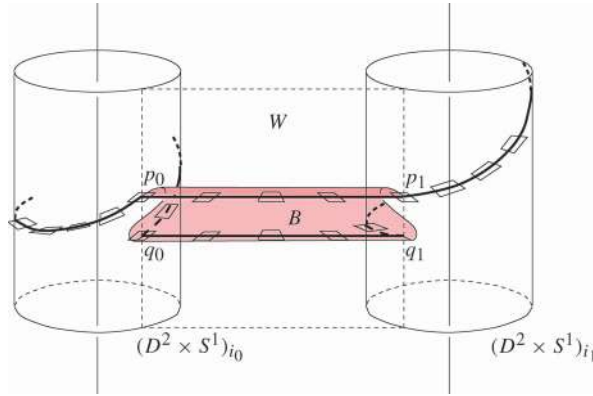

FIGURE 15. The band B.

LEMMA 5.5. *Suppose $A < 0$ and let $m_{i_0} S_{i_0}$ be a negative component of $L(\underline{m})$. Suppose further that there exist a_{i_1} and a_{i_2} satisfying $|a_{i_0}| < |a_{i_2}| < |a_{i_1}|$. Then the contact structure in Proposition 5.2 is overtwisted.*

PROOF. We have the inequality

$$-\frac{|a_{i_1}|}{A} \le \frac{1}{|a_{i_0} a_{i_2}|} = \left(\frac{1}{|a_{i_0}|} - \frac{1}{|a_{i_2}|}\right) \frac{1}{|a_{i_2}| - |a_{i_0}|} \le \frac{1}{|a_{i_0}|} - \frac{1}{|a_{i_2}|} < \frac{1}{|a_{i_0}|} - \frac{1}{|a_{i_1}|}$$

and hence the assertion follows from Lemma 5.4. $\qquad\square$

EXAMPLE 5.6. Suppose that $\gcd(|p|, |q|) = 1$ and $pq < 0$.

(1) $(\Sigma, L) = (\Sigma(1, p, q), -S_1)$ is a (p, q)-torus knot in S^3. Here the component $-S_1$ must be negative because of Lemma 5.1. If $|p|, |q| \geq 2$ then there exists an overtwisted disk by Lemma 5.5. If either $|p| = 1$ or $|q| = 1$ then L is a trivial knot in S^3 and its compatible contact structure is tight. Actually, this does not satisfy the condition in Lemma 5.4.

(2) $(\Sigma, L) = (\Sigma(p, q), S_1 \cup -S_2)$ is a positive Hopf link in S^3. It is well-known that its compatible contact structure is tight, and this actually does not satisfy the condition in Lemma 5.4.

6.   Fibered Seifert links in S^3.   In this section, we study Seifert links in S^3. The classification of Seifert links in S^3 was done by Burde and Murasugi [3], in which they proved that a link is a Seifert link in S^3 if and only if it is either a union of a finite number of fibers of the Seifert fibration in $\Sigma(p, q)$ with $pq \neq 0$ or a link described in Figure 1 with arbitrary orientations (cf. [5, p. 62]). The classification of contact structures on S^3 had been done by Eliashberg [6, 8]. In particular, it is known that S^3 admits a unique tight contact structure up to contactomorphism, so-called the *standard contact structure*.

PROOF OF THEOREM 1.2.   In this proof, we use the letters p and q instead of a_1 and a_2, respectively. The assertion in the case $pq > 0$ follows from Theorem 1.1. Suppose $pq < 0$. We first prove the assertion in the case where all components of L are negative. In this case, (PTP) is satisfied by Lemma 5.1. If L has more than one link components then the contact structure is overtwisted by the last assertion in Proposition 5.2. Suppose that L consists of only one component, then L is either a trivial knot or a (p, q)-torus knot with $pq < 0$. It is well-known that the contact structure of a trivial knot is tight, and that the contact structure of a (p, q)-torus knot with $pq < 0$ is overtwisted if and only if it is not a trivial knot. Thus the assertion follows in this case.

Next we consider the case where L has at least one positive component. Note that L also has one negative component by Lemma 5.1. We can assume that the number of negative components of L is one, otherwise the contact structure is overtwisted by the last assertion in Proposition 5.2.

We decompose the argument into three cases:

(1)   The two exceptional fibers of $\Sigma(p, q)$ are both components of L. That is,

$$L = (\Sigma(\underbrace{1, \ldots, 1}_{n-2}, p, q), m_1 S_1 \cup \cdots \cup m_{n-2} S_{n-2} \cup m_{n-1} S_{n-1} \cup m_n S_n).$$

(2)   One of the two exceptional fibers of $\Sigma(p, q)$ is a component of L. That is,

$$L = (\Sigma(\underbrace{1, \ldots, 1}_{n-1}, p, q), m_1 S_1 \cup \cdots \cup m_{n-1} S_{n-1} \cup m_n S_n).$$

(3)   Neither of the two exceptional fibers of $\Sigma(p, q)$ is a component of L. That is,

$$L = (\Sigma(\underbrace{1, \ldots, 1}_{n}, p, q), m_1 S_1 \cup \cdots \cup m_n S_n).$$

Here $m_i \in \{-1, +1\}$ since L is a fibered link.

