Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 45 (1995), No. 4, 685–698

Persistent URL: http://dml.cz/dmlcz/128555

Czechoslovak Mathematical Journal, 45 (120) 1995, Praha

COMPATIBLE MAPPINGS OF TYPE (B) AND COMMON FIXED POINT THEOREMS OF GREGUŠ TYPE

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(Received February 25, 1994)

I. INTRODUCTION

G. Jungck [4] introduced more generalized commuting mappings, called compatible mappings, which are more general than those of weakly commuting mappings [12]. Several authors proved common fixed point theorems using this concept ([5]–[6] and [8]–[10]). In general, commuting mappings are weakly commuting and weakly commuting mappings are compatible, but the converses are not necessarily true.

Recently, G. Jungck, P. P. Murthy and Y. J. Cho [7] defined the concept of compatible mappings of type (A) which is equivalent to the concept of compatible mappings under some conditions and proved a common fixed point theorem for compatible mappings of type (A) in a metric space.

Further, P.P. Murthy, Y. J. Cho and B. Fisher [10] proved some fixed point theorems of Greguš type (see [1]–[3]) for compatible mappings of type (A) in Banach spaces.

In this paper we introduce the concept of compatible mappings of type (B) and compare these mappings with compatible mappings and compatible mappings of type (A) in normed spaces. In the sequel, we derive some relations between these mappings. Also, we prove a common fixed point theorem of Greguš type for compatible mappings of type (B) in Banach spaces.

* Research partially supported by U.G.C., New Delhi, India. 1991 AMS Subject Classification Code: 54H25. Key words and phrases: Compatible mappings, compatible mappings of type (A), compatible mappings of type (B) and common fixed points.

II. Compatible mappings of type (B)

In this section we introduce the concept of compatible mappings of type (B) and show that under some conditions these mappings are equivalent to compatible mappings and compatible mappings of type (A) in a normed space. Throughout this paper, X denotes a normed space $(X, \|\cdot\|)$ with the norm $\|\cdot\|$.

We state two definitions ([11]), which are motivated by [4] and [7].

Definition 2.1. Let S and T be mappings from a normed space X into itself. The mappings S and T are said to be compatible if

$$\lim_{n\to\infty} \|STx_n - TSx_n\| = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 2.2. Let S and T be mappings from a normed space X into itself. The mappings S and T are said to be compatible of type (A) if

$$\lim_{n\to\infty} \|TSx_n - SSx_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|STx_n - TTx_n\| = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Further, we introduce

Definition 2.3. Let S and T be mappings from a normed space X into itself. The mappings S and T are said to be compatible of type (B) if

$$\lim_{n\to\infty} \|STx_n - TTx_n\| \leqslant \tfrac{1}{2}\big[\lim_{n\to\infty} \|STx_n - St\| + \lim_{n\to\infty} \|St - SSx_n\|\big]$$

and

$$\lim_{n\to\infty} \|TSx_n - SSx_n\| \leqslant \tfrac{1}{2}\big[\lim_{n\to\infty} \|TSx_n - Tt\| + \lim_{n\to\infty} \|Tt - TTx_n\|\big]$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

The following Propositions 2.1–2.3 show that Definition 2.1 and 2.2 are equivalent under some conditions [11]:

Proposition 2.1. *Let S and T be continuous mappings of a normed space X into itself. If S and T are compatible, then they are compatible of type* (A).

Proposition 2.2. *Let S and T be compatible mappings of type* (A) *from a normed space X into itself. If one of S and T is continuous, then S and T are compatible.*

From Propositions 2.1 and 2.2 we have:

Proposition 2.3. *Let S and T be continuous mappings from a normed space X into itself. Then S and T are compatible if and only if they are compatible of type* (A).

By suitable examples, P. P. Murthy, Y. J. Cho and B. Fisher [11] have shown that Proposition 2.3 is not true if S and T are not continuous.

The following propositions show that Definitions 2.1, 2.2 and 2.3 are equivalent under some conditions.

Proposition 2.4. *Every pair of compatible mappings of type* (A) *is compatible of type* (B).

