

COMPATIBLE TIGHT RIESZ ORDERS

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(Received 22 September 1971)

Communicated by J. B. Miller

1. Introduction

Loy and Miller [4] have studied tight Riesz groups (G, \leq) , without pseudo-zeros, and with (G, \leq) an l -group. In this paper we study abelian l -groups (G, \leq) on which a tight Riesz order can be defined, such that \leq is precisely the associated order. Such an order we call a CTRO (compatible tight Riesz order).

We give certain sufficient conditions for an l -group to have a CTRO, a useful necessary and sufficient condition, and some negative results concerning necessary conditions. The class of CTRO of a given l -group, ordered by set inclusion of positive cones, is directed downwards, has a maximal element, but usually is not directed upwards and has no smallest or greatest elements.

2. Preliminaries

DEFINITION [5]. *A tight Riesz group (G, \leq) is a partially ordered abelian group which is directed and satisfies the following interpolation property:*

if $a_1, a_2, b_1, b_2 \in G$ are such that

$$a_i < b_j \text{ for } i, j = 1, 2$$

then there exists $c \in G$ such that

$$a_i < c < b_j \text{ for } i, j = 1, 2.$$

We write TRG to mean tight Riesz group.

Let (G, \leq) be a partially ordered abelian group. We say $x \in G$ is *pseudopositive* if $x \not\geq 0$ and $x + p > 0$ for all $p > 0$. We say x is a *pseudozero* if x and $-x$ are pseudopositive. If (G, \leq) has no pseudozeros we write $x \succ 0$ to mean $x > 0$ or x is pseudopositive. Then (G, \leq) is a partially ordered group and we say \leq is the *associated order*. We note that if $a \leq b < c$ then $a < c$. By taking intervals $(a, b) = \{x: a < x < b\}$, where $a, b \in G$ $a < b$, as a subbase we define the *open-interval topology* \mathcal{U} on G .

THEOREM 1 [4]. *Let (G, \leq) be a TRG. Then (G, \leq, \mathcal{U}) is a topological group, which is Hausdorff if and only if (G, \leq) has no pseudozeros. The family $\{(-a, a) : a > 0\}$ form a base for \mathcal{U} at 0. (G, \leq) is an antilattice (i.e. $a \wedge b$ exists if and only if a and b are comparable), and contains small elements (i.e. if $a > 0$ and $n > 0$ then there exists $b > 0$ such that $a > nb > 0$). If (G, \leq) is an l -group then \leq is isolated, and $a, b > 0$ implies $a \wedge b > 0$.*

We now present some elementary results about archimedean classes, following Loonstra’s work in [3]. We assume that (G, \leq) is an abelian l -group.

We define an equivalence relation \sim on $G^+ = \{x : x \geq 0\}$. If $a, b \geq 0$ we write $a \sim b$ if there exist positive integers m, n such that

$$a \leq mb \text{ and } b \leq na.$$

If $a \geq 0$ we write a to mean the set $\{x : x \sim a\}$. The family of these *archimedean classes* will be denoted by \mathcal{A} . We write $a^0 \leq b^0$, if there exists some positive n such that $a \leq nb$. We write $a^0 \ll b^0$ if $na \leq b$ for each positive n .

LEMMA 1 [3]. *(\mathcal{A}, \leq) is a distributive lattice with smallest element 0^0 , $a^0 \vee b^0 = (a \vee b)^0$ and $a^0 \wedge b^0 = (a \wedge b)^0$. (\mathcal{A}, \ll) is a poset.*

We say a^0 is *divisible* if $b/n \in G$ for each $b \in a^0$ and each positive n . We say a^0 is *dense* if $a^0 = a^0 + a^0$. An l -group (G, \leq) is said to be *archimedean* if $nx \leq y$ for each integer n implies that $x = 0$. We denote $\mathcal{A} \setminus \{0^0\}$ by \mathcal{A}^* .

3. Compatible tight Riesz orders

From now on we assume that (G, \leq) is a non-trivial abelian l -group.

DEFINITION. *A compatible tight Riesz order (abbreviated CTRO) on (G, \leq) is a non-trivial partial order \leq making (G, \leq) a TRG without pseudozeros, and having \ll as its associated order.*

LEMMA 2. *Let \leq be a CTRO. Then*

$$t + x \geq 0 \text{ for all } t > 0 \text{ implies that } x \geq 0.$$

PROOF. Let $t > 0$, then by Theorem 1 there exists $u > 0$ such that $t > 2u > 0$. So $t + x > 2u + x = u + (u + x) \geq u > 0$, hence $t + x > 0$ for all $t > 0$, and thus by the definition of the associated order $x \geq 0$.