We first consider Case (1). If $n = 2$ then L is a positive Hopf link in S^3. Suppose $n \geq 3$ and that either S_{n-1} or S_n, say S_{n-1}, is a negative component. The linking number of $\mathfrak{m}_{n-1}S_{n-1}$ and all the other components of L is $(n-2)|q| + 1$. Note that $n-2$ is the number of the link components of L along non-exceptional fibers. For a fiber surface F of L, the oriented boundary $\partial(F \cap (D^2 \times S^1)_{n-1}) \setminus \mathfrak{m}_{n-1}S_{n-1}$ on $\partial(D^2 \times S^1)_{n-1}$ is given as $\gamma = \pm(-((n-2)|q|+1)\mathfrak{m}_{n-1} + \mathfrak{l}_{n-1})$, where the sign $\pm$ is $+$ if $p > 0$ and $-$ otherwise (see Figure 16). Here the surface on the right is described by applying the Seifert's algorithm to the diagram on the left.
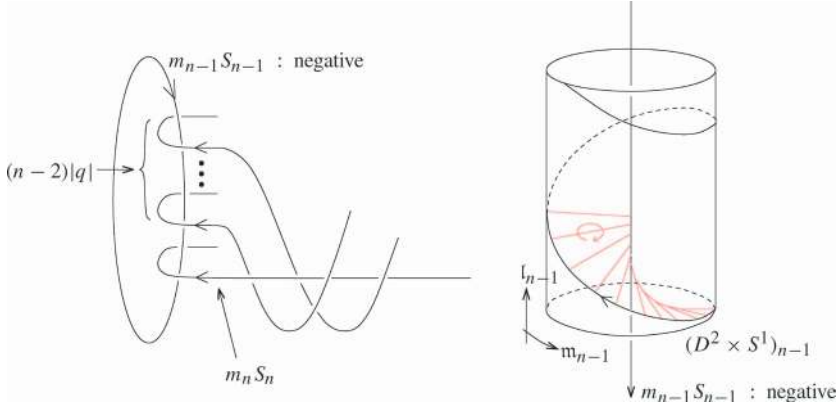

FIGURE 16. The framing of the Seifert surface in Case (1) with negative component $\mathfrak{m}_{n-1}S_{n-1}$ and $p > 0$.

Since $H = q\mathfrak{m}_{n-1} + p\mathfrak{l}_{n-1}$, (PTP) implies the inequality $I(\gamma, H) = \mp(((n-2)|q|+1)p + q) > 0$, where $I(\gamma, H)$ is the algebraic intersection number of γ and H on $\partial(D^2 \times S^1)_{n-1}$. However,

$$I(\gamma, H) = \mp(((n-2)|q|+1)p + q) = (n-2)pq \mp (p+q)$$
$$= (p \mp 1)(q \mp 1) + (n-3)pq - 1 < 0$$

since $(p \mp 1)(q \mp 1) \leq 0$ and $(n-3)pq \leq 0$ for $n \geq 3$. This is a contradiction.

Suppose $n \geq 3$ and a regular fiber is a negative component of L. The linking number of $\mathfrak{m}_{n-1}S_{n-1}$ and all the other components of L is $-(n-4)|q| - 1$ and the oriented boundary $\partial(F \cap (D^2 \times S^1)_{n-1}) \setminus \mathfrak{m}_{n-1}S_{n-1}$ on $\partial(D^2 \times S^1)_{n-1}$ becomes $\gamma = \pm((-(n-4)|q|-1)\mathfrak{m}_{n-1} - \mathfrak{l}_{n-1})$ (see Figure 17). Thus, $I(\gamma, H) = \mp(((n-4)|q|+1)p - q) = (n-4)pq \mp p \pm q$. If $|p|, |q| \geq 2$ then the contact structure of L is overtwisted by Lemma 5.5. If either $|p|$ or $|q|$ equals 1 then

$$(n-4)pq \mp p \pm q = (n-3)pq - (p \mp 1)(q \pm 1) - 1 < 0$$

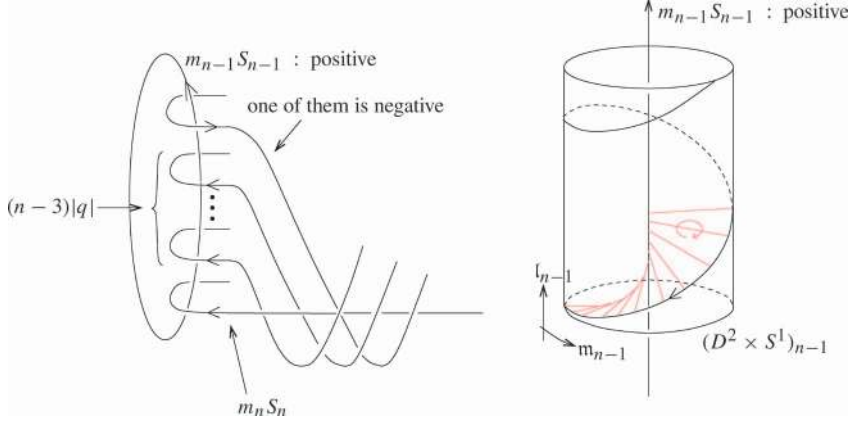since $(p \mp 1)(q \pm 1) = 0$. Hence (PTP) does not hold.

FIGURE 17.   The framing of the Seifert surface in Case (1) with a non-exceptional fiber being the negative component.