Proof. Suppose that S and T are compatible mappings of type (A), then we have

$$0=\lim_{n\to\infty}\|STx_n-TTx_n\|\leqslant\tfrac{1}{2}\Big[\lim_{n\to\infty}\|STx_n-St\|+\lim_{n\to\infty}\|St-SSx_n\|\Big]$$

and

$$0=\lim_{n\to\infty}\|TSx_n-SSx_n\|\leqslant\tfrac{1}{2}\Big[\lim_{n\to\infty}\|TSx_n-Tt\|+\lim_{n\to\infty}\|Tt-TTx_n\|\Big]$$

as derived. □

Proposition 2.5. *Let S and T be continuous mappings of a normed space X into itself. If S and T are compatible of type* (B), *then they are compatible of type* (A)

Proof. Let $\{x_n\}$ be a sequence in X such that $\lim\limits_{n\to\infty}Sx_n=\lim\limits_{n\to\infty}Tx_n=t$ for some $t\in X$. Since S and T are continuous, we have

$$\begin{aligned}\lim_{n\to\infty}\|STx_n-TTx_n\|&\leqslant\tfrac{1}{2}\Big[\lim_{n\to\infty}\|STx_n-St\|+\lim_{n\to\infty}\|St-SSx_n\|\Big]\\&=\|St-St\|=0\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\|TSx_n-SSx_n\|&\leqslant\Big[\tfrac{1}{2}\lim_{n\to\infty}\|TSx_n-Tt\|+\lim_{n\to\infty}\|Tt-TTx_n\|\Big]\\&=\|Tt-Tt\|=0.\end{aligned}$$

Therefore, S and T are compatible mappings of type (A). This completes the proof. □

Proposition 2.6. *Let S and T be continuous mappings of a normed space X into itself. If S and T are compatible of type* (B), *then they are compatible.*

Proof. Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Since S and T are continuous, we have

$$\lim_{n\to\infty} SSx_n = St = \lim_{n\to\infty} STx_n$$

and

$$\lim_{n\to\infty} TSx_n = Tt = \lim_{n\to\infty} TTx_n.$$

By triangle inequality, we have

$$\|STx_n - TSx_n\| \leqslant \|STx_n - TTx_n\| + \|TTx_n - TSx_n\|.$$

Letting $n \to \infty$ and taking into account that S and T are compatible of type (B), we have

$$\begin{aligned}\lim_{n\to\infty} \|STx_n - TSx_n\| &\leqslant \lim_{n\to\infty} \|STx_n - TTx_n\| + \lim_{n\to\infty} \|TTx_n - TSx_n\| \\ &\leqslant \tfrac{1}{2}\big[\lim_{n\to\infty} \|STx_n - St\| + \lim_{n\to\infty} \|St - SSx_n\| \\ &\quad + \lim_{n\to\infty} \|TTx_n - TSx_n\|\big] \\ &\leqslant 0.\end{aligned}$$

Therefore, S and T are compatible. This completes the proof. □

Proposition 2.7. *Let S and T be continuous mappings from a normed space X into itself. If S and T are compatible, then they are compatible of type* (B).

By unifying Proposition 2.4–2.7, we have

Proposition 2.8. *Let S and T be continuous mappings from a normed space X into itself. Then*

(1) *S and T are compatible if and only if they are compatible of type* (B);

(2) *S and T are compatible of type* (A) *if and only if they are compatible of type* (B).

The following examples show that Proposition 2.8 is not true if S and T are not continuous.

Example 2.1. Let $X = R$, the set of all real numbers, with the Euclidean norm $\|\cdot\|$. Define S and $T\colon X \to X$ as follows:

$$S(x) = \begin{cases} \frac{1}{x^4} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases} \quad \text{and} \quad T(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0, \\ 2 & \text{if } x = 0. \end{cases}$$

Then S and T are not continuous at $t = 0$. Consider a sequence $\{x_n\}$ in X defined by $x_n = n$, $n = 1, 2, \ldots$. Then for $n \to \infty$ we have

$$Sx_n = \frac{1}{n^4} \to t = 0, \quad Tx_n = \frac{1}{x^2} \to t = 0$$

and

$$\lim_{n\to\infty} \|STx_n - TSx_n\| = \lim_{n\to\infty} \|n^8 - n^8\| = 0.$$

However, the following limits do not exist:

$$\begin{aligned} &\lim_{n\to\infty} \|STx_n - TTx_n\| = \lim_{n\to\infty} \|n^8 - n^4\| = \infty, \\ &\frac{1}{2}\Big[\lim_{n\to\infty} \|STx_n - S0\| + \lim_{n\to\infty} \|S0 - SSx_n\|\Big] \\ &= \tfrac{1}{2}\Big[\lim_{n\to\infty} \|n^8 - 1\| + \lim_{n\to\infty} \|1 - n^{16}\|\Big] = \infty \end{aligned}$$

and

$$\begin{aligned} &\lim_{n\to\infty} \|TSx_n - SSx_n\| = \lim_{n\to\infty} \|n^8 - n^{16}\| = \infty, \\ &\frac{1}{2}\Big[\lim_{n\to\infty} \|TSx_n - T0\| + \lim_{n\to\infty} \|T0 - TTx_n\|\Big] \\ &\qquad = \tfrac{1}{2}\Big[\lim_{n\to\infty} \|n^8 - 2\| + \lim_{n\to\infty} \|2 - n^4\|\Big] = \infty. \end{aligned}$$

Therefore S and T are compatible but they are neither compatible of type (A) nor compatible of type (B).

Example 2.2. Let $X = [0, 6]$ with the Euclidean norm $\|\cdot\|$. Define S and T: $X \to X$ by

$$S(x) = \begin{cases} x & \text{if } x \in [0, 3), \\ 6 & \text{if } x \in [3, 6], \end{cases} \quad \text{and} \quad T(x) = \begin{cases} 6 - x & \text{if } x \in [0, 3), \\ 6 & \text{if } x \in [3, 6]. \end{cases}$$

Then S and T are not continuous at $t = 3$. Now, we assert that S and T are not compatible but they are compatible of type (A) and hence compatible of type (B). To see this, suppose that $\{x_n\} \subseteq [0, 6]$ and that Sx_n, $T_n \to t$. By definition of S and T, $t \in [3, 6]$. Since S and T agree on $[3, 6]$, we have only to consider $t = 3$. So we can suppose that $x_n \to 3$ and that $x_n < 3$ for all n. Then $Tx_n = 6 - x_n \to 3$ from the right and $Sx_n = x_n \to 3$ from the left. Thus, since $x_n < 3$ and $6 - x_n > 3$, for all n,

$$\|STx_n - TSx_n\| = \|6 - (6 - x_n)\| \to 3.$$

Further, we have

$$\lim_{n\to\infty} \|STx_n - TTx_n\| = \|6 - 6\| \to 0,$$

$$\tfrac{1}{2}\big[\lim_{n\to\infty} \|STx_n - S3\| + \lim_{n\to\infty} \|S3 - SSx_n\|\big] = \tfrac{1}{2}\big[\|6-6\| + \|6 - x_n\| \to \frac{3}{2}$$

and

$$\lim_{n\to\infty} \|TSx_n - SSx_n\| = \|(6 - x_n) - x_n\| \to 0,$$

$$\tfrac{1}{2}\big[\lim_{n\to\infty} \|TSx_n - T3\| + \lim_{n\to\infty} \|T3 - TTx_n\|\big] = \tfrac{1}{2}\big[\|(6 - x_n) - 6\| + \|6 - 6\|\big] \to \frac{3}{2}$$

as $x_n \to 3$. Therefore, S and T are both compatible mappings of type (A) and compatible mappings of type (B) but they are not compatible.

Example 2.3. Let $X = [0, \infty)$ with the Euclidean norm. Define S and T: $X \to X$ by

$$S(x) = \begin{cases} 1 + x & \text{if } x \in [0,1), \\ 1 & \text{if } x \in [1,\infty), \end{cases} \quad \text{and} \quad T(x) = \begin{cases} 1 - x & \text{if } x \in [0,1), \\ 2 & \text{if } x \in [1,\infty). \end{cases}$$

Then S and T are not continuous at $t = 1$. Now, we assert that S and T are neither compatible of type (A) nor compatible of type (B), but they are compatible. To verify this, we consider that $\{x_n\} \subseteq [0, \infty)$ converges to zero, as we know from the definition of S and T, and that $Sx_n, Tx_n \to t = 1$. Then $Sx_n = 1 + x_n \to 1$ from the right and $Tx_n = 1 - x_n \to 1$ from the left. Thus, since $1 + x_n > 1$ and $1 - x_n < 1$ for all n,