THEOREM 2. *There is a one-one correspondence between CTROs on (G, \leq) and sets T with the properties:*

- (i) T is a proper dual ideal of (G^+, \ll)
- (ii) $T = T + T$
- (iii) $\wedge T = 0$.

In fact the set of strictly positive elements of a CTRO satisfies conditions (i)–(iii) and vice versa.

PROOF. Suppose that T is the strictly positive cone of a CTRO \leq , then by Theorem 1 $a, b \in T$ implies that $a \wedge b \in T$. If $a \leq b$ and $a \in T$ then $b \in T$, also $0 \notin T$ and by definition T is not empty, hence (i) follows. If $a \in T$ then there exists $b \in T$ such that $a > b > 0$, by Theorem 1. Since $a = (a - b) + b$ (ii) is satisfied. If $x \leq t$ for all $t \in T$ then by Lemma 2 $-x \geq 0$, and so (iii) is satisfied.

Now suppose that conditions (i)–(iii) are satisfied, and write $x > 0$ to mean $x \in T$. By (i) (G, \leq) is a partially ordered group. If $x, y \in G$, choose some $a > 0$, then $x, y \leq |x| + |y| < |x| + |y| + a$, so (G, \leq) is directed. If $a_1, a_2 < b_1, b_2$ then

$$b_1 \wedge b_2 - a_1 \vee a_2 = (b_1 - a_1) \wedge (b_1 - a_2) \wedge (b_2 - a_1) \wedge (b_2 - a_2)$$

belongs to T , by (i). So by (ii) there exist $c, d > 0$ such that $b_1 \wedge b_2 - a_1 \vee a_2 = c + d$. Hence

$$a_1, a_2 \leq a_1 \vee a_2 < a_1 \vee a_2 + c < b_1 \wedge b_2 \leq b_1, b_2$$

and so (G, \leq) is a TRG. If $x + t > 0$ for all $t \in T$, then by (i) and (iii) $x \geq 0$. Hence (G, \leq) has no pseudozeros and \leq is its associated order.

Fuchs [1] has studied, in another context, the case where (G, \leq) is divisible.

It will prove useful to restate the above theorem in terms of archimedean classes.

THEOREM 2'. *There is a one-one correspondence between CTROs on (G, \leq) and sets \mathcal{F} with the properties:*

- (i) \mathcal{F} is a proper dual ideal of (\mathcal{A}, \leq)
- (ii) if $a^0 \in \mathcal{F}$ then there exist $b^0, c^0 \in \mathcal{F}$ such that $a = b + c$
- (iii) if $x^0 \in \mathcal{F}$ (i.e. if $x^0 \leq y^0$ for all $y^0 \in \mathcal{F}$) then $x = 0$.

In fact the set of archimedean classes of the strictly positive elements of a CTRO satisfies conditions (i)–(iii) and vice versa.

PROOF. Suppose that \leq is a CTRO. If $a^0 = b^0$ and $a > 0$ then for some positive n , $nb \geq a > 0$, so by Theorem 1 $b > 0$. Also if $x \leq t$ for all $t > 0$, then by the existence of small elements $nx \leq t$ for all $t > 0$ and all positive n . The rest of the proof follows from that of Theorem 2.

NOTE. In Theorem 2' (iii) cannot be replaced by ‘‘if $x^0 \in \mathcal{F}$ then $x = 0$ ’’. For example, in R^2 with $(x, y) \geq 0$ if and only if $x, y \geq 0$ let $(x, y) > 0$ mean that $x > 0$ and $y > 0$. Then \leq is a CTRO and $\mathcal{F} = \{(1, 1)^0\}$, but $(0, 0)^0 \neq (0, 1)^0 \leq (1, 1)^0$.

By an abuse of language we shall call \leq, T, \mathcal{F} a CTRO as the need arises.

THEOREM 3. *Each of the following is separately a sufficient condition for (G, \preceq) to possess a CTRO:*

- (i) *G is divisible*
- (ii) *(\mathcal{A}, \preceq) contains a dense atom*
- (iii) *(\mathcal{A}^*, \preceq) contains a maximal descending chain each of whose elements is dense.*

PROOF. Firstly we show that (iii) is sufficient. Let \mathcal{C} be the chain and let $\mathcal{T} = \{x^0 : x^0 \succcurlyeq c^0 \text{ for some } c^0 \in \mathcal{C}\}$. Then \mathcal{T} satisfies conditions (i) and (iii) of Theorem 2'. If $x^0 \in \mathcal{T}$ then for some positive n and some $c^0 \in \mathcal{C}$, $nx \succcurlyeq c$. By Lemma 1 and the denseness of c^0 there exists $c_1 \in c^0$ such that $c \succcurlyeq 2nc_1$. So $x \succcurlyeq 2c_1$, and $x = (x - c_1) + c_1$, hence condition (ii) of Theorem 2' is satisfied.