Next we consider Case (2). If $n = 1$ then L is a trivial knot in S^3. Suppose $n \geq 2$ and that S_n is a negative component. Since

$$I(\gamma, H) = \mp((n-1)|q|p+q) = (n-1)pq \mp q = (n-1)pq + |q| \leq 0\,,$$

(PTP) does not hold (cf. Figure 16 with deleting the component $m_n S_n$ and replacing the number $(n-2)|q|$ by $(n-1)|q|$ and the indices $n-1$ by n). We remark that the equality holds when $n = 2$ and $|p| = 1$, and if $|q| = 1$ in addition then L becomes a positive Hopf link. Nevertheless, we can ignore this case because the fibration of a positive Hopf link is not given by this Seifert fibration.

Suppose $n \geq 2$ and a regular fiber is a negative component of L, then

$$I(\gamma, H) = \mp((n-3)|q|p-q) = (n-3)pq \pm q = (n-3)pq - |q|$$

(cf. Figure 17 with deleting the component $m_n S_n$ and replacing the number $(n-3)|q|$ by $(n-2)|q|$ and the indices $n-1$ by n). This is positive if and only if $n = 2$ and $|p| \geq 2$, in which case if $|q| \geq 2$ then the contact structure of L is overtwisted by Lemma 5.5, and if $|q| = 1$ then L is a positive Hopf link and its contact structure is tight.

Finally we consider Case (3). If $n = 1$ then it is a (p, q)-torus knot and we know that its contact structure is tight if and only if it is a trivial knot. If $n = 2$ then L is a positive Hopf link, otherwise L is not fibered. If $n \geq 3$ and $|p|, |q| \geq 2$ then its contact structure is overtwisted by Lemma 5.5. So, we can suppose that $n \geq 3$ and either $|p|$ or $|q|$ equals 1. Choose a positive component $m_{i_1} S_{i_1}$ of L, then the oriented boundary $\partial(F \cap (D^2 \times S^1)_{i_1}) \setminus m_{i_1} S_{i_1}$ on $\partial(D^2 \times S^1)_{i_1}$ is given as $\gamma = -(n-3)|q|m_{i_1} - \mathfrak{l}_{i_1}$ (see Figure 18). Since $I(\gamma, H) = -(n-3)|q| + pq < 0$, (PTP) does not hold.
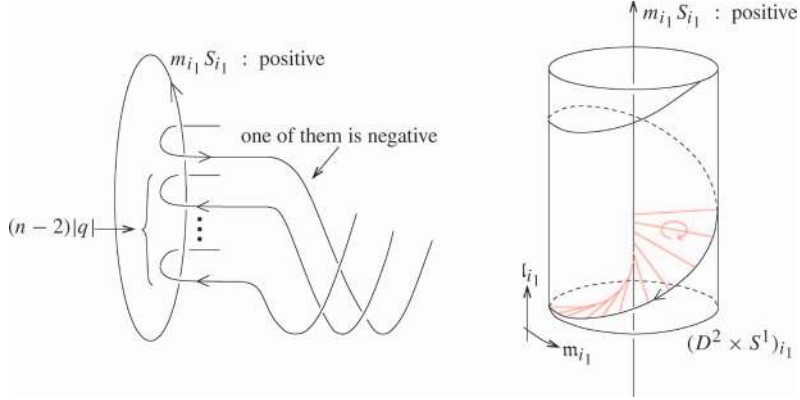
FIGURE 18.   The framing of the Seifert surface in Case (3).

If $pq = 0$ then L is as shown in Figure 1, which is a connected sum of a finite number of Hopf links. The plumbing argument in [32] ensures that the contact structure of such a link is tight if and only if every summand is a positive Hopf link. This completes the proof.   □

7.   Seifert links in S^3 and their strongly quasipositivity.   A Seifert surface in S^3 is called *quasipositive* if it is obtained from a finite number of parallel copies of a disk by attaching positive bands. A link is called *strongly quasipositive* if it is realized as the boundary of some quasipositive surface. In other words, a strongly quasipositive link is the closure of a braid given by the product of words of the form

$$\sigma_{i,j} = (\sigma_i \cdots \sigma_{j-2})\sigma_{j-1}(\sigma_i \cdots \sigma_{j-2})^{-1},$$

where σ_i is a positive generator of braid. See [23, 24, 25, 26, 27, 28] for further studies of quasipositive surfaces.

It is known by Hedden [15], and Baader and the author [1] in a different way, that the compatible contact structure of a fibered link in S^3 is tight if and only if its fiber surface is quasipositive. Hence, Theorem 1.2 can be generalized into the non-fibered case as stated in Corollary 1.3.

PROOF OF COROLLARY 1.3.   The assertion had been proved in Theorem 1.2 if L is fibered. So, hereafter we assume that L is non-fibered. If $(a_1, a_2) = (0, 1)$ then L must be a trivial link with several components, which is excluded by the assumption. Suppose that $a_1 a_2 \neq 0$. We now prove that L is not fibered if and only if it is a positive or negative torus link, other than a Hopf link, which consists of even number of link components half of which have reversed orientations.

We first consider the case where both of the fibers of the Seifert fibration corresponding to a_1 and a_2 are not components of L. Let k and k' be the numbers of positive and negative

components, respectively. Since $\sum_{i=1}^{n} m_i \sigma_i = (k - k') a_1 a_2$, L is fibered if and only if $k = k'$ by Theorem 2.1. Thus the assertion follows.