$$\|STx_n - TSx_n\| = \|(2 - x_n) - 2\| \to 0.$$

Further, we have

$$\lim_{n\to\infty} \|STx_n - TTx_n\| = \|(2 - x_n) - x_n\| \to 2 \neq 0,$$

$$\tfrac{1}{2}\big[\lim_{n\to\infty} \|STx_n - S1\| + \lim_{n\to\infty} \|S1 - SSx_n\| = \tfrac{1}{2}\big[\|(2 - x_n) - 1\| + \|1 - 1\|\big] \to \frac{1}{2}$$

and

$$\lim_{n\to\infty} \|TSx_n - SSx_n\| = \|2 - 1\| = 1 \neq 0,$$

$$\tfrac{1}{2}\big[\lim_{n\to\infty} \|TSx_n - T1\| + \lim_{n\to\infty} \|T1 - TTx_n\| = \tfrac{1}{2}\big[\|2 - 2\| + \|2 - x_n\|\big] \to 1$$

as $x_n \to 0$. Therefore S and T are compatible but they are neither compatible mappings of type (A) nor compatible of type (B).

Example 2.4. Let $X = [0,2]$ with the Euclidean norm $\|.\|$. Define S and T: $X \to X$ by

$$S(x) = \begin{cases} \frac{1}{2} + x & \text{if } x \in [0, \frac{1}{2}), \\ 2 & \text{if } x = \frac{1}{2}, \\ 1 & \text{if } x \in (\frac{1}{2}, 2], \end{cases} \quad \text{and} \quad T(x) = \begin{cases} \frac{1}{2} - x & \text{if } x \in [0, \frac{1}{2}), \\ 1 & \text{if } x = \frac{1}{2}, \\ 0 & \text{if } x \in (\frac{1}{2}, 2]. \end{cases}$$

Then S and T are not continuous at $t = \frac{1}{2}$. Now we assert that S and T are compatible of type (B) but they are neither compatible nor compatible of type (A). For, suppose that $\{x_n\} \subseteq [0,2]$ and that $Sx_n, Tx_n \to t = \frac{1}{2}$. By definition of S and T, $t \in \{\frac{1}{2}\}$. So we can suppose $x_n \to 0$. Then $Sx_n = \frac{1}{2} + x_n \to \frac{1}{2}$ from the right and $Tx_n = \frac{1}{2} - x_n \to \frac{1}{2}$ from the left. Also,

$$\|STx_n - TSx_n\| = \|(1 - x_n) - 0\| \to 1 \neq 0.$$

Further, we have

$$\begin{aligned} \lim_{n\to\infty} \|STx_n - TTx_n\| &= \lim_{n\to\infty} \|(1 - x_n) - x_n\| = 1, \\ \tfrac{1}{2}\big[\lim_{n\to\infty} \|STx_n - S\tfrac{1}{2}\| &+ \lim_{n\to\infty} \|S\tfrac{1}{2} - SSx_n\|\big] \\ &= \tfrac{1}{2}\big[\lim_{n\to\infty} \|(1 - x_n) - 2\| + \lim_{n\to\infty} \|2 - 1\|\big] = 1 \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty} \|TSx_n - SSx_n\| &= \lim_{n\to\infty} \|0 - 1\| = 1, \\ \tfrac{1}{2}\big[\lim_{n\to\infty} \|TSx_n - T\tfrac{1}{2}\| &+ \lim_{n\to\infty} \|T\tfrac{1}{2} - TTx_n\|\big] \\ &= \tfrac{1}{2}\big[\lim_{n\to\infty} \|0 - 1\| + \lim_{n\to\infty} \|1 - x_n\|\big] = 1. \end{aligned}$$

Therefore, S and T are neither compatible nor compatible of type (A) but they are compatible of type (B).

We need the following properties of * compatible mappings of type (B) for our main theorems:

Proposition 2.9. *Let S and T be compatible mappings of type* (B) *from a normed space X into itself. If $St = Tt$ for some $t \in X$, then $STt = SSt = TTt = TSt$.*

Proof. Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = t$, $n = 1, 2, \ldots$ for some $t \in X$ and $St = Tt = z$, say. Then we have $Sx_n, Tx_n \to St$ as $n \to \infty$.