The sufficiency of (ii) follows immediately from the above. If G is divisible then (iii) is satisfied, by Hausdorff's maximal principle.

COROLLARY 1. *If G is divisible and $a \succ 0$, there exists a CTRO, \preceq , with $a > 0$.*

PROOF. There exists a maximal descending chain in (\mathcal{A}^*, \preceq) through a^0 ; define the CTRO as in the proof above.

COROLLARY 2. *If (G, \preceq) is archimedean and divisible, then every proper dual ideal of (\mathcal{A}, \preceq) is a CTRO. In particular if $a \succ 0$, then $\mathcal{T} = \{x^0 : x^0 \succcurlyeq a^0\}$ is a CTRO.*

We will show in Lemma 3 that there exists an archimedean l -group with a CTRO, which does not have any dense archimedean classes, except for 0^0 .

Let $\{(F_i, \preceq) : i \in I\}$ be a non-empty family of fully ordered non-trivial abelian groups. We denote the full direct product by $\prod F_i$ and the direct sum by $\sum F_i$. If $x \in \prod F_i$ we write its i th component as $x(i)$ and we write $x \succcurlyeq 0$ to mean $x(i) \succcurlyeq 0$ for all i . We call this the *pointwise order*.

THEOREM 4. *The l -group $(\prod F_i, \preceq)$ has a CTRO if and only if either I is infinite or at least one (F_i, \preceq) is dense. The l -group $(\sum F_i, \preceq)$ has a CTRO if and only if at least one (F_i, \preceq) is dense.*

PROOF. Suppose I is infinite, then select a countably infinite subset $J \subseteq I$, and write $J = \{1, 2, 3, \dots\}$. Choose any $k_n \succ 0$, $k_n \in F_n$, and if $x \in \prod F_i$, write $x \succ 0$ to mean that $x \succcurlyeq 0$ and there exist positive integers m_n such that $x(n) \succcurlyeq m_n k_n$ and $\lim_n m_n = \infty$. Then \preceq is a CTRO on $(\prod F_i, \preceq)$, by Theorem 2.

Suppose (F_{i_0}, \preceq) is dense and for $x \in \prod F_i$ write $x \succ 0$ to mean that $x \succcurlyeq 0$ and $x(i_0) \succ 0$. Then \preceq is a CTRO on $(\prod F_i, \preceq)$.

Suppose that I is finite, none of the (F_i, \preceq) are dense, and \preceq is a CTRO on

$(\Pi F_i, \preccurlyeq)$. There exist atoms a_i in (F_i^+, \preccurlyeq) . Suppose that for each i there exists $x_i > 0$ with $x_i(i)^0 \preccurlyeq a_i^0$. Then by the existence of small elements there exist $y_i > 0$ with $y_i(i) = 0$. So $\bigwedge y_i = 0$, but by Theorem 1 $\bigwedge y_i > 0$. So in fact there exists i_0 such that, if $x > 0$ then $x(i_0) \succcurlyeq a_{i_0}$. Now define $q > 0$ by, $q(i) = 0$ if $i \neq i_0$ and $q(i_0) = a_{i_0}$. Then q is a pseudozero, and so \preccurlyeq is not a CTRO.

The result for ΣF_i follows easily from the above.

COROLLARY. *A fully ordered group (G, \preccurlyeq) has either no CTRO, or is dense and has exactly one CTRO, \preccurlyeq itself.*

LEMMA 3. *There exists an archimedean l -group with a CTRO but without a dense archimedean class, save 0^0 . There exists an l -group with the set $\{x^0 : x^0 \succcurlyeq a^0\}$ a CTRO but a^0 not divisible.*

PROOF. Let (G, \preccurlyeq) be the l -group of all sequences of integers with pointwise order. By Theorem 4 (G, \preccurlyeq) has a CTRO. If $a > 0$ and $a(i) > 0$, say, let b be defined by $b(j) = a(j)$ if $j \neq i$, and $b(i) = 1$. Then $b \in a^0$, but clearly if $b = c + d$ then either $c \notin b^0$ or $d \notin b^0$.

Let (G, \preccurlyeq) be the subgroup of R , with the usual order, generated by $\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\}$. By Theorem 4 Corollary (G, \preccurlyeq) has a CTRO but $\frac{1}{2} \cdot \frac{1}{3}$ does not belong to G .