Next we consider the case where one of the fibers corresponding to a_1 and a_2, say a_1, is a component of L and the other is not. If $|a_1| = 1$ then we can regard such a Seifert link

$$L = (\Sigma \underbrace{(1, \ldots, 1}_{n-1}, \pm 1, a_2), m_1 S_1 \cup \cdots \cup m_{n-1} S_{n-1} \cup m_n S_n)$$

as

$$L = (\Sigma \underbrace{(1, \ldots, 1}_{n}, \pm 1, a_2), m_1 S_1 \cup \cdots \cup m_{n-1} S_{n-1} \cup m_n S_n),$$

where $m_i \in \{-1, +1\}$. Hence, this is included in the previous case. Assume that $|a_1| > 1$. Then we have $\sum_{i=1}^{n} m_i \sigma_i = k a_1 a_2 + m_n a_2 \neq 0$, where k is some integer and $|m_n| = 1$, and hence L is always fibered in this case by Theorem 2.1.

Finally we consider the case where both of the fibers corresponding to a_1 and a_2 are components of L. In this case, $\sum_{i=1}^{n} m_i \sigma_i = k a_1 a_2 + m_{n-1} a_2 + m_n a_1$, where k is some integer and $|m_{n-1}| = |m_n| = 1$. As in the previous case, we can assume that $|a_1| > 1$ and $|a_2| > 1$. Since $\gcd(|a_1|, |a_2|) = 1$, the inequalities $|a_1| \neq |a_2|$ and $|a_1 a_2| > |a_1| + |a_2|$ hold. Hence $k a_1 a_2 + m_{n-1} a_2 + m_n a_1 \neq 0$, i.e., L is always fibered in this case also.

Thus the non-fibered link L is a positive or negative torus link, other than a Hopf link, which consists of even number of link components, say $2k$, half of which have reversed orientations. Such an L is realized as the boundary of a Seifert surface F consisting of k annuli.

Suppose $a_1 a_2 > 0$ and let F' be one of the annuli of F. The core curve of F' constitutes a positive torus knot, say a (p, q) torus knot with $p, q > 0$. It is known in [1, Lemma 6.1] that if F' is quasipositive then -1 times the linking number $lk(F')$ of the two boundary components of F' is at most the maximal Thurston-Bennequin number $TB(K)$ of the core curve K of the annulus, i.e., $-lk(F') \leq TB(K)$. It is known in [30] that

$$TB(K) = (p - 1)q - p = pq - p - q,$$

where p is regarded as the number of Seifert circles, which equals the braid index. However, we can easily check $lk(F') = -pq$, which does not satisfy the inequality $-lk(F') \leq TB(K)$. Thus F' is not quasipositive. Now assume that L is strongly quasipositive. Then, by definition, there exists a quasipositive surface bounded by L. However this surface contains the above non-quasipositive annulus as an essential subsurface, which contradicts the Characterization Theorem of quasipositive surfaces in [23]. Thus L is not strongly quasipositive.

If $a_1 a_2 < 0$ then the link L is in Case (3) in the assertion. Suppose that the core curves of annuli of F constitute a (kp, kq) torus link with $p > 0$ and $q < 0$. Using ambient isotopy move in S^3, we can assume that $p \leq |q|$. In the case where $p = |q|$, we set the surface F in the position as shown in Figure 19, which shows that the surface is quasipositive. If $p < |q|$, we need to add more crossings, though we can check that the surface is still quasipositive as shown in Figure 20. This completes the proof. □

We close this section with a conjecture arising from the fact in Corollary 1.3.
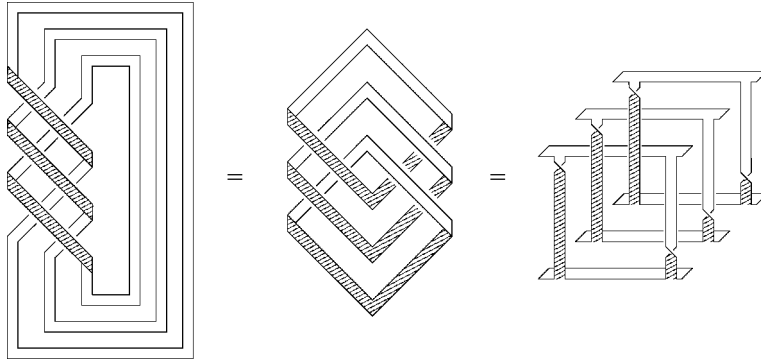
FIGURE 19.   The surface F in the case $(p, q) = (3, -3)$.
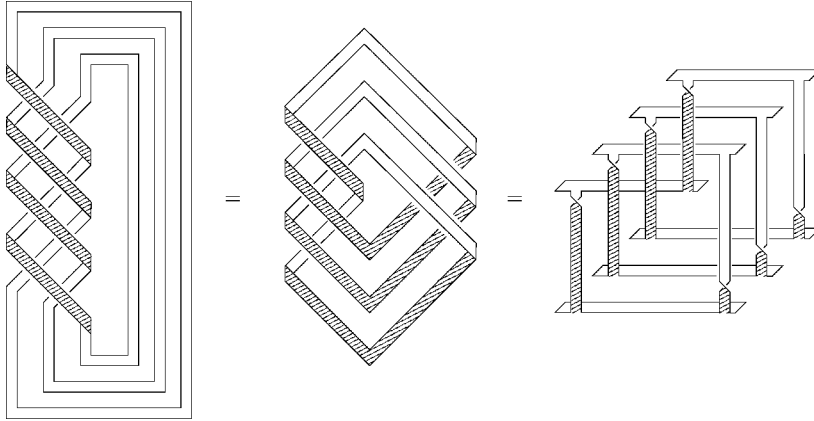
FIGURE 20.   The surface F in the case $(p, q) = (3, -4)$.