Since S and T are compatible of type (B), we have

$$\begin{aligned}\|STt - TTt\| &= \lim_{n\to\infty} \|STx_n - TTx_n\| \\ &\leqslant \tfrac{1}{2}\big[\lim_{n\to\infty} \|STx_n - SSt\| + \lim_{n\to\infty} \|SSt - SSx_n\|\big] \\ &= \|Sz - Sz\| = 0.\end{aligned}$$

Hence we have $STt = TTt$. Therefore, we have $STt = SSt = TTt = TSt$ since $St = Tt$. This completes the proof. □

From Proposition 2.6 and Proposition 2.2 of G. Jungck [5] we immediately have

Proposition 2.10. *Let S and T be compatible mappings of type* (B) *from a normed space X into itself. Suppose that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Then*

(1) $\lim_{n\to\infty} TTx_n = St$ *if S is continuous at t.*

(2) $\lim_{n\to\infty} SSx_n = Tt$ *if T is continuous at t.*

(3) $STt = TSt$ *and* $St = Tt$ *if S and T are continuous at t.*

Proof. (1) Suppose that S is continuous at t. Since $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$, we have $SSx_n, STx_n \to St$ as $n \to \infty$. Since S and T are compatible of type (B), we have

$$\begin{aligned}\lim_{n\to\infty} \|St - TTx_n\| &= \lim_{n\to\infty} \|STx_n - TTx_n\| \\ &\leqslant \tfrac{1}{2}\big[\lim_{n\to\infty} \|STx_n - St\| + \lim_{n\to\infty} \|St - SSx_n\|\big] \\ &= \|St - St\| = 0.\end{aligned}$$

Therefore, $\lim_{n\to\infty} TTx_n = St$. This completes the proof.

(2) The proof of $\lim_{n\to\infty} SSx_n = Tt$ follows by similar arguments as in (1).

(3) Suppose that S and T are continuous at t. Since $Tx_n \to t$ as $n \to \infty$ and S is continuous at t, by Proposition 2.10 (1), $TTx_n \to St$ as $n \to \infty$. On the other hand, T is also continuous at t, $TTx_n \to Tt$. Thus, we have $St = Tt$ by the uniqueness of the limit and so by Proposition 2.9, $STt = TSt$. This completes the proof. □

III. A common fixed point theorem of Greguš type

Let R^+ be the set of non-negative real numbers and F the family of mappings φ from R^+ into R^+ such that each φ is upper semicontinuous, nondecreasing in each coordinate variable, and $\varphi(t) < t$ for any $t > 0$.

Let A, B, S and T be mappings from a normed space X into itself such that

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X), \tag{3.1}$$

$$\begin{aligned} \|Ax - By\|^p \leqslant \varphi\big(a\|Sx - Ty\|^p + (1-a)\max\{&\|Sx - Ax\|^p, \|Ty - By\|^p, \\ &\|Sx - Ax\|^{\frac{p}{2}}\|Ty - By\|^{\frac{p}{2}}, \|Ty - Ax\|^{\frac{p}{2}}\|Sx - By\|^{\frac{p}{2}}, \\ &\tfrac{1}{2}[\|Ty - Ax\|^p + \|Sx - By\|^p]\}\big) \end{aligned} \tag{3.2}$$

for all x, y in X, where $0 < a < 1$, $p \geqslant 1$ and $\varphi \in F$.

Then, by (3.1), since $A(X) \subset T(X)$, for an arbitrary point $x_0 \in X$ there exists a point $x_1 \in X$ such that $Ax_0 = Tx_1$. Since $B(X) \subset S(X)$, for this point x_1 we can choose a point $x_2 \in X$ such that $Bx_1 = Sx_2$, and so on. Inductively, we can define a sequence $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n} \quad \text{and} \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \tag{3.3}$$

for every $n = 0, 1, 2, \ldots$.

For our main theorems, we need the following lemmas:

Lemma 3.1. ([13]). *For any $t > 0$, $\varphi(t) < t$ if and only if $\lim\limits_{n\to\infty} \varphi^n(t) = 0$ where φ^n denotes the n-times repeated composition of φ with itself.*

Lemma 3.2. *Let A, B, S and T be mappings from a normed space X into itself satisfying the conditions* (3.1) *and* (3.2). *Then $\lim\limits_{n\to\infty} \|y_n - y_{n+1}\| = 0$, where $\{y_n\}$ is the sequence in X defined by* (3.3).