CONJECTURE 7.1.   *Any non-splittable unoriented link in S^3 has at most two strongly quasipositive orientations.*

Here a strongly quasipositive orientation means an orientation assigned to the unoriented link such that the obtained oriented link becomes strongly quasipositive. As in Corollary 1.3, this conjecture is true for all Seifert links in S^3. We will prove the same assertion for fibered, positively-twisted graph links in S^3 in the subsequent paper [17].

8. Cablings.

8.1. Definition of positive and negative cablings.   A cabling is an operation which produces a new link L in a 3-manifold from an old one L' by replacing a small tubular neighborhood of a component of L' by a solid torus containing a torus link parallel to the boundary of the solid torus. We define this operation for multilinks as follows: Let M be an oriented,

closed, smooth 3-manifold and $L(\underline{m})$ a fibered multilink in M. Suppose that there exists a solid torus N in M such that each $L(\underline{m}) \cap N$ is a torus multilink in N with consistent orientation, i.e., a multilink in N lying on a torus parallel to the boundary ∂N all of whose link components have consistent orientations. We replace the torus multilink components of $L(\underline{m})$ in N by its core curve S, extend the fiber surfaces of $L(\underline{m})$ by the retraction of N to S, and define the multiplicity of S from these fiber surfaces canonically. We denote the obtained multilink in M by $L'(\underline{m}')$. Note that $L'(\underline{m}')$ is always fibered. In other words, $L(\underline{m})$ is a multilink obtained from $L'(\underline{m}')$ by replacing a small tubular neighborhood of a link component S of $L'(\underline{m}')$ by a solid torus N containing a torus multilink with suitable multiplicities.

Next we define the notion of positive and negative cablings. We set $L(\underline{m}) \cap N$ and $L'(\underline{m}')$ in M simultaneously such that the core curve of N coincides with the link component of $L'(\underline{m}')$ in N, and check the intersection of $L(\underline{m}) \cap N$ with the fiber surface of $L'(\underline{m}')$.

LEMMA 8.1. $L(\underline{m}) \cap N$ intersects the fiber surface of $L'(\underline{m}')$ transversely.

PROOF. By [5, Theorem 4.2], the fibration of $L(\underline{m})$ is decomposed into two fibered multilinks $L'(\underline{m}')$ and $L''(\underline{m}'')$ by the splice decomposition, each of whose fibrations is induced from that of $L(\underline{m})$. The multilink $L(\underline{m}) \cap N$ is parallel to the fibers H of the Seifert fibration of $L''(\underline{m}'')$. Hence, it is enough to show that H is transverse to the fiber surface of $L'(\underline{m}')$, and this follows immediately since H is transverse to the fiber surfaces of $L''(\underline{m}'')$. □

DEFINITION 8.2. A cabling is called *positive* if $L(\underline{m}) \cap N$ intersects the fiber surface of $L'(\underline{m}')$ positively transversely. If the intersection is negative then the cabling is called *negative*.

To discuss the framing of the cabling, we fix a basis of ∂N as follows: Let $\mathfrak{m}$ be an oriented meridian on ∂N positively transverse to the fiber surface F of $L(\underline{m})$, and $\mathfrak{l}$ an oriented simple closed curve on ∂N such that $I(\mathfrak{m}, \mathfrak{l}) = 1$, where $I(\mathfrak{m}, \mathfrak{l})$ is the algebraic intersection number of $\mathfrak{m}$ and $\mathfrak{l}$ on ∂N. Each connected component of the oriented boundary of $F \setminus \mathrm{int} N$ on $\partial(M \setminus \mathrm{int} N)$ is given as $\gamma = u\mathfrak{m} + v\mathfrak{l}$, where $(u, v) \in \mathbf{Z} \times \mathbf{N}$ are assumed to be coprime.

Now we embed N into S^3 along a trivial knot such that $(\mathfrak{m}, \mathfrak{l})$ becomes the preferred meridian-longitude pair of this trivial knot. We then add the core curve S_n of $S^3 \setminus \mathrm{int} N$ as an additional link component to $L(\underline{m}) \cap N$ embedded in S^3, extend the fiber surfaces of $L(\underline{m})$ by the retraction of $S^3 \setminus \mathrm{int} N$ to S_n, and define the multiplicity m_n of S_n from these fiber surfaces canonically. The obtained multilink can be represented as

$$L_{p,q}(\underline{m}_{p,q}) = (\Sigma(\underbrace{1, \ldots, 1}_{n-1}, \varepsilon q, \varepsilon p), m_1 S_1 \cup \cdots \cup m_{n-1} S_{n-1} \cup \varepsilon_n m_n S_n),$$

where $p > 0$,

$$\varepsilon = \begin{cases} 1 & \text{if the cabling is positive} \\ -1 & \text{if the cabling is negative}, \end{cases}$$

and

$$\varepsilon_n = \begin{cases} -1 & \text{if the cabling is negative and } q > 0 \\ 1 & \text{otherwise} \end{cases}$$

(cf. Figure 21). The sign ε is chosen such that $I(H, \gamma) > 0$, where H is the fibers of the Seifert fibration on $\partial(D^2 \times S^1)_n$ and $I(H, \gamma)$ is the algebraic intersection number of H and γ on ∂N. This is checked as follows: $H = \varepsilon\, p\, \mathfrak{m}_n + \varepsilon\, q\, \mathfrak{l}_n = \varepsilon\, q\, \mathfrak{m} + \varepsilon\, p\, \mathfrak{l}$ on ∂N and $I(H, \gamma) = \varepsilon\,(qv - pu)$. If the cabling is positive then we have $qv - pu > 0$. If it is negative then $qv - pu < 0$. In either case, we have $I(H, \gamma) > 0$. This inequality means that H intersects F positively transversely. The sign ε_n is needed since the working orientation of S_n changes depending on the mutual positions of 0, q/p and u/v, where 0 is the slope of the longitude, q/p is the slope of the cabling, and u/v is the slope of the fiber surface.

Let $\mathfrak{L}$ be the set of longitudes with $u \geq 0$ and $q \neq 0$, then there exists a longitude $\mathfrak{l}$ in $\mathfrak{L}$ such that u becomes minimal among them. We always use this meridian-longitude pair $(\mathfrak{m}, \mathfrak{l})$ in the discussion below. In particular, the case $q = 0$ is excluded.
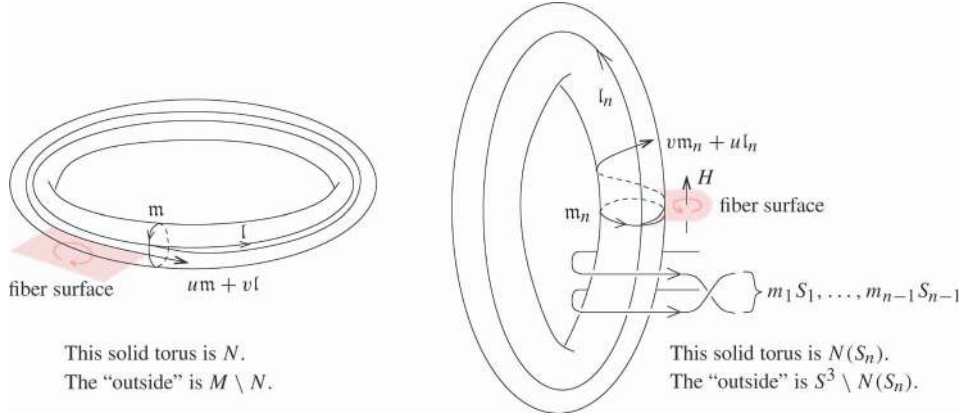

FIGURE 21. The left figure shows the fiber surface F in $M \setminus \text{int } N$ and the right one shows $L(\underline{m}) \cap N$ in $N \subset S^3$.

LEMMA 8.3. *For each $i = 1, \ldots, n - 1$, m_i is positive if and only if the cabling is positive.*

PROOF. Recall that the orientation of $m_i S_i$ is consistent with that of $\mathfrak{l}$. If the cabling is positive then the working orientation of S_i is consistent with that of $\mathfrak{l}$. Hence $m_i > 0$. If it is negative then, since we change the orientation of the fibers of the Seifert fibration by multiplying ε, the working orientation becomes opposite to that of $\mathfrak{l}$. Hence $m_i < 0$. □

8.2. Proof of Theorem 1.4.

LEMMA 8.4. *Let $L(\underline{m})$ be a fibered multilink in an oriented, closed, smooth 3-manifold M with a cabling in a solid torus N. Then there exists a positive contact form α on M with the following properties:*

(1) $L(\underline{m})$ is compatible with the contact structure $\xi = \ker \alpha$.

(2)   On a neighborhood of ∂N, α is given as $\alpha = h_2(r)d\mu - h_1(r)d\lambda$, where (r, μ, λ) are coordinates of $N = D^2 \times S^1$ chosen such that (r, μ) are the polar coordinates of D^2 of radius 1 and (μ, λ) are the coordinates of ∂N with respect to the meridian-longitude pair $(\mathfrak{m}, \mathfrak{l})$, and h_1 and h_2 are real-valued smooth functions with a parameter $r \in [0, 1]$ such that $u/v - h_1(1)/h_2(1) > 0$ is sufficiently small.

(3) α on N is the restriction of the contact form compatible with the Seifert multilink $L_{p,q}(\underline{m}_{p,q})$ to $S^3 \setminus \mathrm{int} N(S_n)$.

PROOF.   Let $L'(\underline{m}')$ be the multilink in M before the cabling and let α' be a contact form obtained in Proposition 3.3, whose kernel is compatible with $L'(\underline{m}')$. On a neighborhood of ∂N, α' is given as

$$\alpha' = Rvd\mu + \left(\frac{1}{r} - Ru\right)d\lambda$$

as in equation (3.1). Hence

$$\frac{u}{v} - \frac{h_1(1)}{h_2(1)} = \frac{u}{v} - \frac{-(1 - Ru)}{Rv} = \frac{1}{Rv} > 0$$

can be sufficiently small since we can choose $R > 0$ sufficiently large.