Proof. By (3.2) and (3.3) we have

$$\begin{aligned} \|y_{2n} - y_{2n+1}\|^p &= \|Ax_{2n} - Bx_{2n+1}\|^p \\ &\leqslant \varphi\big(a\|y_{2n-1} - y_{2n}\|^p + (1-a)\max\{\|y_{2n-1} - y_{2n}\|^p, \\ &\quad \|y_{2n} - y_{2n+1}\|^p, \|y_{2n-1} - y_{2n}\|^{\frac{p}{2}}\|y_{2n} - y_{2n+1}\|^{\frac{p}{2}}, \\ &\quad \|y_{2n} - y_{2n}\|^{\frac{p}{2}}\|y_{2n-1} - y_{2n+1}\|^{\frac{p}{2}} \\ &\quad + \tfrac{1}{2}[\|y_{2n} - y_{2n}\|^p + \|y_{2n-1} - y_{2n+1}\|^p]\}\big). \end{aligned}$$

If $\|y_{2n} - y_{2n+1}\| > \|y_{2n-1} - y_{2n}\|$ in the above inequality, then

$$\begin{aligned}\|y_{2n} - y_{2n+1}\|^p &\leqslant \varphi\big(a\|y_{2n} - y_{2n+1}\|^p + (1-a)\max\{\|y_{2n} - y_{2n+1}\|^p, \\ &\qquad \|y_{2n} - y_{2n+1}\|^p, \|y_{2n} - y_{2n+1}\|^p, 0, \tfrac{1}{2}\big[\|y_{2n} - y_{2n+1}\|^p \\ &\qquad + \|y_{2n} - y_{2n+1}\|^p\big]\}\big) \\ &< \|y_{2n} - y_{2n+1}\|^p,\end{aligned}$$

which is a contradiction. Thus

$$\|y_{2n} - y_{2n+1}\|^p \leqslant \varphi(\|y_{2n-1} - y_{2n}\|^p).$$

Similarly, we have

$$\|y_{2n+1} - y_{2n+2}\|^p \leqslant \varphi(\|y_{2n} - y_{2n+1}\|^p).$$

It follows that

$$\|y_n - y_{n+1}\|^p \leqslant \varphi(\|y_{n-1} - y_n\|^p) \leqslant \ldots \leqslant \varphi^n(\|y_0 - y_1\|^p).$$

It follows from Lemma 3.1 that

$$\lim_{n\to\infty} \|y_n - y_{n+1}\| = 0.$$

This completes the proof. □

Lemma 3.3. *Let A, B, S and T be mappings from a normed space X into itself satisfying the conditions* (3.1) *and* (3.2). *Then the sequence $\{y_n\}$ defined by* (3.3) *is a Cauchy sequence in X.*

Proof. By virtue of Lemma 3.2 it is sufficient to show that a subsequence $\{y_{2n}\}$ of $\{y_n\}$ is not a Cauchy sequence in X. Then there is an $\varepsilon > 0$ such that for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) \geqslant 2k$ such that

$$(3.4) \qquad \|y_{2m(k)} - y_{2n(k)}\| > \varepsilon.$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (3.4), that is,

$$(3.5) \qquad \|y_{2n(k)} - y_{2m(k)-2}\| \leqslant \varepsilon \quad \text{and} \quad \|y_{2n(k)} - y_{2m(k)}\| > \varepsilon.$$

Then for each even integer $2k$ we have

$$\begin{aligned}\varepsilon &< \|y_{2n(k)} - y_{2m(k)}\| \\ &\leqslant \|y_{2n(k)} - y_{2m(k)-2}\| + \|y_{2m(k)-2} - y_{2n(k)-1}\| + \|y_{2m(k)-1} - y_{2m(k)}\|.\end{aligned}$$

It follows from Lemma 3.2 and (3.5) that

$$\|y_{2n(k)} - y_{2m(k)}\| \to \varepsilon \quad \text{as } k \to \infty. \tag{3.6}$$

By the triangle inequality, we have

$$\big|\|y_{2n(k)} - y_{2m(k)-1}\| - \|y_{2n(k)} - y_{2m(k)}\|\big| \leqslant \|y_{2m(k)-1} - y_{2m(k)}\|$$

and

$$\begin{aligned}&\big|\|y_{2n(k)+1} - y_{2m(k)-1}\| - \|y_{2n(k)} - y_{2m(k)}\|\big| \\ &\qquad\leqslant \|y_{2m(k)-1} - y_{2m(k)}\| + \|y_{2n(k)} - y_{2n(k)+1}\|.\end{aligned}$$

From Lemma 3.2 and (3.6), as $k \to \infty$,

$$\|y_{2n(k)} - y_{2m(k)-1}\| \to \varepsilon \quad \text{and} \quad \|y_{2n(k)+1} - y_{2m(k)-1}\| \to \varepsilon. \tag{3.7}$$

Therefore, by (3.2) and (3.3), we have

$$\begin{aligned}\|y_{2n(k)} - y_{2m(k)}\| &\leqslant \|y_{2n(k)} - y_{2n(k)+1}\| + \|Ax_{2m(k)} - B_{2m(k)}\| \\ &\leqslant \|y_{2n(k)} - y_{2n(k)+1}\| + \big[\varphi(a\|y_{2m(k)-1} - y_{2n(k)}\|^p \\ &\quad + (1-a)\max\{\|y_{2m(k)-1} - y_{2m(k)}\|^p, \|y_{2n(k)} - y_{2n(k)+1}\|^p, \\ &\quad \|y_{2m(k)-1} - y_{2m(k)}\|^{\frac{p}{2}}\|y_{2n(k)} - y_{2n(k)+1}\|^{\frac{p}{2}}, \\ &\quad \|y_{2n(k)} - y_{2m(k)}\|^{\frac{p}{2}}\|y_{2m(k)-1} - y_{2n(k)+1}\|^{\frac{p}{2}}, \\ &\quad \tfrac{1}{2}\|y_{2n(k)} - y_{2m(k)}\|^p + \|y_{2m(k)-1} - y_{2n(k)+1}\|^p\big]\})^{\frac{1}{p}}.\end{aligned}$$

Since $\varphi \in F$, by Lemma 3.2, (3.6) and (3.7) we have

$$\varepsilon \leqslant \big[\varphi(a\varepsilon^p + (1-a)\max\{0, 0, \varepsilon^p\})\big]^{\frac{1}{p}} < \varepsilon$$

as $k \to \infty$ in (3.8), which is a contradiction. Therefore, $\{y_{2n}\}$ is a Cauchy sequence in X. This completes the proof. □

Now, we are ready to present our main theorem.

Theorem 3.4. *Let A, B, S and T be mappings from a Banach space (X, d) into itself satisfying the conditions (3.1) and (3.2). Suppose that one of A, B, S and T is continuous, and the pairs A, S and B, T are compatible of type (B).*

Then A, B, S and T have a unique common fixed point z in X.

Proof. Let $\{y_n\}$ be the sequence in X defined by (3.3). By Lemma 3.3, $\{y_n\}$ is a Cauchy sequence in X and hence it converges to some point z in X. Consequently, subsequences $\{Ax_{2n}\}$, $\{Bx_{2n+1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n+1}\}$ of $\{y_n\}$ also converge to z.

Now, suppose that A is continuous. Since A and S are compatible of type (B), it follows from Proposition 2.10 that

$$ASx_n \text{ and } SSx_{2n} \to Az \quad \text{as } n \to \infty.$$

By (3.2) we have

$$\begin{aligned}\|ASx_{2n} - Bx_{2n+1}\|^p \leqslant \varphi(a\|&SSx_{2n} - Tx_{2n+1}\|^p \\ &+ (1-a)\max\{\|SSx_{2n} - ASx_{2n}\|^p, \|Tx_{2n+1} - Bx_{2n+1}\|^p, \\ &\|SSx_{2n} - ASx_{2n}\|^{\frac{p}{2}}\|Tx_{2n+1} - Bx_{2n+1}\|^{\frac{p}{2}}, \\ &\|Tx_{2n+1} - ASx_{2n}\|^{\frac{p}{2}}\|SSx_{2n} - Bx_{2n+1}\|^{\frac{p}{2}}, \\ &\tfrac{1}{2}[\|Tx_{2n+1} - ASx_{2n}\|^p + \|SSx_{2n} - Bx_{2n+1}\|^p]\}).\end{aligned}$$