Next we make a contact form compatible with $L(\underline{m})$ from α' by replacing the form on N suitably. Let $\alpha_{p,q}$ be a positive contact form on S^3 whose kernel is compatible with the fibered Seifert multilink $L_{p,q}(\underline{m}_{p,q})$ of the cabling. Such a contact form is given explicitly in Proposition 4.1 and Proposition 5.2. Let (r_n, μ_n, λ_n) be the coordinates on $(D^2 \times S^1)_n$, then in a small neighborhood of ∂N, the gluing map of the cabling is given as $(r, \mu, \lambda) = (2 - r_n, \lambda_n, \mu_n)$. Hence, on this neighborhood, we have

$$\alpha = h_2(r)d\mu - h_1(r)d\lambda = -h_1(2 - r_n)d\mu_n + h_2(2 - r_n)d\lambda_n.$$

First consider the case where the cabling in N is positive. In this case, we have $H = \varepsilon q\, \mathfrak{m}_n + \varepsilon p\, \mathfrak{l}_n = q\mathfrak{m} + p\mathfrak{l}$ since $\varepsilon = 1$, $q > 0$, $u \geq 0$, $v > 0$ and $qv - pu > 0$. By choosing $R > 0$ sufficiently large, we can assume that H, γ, α' and $\alpha_{p,q}$ are as shown in Figure 22. Remark that the contact forms α' and $\alpha_{p,q}$ in the figure are given with the coordinates (r_n, μ_n, λ_n), hence the x-axis represents $-h_2(2-r_n)$ and the y-axis does $-h_1(2-r_n)$. By multiplying a positive constant to $\alpha_{p,q}$ if necessary, we can connect the two contact forms α' and $\alpha_{p,q}$ smoothly with keeping the positive transversality of the Reeb vector field and the interiors of the fiber surfaces.

Next we consider the case where the cabling is negative. Recall that the contact form constructed according to Lemma 5.3 and Proposition 5.2 depends on the choice of $b_1, \ldots, b_k$. By Lemma 8.3 we have $m_i < 0$ for $i = 1, \ldots, n - 1$. We now choose for instance $m_1 S_1$ as the negative component with index i_0 specified in Lemma 5.3. In this setting, we re-choose these b_i's such that $b_n/a_n \leq 0$, and then choose R_n in Lemma 5.3 (2) sufficiently large so that the line representing $\ker \alpha_{p,q}$ is sufficiently close to H on the xy-plane.

If $q < 0$ then we have $H = \varepsilon q\, \mathfrak{m}_n + \varepsilon p\, \mathfrak{l}_n = q\mathfrak{m} + p\mathfrak{l}$ since $\varepsilon = 1$, $u \geq 0$, $v > 0$ and $qv - pu > 0$. By choosing $R > 0$ sufficiently large, we can assume that H, γ, α' and $\alpha_{p,q}$
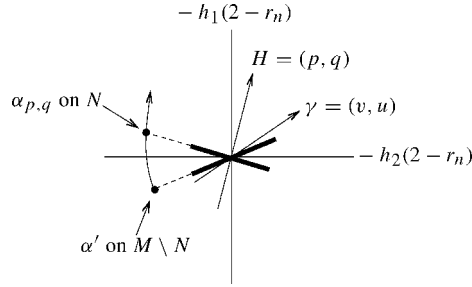
FIGURE 22.   Connect α' and $\alpha_{p,q}$ smoothly (case of positive cabling).

are as shown on the left in Figure 23. If $q > 0$ then $H = \varepsilon q\, \mathfrak{m}_n + \varepsilon p\, \mathfrak{l}_n = -q\, \mathfrak{m} - p\, \mathfrak{l}$ since $\varepsilon = -1$. Thus they are in the positions as shown on the right in Figure 23. In either case, by multiplying a positive constant to $\alpha_{p,q}$ if necessary, we can connect the contact forms α' and $\alpha_{p,q}$ smoothly as shown in the figures. Thus we obtain the contact form required.   □
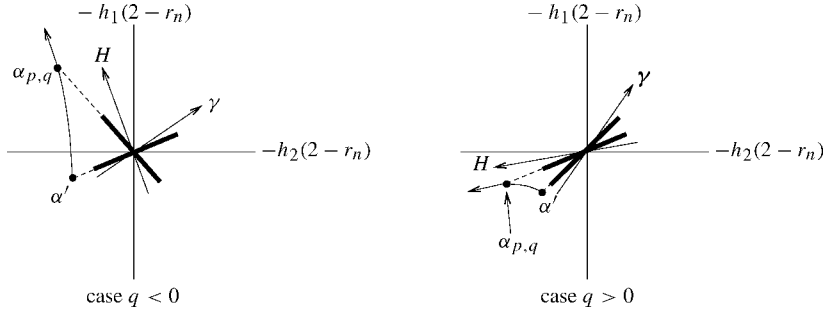

FIGURE 23.   Connect α' and $\alpha_{p,q}$ smoothly (case of negative cabling).

Now we prove Theorem 1.4. We first recall the statement.

THEOREM 1.4.   *Let $L(\underline{m})$ be a fibered multilink in an oriented, closed, smooth 3-manifold M with cabling in a solid torus N in M, and $L'(\underline{m'})$ the fibered multilink obtained from $L(\underline{m})$ by retracting N to its core curve. Let ξ and ξ' denote the contact structures on M compatible with $L(\underline{m})$ and $L'(\underline{m'})$, respectively.*

(1)   *If ξ' is tight and the cabling is positive, then ξ is tight.*

(2)   *If ξ' is tight, the cabling is negative and $L(\underline{m}) \cap N$ has at least two components, then ξ is overtwisted.*

(3)   *If ξ' is tight, the cabling is negative, $L(\underline{m}) \cap N$ is connected, $p \geq 2$ and $q \leq -2$, then ξ is overtwisted.*

(4)   *If ξ' is overtwisted then ξ is also overtwisted.*

PROOF. We use the contact structure constructed in Lemma 8.4. If ξ' is in Case (1) in the assertion then there exists a one-parameter family which connects ξ and ξ'. Hence ξ and ξ' are contactomorphic by Gray's theorem [14]. Suppose that ξ' is in Case (2). In this case, each $m_i S_i$ for $i = 1, \ldots, n-1$ is a negative component of $L_{p,q}(\underline{m}_{p,q})$ by Lemma 8.3. Thus, Proposition 5.2 and Lemma 8.4 ensure that there exists a negative component which contains an overtwisted disk. Suppose ξ' is in Case (3). We will use Lemma 5.4 to detect an overtwisted disk. We assign the index i_0 to the link component S_1 and the index i_1 to the singular fiber of the Seifert fibration other than S_n. From Figure 22, we can make sure that the proof of Lemma 8.4 works even if the point representing α' is sufficiently close to the horizontal axis. This means that we can choose R_n to be any value in $(-b_n/a_n, \infty)$. This is important since, in the proof of Lemma 5.4, R_n is some value with $-b_n/a_n < R_n$ and we do not know at which value the overtwisted disk is detected. Since $a_{i_0} = 1$, we have $1 > 1/|p| + 1/|q|$. Hence, we can detect an overtwisted disk between $(D^2 \times S^1)_{i_0}$ and $(D^2 \times S^1)_{i_1}$ by Lemma 5.4, which is outside of $(D^2 \times S^1)_n$. In Case (4), let D denote an overtwisted disk in (M, ξ'). Since we can choose N sufficiently small so that $\partial D \cap N = \emptyset$, the overtwisted disk still remains in (M, ξ) after the cabling. $\square$

REMARK 8.5. If $p = 1$ then $L(\underline{m})$ is ambient isotopic to $L'(\underline{m}')$. Suppose $p \geq 2$. We have chosen $(\mathfrak{m}, \mathfrak{l})$ such that $u \geq 0$ is minimal among $\mathcal{L}$. If the cabling is negative and $q \geq 2$ then we can change $\mathfrak{l} \mapsto \mathfrak{l} - (q-1)\mathfrak{m}$ such that the cabling is negative and $q = 1$. Hence this case is excluded since u is not minimal in $\mathcal{L}$. Now, the remaining case is when ξ' is tight, the cabling is negative, $L(\underline{m}) \cap N$ is connected, $p \geq 2$ and $q \in \{-1, 1\}$.

8.3. Cabling along fibered knots.

In this last subsection, we apply our result to the usual cabling without multiplicities. A usual cabling is an operation producing a new link L in a 3-manifold M from an old one L' by replacing a small tubular neighborhood of a link component of L' by a solid torus containing a torus link parallel to its boundary. We can apply our result to the case where L' is a fibered knot, i.e., it has only one link component.

Let L' be a fibered knot in M and $N(L')$ its small tubular neighborhood with the meridian-longitude pair $(\mathfrak{m}', \mathfrak{l}')$ induced from the fiber surface, namely $\mathfrak{m}'$ is the boundary of a meridional disk and $\mathfrak{l}'$ is the oriented boundary of a fiber surface of L'.

COROLLARY 8.6. *Let L' be a fibered knot in an oriented, closed, smooth 3-manifold M and L the link obtained from L' by cabling a (np, nq)-torus link with respect to $(\mathfrak{m}', \mathfrak{l}')$, i.e., the cabling with slope $q\,\mathfrak{m}' + p\,\mathfrak{l}'$ and having n components, where $p > 0$ and $\gcd(p, |q|) = 1$. Let ξ and ξ' denote the contact structure on M compatible with L and L', respectively.*

    (1) *If ξ' is tight and $q > 0$ then ξ is tight.*
    (2) *If ξ' is tight, $q < 0$ and $n \geq 2$ then ξ is overtwisted.*
    (3) *If ξ' is tight, $p \geq 2$ and $q \leq -2$ then ξ is overtwisted.*
    (4) *If ξ' is overtwisted then ξ is also overtwisted.*

PROOF. Let $L'(\underline{m}')$ be the fibered multilink obtained from L by retracting the solid torus $N(L')$ of the cabling to its core curve. Since L' is a knot, the framing of the fiber surfaces of $L'(\underline{m}')$ coincides with that of the fiber surface of L'. This means that $\gamma = \mathfrak{l}$, i.e., $(u, v) = (0, 1)$. Hence the cabling is positive in the sense in Definition 8.2 if and only if $q > 0$. Note that the case $q = 0$ is excluded by Lemma 8.1. The assertion is just a restatement of Theorem 1.4 in this special case. $\qquad\square$

REMARK 8.7. In the remaining case, i.e., the case where ξ' is tight, $n = 1$, $p \geq 2$ and $q = -1$, it is known in [2] that the contact structure ξ is tight if and only if $M = S^3$ and L is a trivial knot (cf. [16] for the case where L' is a fibered knot in S^3).

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