By letting $n \to \infty$, we have

$$\begin{aligned}\|Az - z\|^p &\leqslant \varphi(a\|Az - z\|^p \\ &\quad + (1-a)\max\{0, 0, 0, \|z - Az\|^p, \tfrac{1}{2}[\|z - Az\|^p + \|Az - z\|^p]\} \\ &< \|Az - z\|^p,\end{aligned}$$

which is a contradiction. Thus we have $Az = z$. Since $A(X) \subset T(X)$, there exists a point $u \in X$ such that $z = Az = Tu$. Again by (3.2), we have

$$\begin{aligned}\|ASx_{2n} - Bu\|^p \leqslant \varphi(a\|&SSx_{2n} - Tu\|^p \\ &+ (1-a)\max\{\|SSx_{2n} - ASx_{2n}\|^p, \|Tu - Bu\|^p, \\ &\|SSx_{2n} - ASx_{2n}\|^{\frac{p}{2}} \cdot \|Tu - Bu\|^{\frac{p}{2}}, \\ &\|Tu - ASx_{2n}\|^{\frac{p}{2}} \cdot \|SSx_{2n} - Bu\|^{\frac{p}{2}} \\ &\tfrac{1}{2}[\|Tu - ASx_{2n}\|^p + \|SSx_{2n} - Bu\|^p]\}).\end{aligned}$$

By letting $n \to \infty$, $\varphi \in F$ we obtain

$$\begin{aligned}\|z - Bu\|^p &\leqslant (\varphi(1-a)\max\{0, \|z - Bu\|^p, 0, 0, \tfrac{1}{2}\|z - Bu\|^p\}) \\ &< \|z - Bu\|^p,\end{aligned}$$

which implies that $z = Bu$. Since B and T are compatible of type (B) and $Tu = Bu = z$, by Proposition 2.9, $TBu = BTu$ and hence $Tz = TBu = BTu = Bz$.

Moreover, by (3.2) we have

$$\begin{aligned}\|Ax_{2n} - Bz\|^p \leqslant \varphi(a\|Sx_{2n} - Tz\|^p + (1-a)\max\{&\|Sx_{2n} - Ax_{2n}\|^p, \\ &\|Tz - Bz\|^p, \|Sx_{2n} - Ax_{2n}\|^{\frac{p}{2}} \cdot \|Tz - Bz\|^{\frac{p}{2}}, \\ &\|Tz - Ax_{2n}\|^{\frac{p}{2}} \cdot \|Sx_{2n} - Bz\|^{\frac{p}{2}}, \\ &\tfrac{1}{2}\big[\|Tz - Ax_{2n}\|^p + \|Sx_{2n} - Bz\|^p\big]\}).\end{aligned}$$

By letting $n \to \infty$, we obtain

$$\begin{aligned}\|z - Bz\|^p &\leqslant \varphi\big(a\|z - Bz\|^p + (1-a)\max\{0, 0, 0, \|z - Bz\|^p, \|z - Bz\|^p\}\big) \\ &< \|z - Bz\|^p,\end{aligned}$$

which means that $z = Bz$. Since $B(X) \subset S(X)$, there exists a point $v \in X$ such that $z = Bz = Sv$. By using (3.2) we have

$$\begin{aligned}\|Av - z\|^p &= \|Av - Bz\|^p \\ &\leqslant \varphi(a\|Sv - Tz\|^p + (1-a)\max\{\|Sv - Av\|^p, \|Tz - Bz\|^p, \\ &\quad \|Sv - Av\|^{\frac{p}{2}} \cdot \|Tz - Bz\|^{\frac{p}{2}}, \|Tz - Av\|^{\frac{p}{2}}\|Sv - Bz\|^{\frac{p}{2}}, \\ &\quad \tfrac{1}{2}\big[\|Tz - Av\|^p + \|Sv - Bz\|^p\big]\}) \\ &= \varphi\big((1-a)\max\{\|z - Av\|^p, 0, 0, 0, \tfrac{1}{2}\|z - Av\|^p\}\big) \\ &< \|z - Av\|^p,\end{aligned}$$

so that $Av = z$. Since A and S are compatible of type (B) and $Av = Sv = z$, $SAv = ASv$ and hence $Sz = SAv = ASv = Az$. Therefore, z is a common fixed point of A, B, S and T. Similarly, we can also complete the proof when B, S and T are continuous.

It follows easily from (3.2) that z is a unique common fixed point of A, B, S and T. This completes the proof. □

Remark. Theorem 3.4 generalizes the result of P. P. Murthy, Y. J. Cho and B. Fisher [11] with the generalized Greguš type [3] mappings.

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