THEOREM 5. *If \mathcal{T} is a CTRO with open-interval topology \mathcal{U} , then the following are equivalent:*

- (i) (G, \mathcal{U}) is first countable.
- (ii) There exists a countable descending chain $a_1^0 \succcurlyeq a_2^0 \succcurlyeq \dots$ in $(\mathcal{T}, \preccurlyeq)$ with $\mathcal{T} = \{x^0 : x^0 \succcurlyeq a_n^0 \text{ for some } n\}$.
- (iii) (G, \mathcal{U}) has a translation-invariant metric, ρ , satisfying:

$$|x| \preccurlyeq |y| \text{ implies } \rho(x, 0) \leq \rho(y, 0).$$

PROOF. (i) implies (ii). If (G, \mathcal{U}) is first countable then by Theorem 1 there exists a sequence $\{p_n\}$ such that, if $x > 0$ then $x > p_n$ for some n , and $p_n > 0$ for all n . Let $a_n = p_1 \wedge \dots \wedge p_n$, then $a_1 \succcurlyeq a_2 \succcurlyeq \dots > 0$, and $\mathcal{T} = \{x^0 : x^0 \succcurlyeq a_n^0 \text{ for some } n\}$.

(ii) implies (i). Suppose $\mathcal{T} = \{x^0 : x^0 \succcurlyeq a_n^0 \text{ for some } n\}$ and $a_1^0 \succcurlyeq a_2^0 \succcurlyeq \dots$, then by Theorem 1, there exist $p_{in} > 0$, such that $a_i > np_{in} > 0$. If $x > 0$, then there exist integers i and n such that $nx \succcurlyeq a_i$, so $x > p_{in}$.

(i) implies (iii). If $|x|^0 \preccurlyeq p_r^0$ define $p_r(x)$ by

$$p_r(x) = \inf \left\{ \frac{m}{n} : -mp_r \preccurlyeq nx \preccurlyeq mp_r, m \geq 0, n > 0 \right\}.$$

Otherwise write $p_r(x) = \infty$. Now let

$$\rho(x, y) = \sum_{n=1}^{\infty} \min \left\{ p_n(x - y), \frac{1}{2^n} \right\},$$

then ρ is the required metric.

Any Hausdorff abelian topological group which is first countable has a translation-invariant metric [2].

NOTE. It can be shown that in a Banach lattice with strong unit the metric, corresponding to the CTRO consisting precisely of the strong units, is equivalent to the original metric.

4. The class of CTROs

Let (G, \leq) be a non-trivial abelian l -group with a CTRO. Denote the family of CTROs by $\{R_x\}$ and partially order it by set inclusion (of the strictly positive cones).

THEOREM 6. *The poset $\{R_x, \subseteq\}$ is directed downwards. If G is divisible then $\{R_x, \subseteq\}$ is directed upwards if and only if (G, \leq) is fully ordered; and $T_1 \wedge T_2 = T_1 \cap T_2$.*

PROOF. Suppose that T_1 and T_2 are CTROs, and let $T_3 = T_1 + T_2 = \{t_3 : t_3 = t_1 + t_2, t_1 \in T_1, t_2 \in T_2\}$. If $x \succcurlyeq t_1 + t_2$, with $t_1 \in T_1, t_2 \in T_2$, then $x - t_1 \succcurlyeq t_2$, so $x - t_1 \in T_2$ and $x \in T_3$. If $t_1, u_1 \in T_1$ and $t_2, u_2 \in T_2$ then $(t_1 + t_2) \wedge (u_1 + u_2) \succcurlyeq t_1 \wedge u_1 + t_2 \wedge u_2$. So T_3 satisfies condition (i) of Theorem 2, and also condition (ii). If $t_1 + t_2 \succcurlyeq x$ for all $t_1 \in T_1, t_2 \in T_2$, then $0 \leq x - t_1$ for all $t_1 \in T_1$, since $\wedge T_2 = 0$. So in fact $x \leq 0$, since $\wedge T_1 = 0$. Hence T_3 satisfies condition (iii) of Theorem 2, and clearly $T_3 \subseteq T_1, T_2$.

Suppose that (G, \leq) is divisible and not fully ordered. Then there exist $a_1, a_2 \succ 0$ such that $a_1 \wedge a_2 = 0$. By Theorem 3 Corollary 1 there exist CTROs T_1, T_2 with $a_1 \in T_1, a_2 \in a_2$. If there exists a CTRO T_0 such that $T_0 \supseteq T_1, T_2$ then $0 = a_1 \wedge a_2 \in T_0$, by Theorem 1. Hence $\{R_x, \subseteq\}$ is not directed upwards.

If (G, \leq) is divisible and fully ordered, then by Theorem 4 Corollary $\{R_x\}$ is a singleton.

Suppose that G is divisible and T_1, T_2 are CTROs. Then $T_1 \cap T_2$ satisfies conditions (i) and (ii) of Theorem 2. Since $T_1 + T_2 \subseteq T_1 \cap T_2$, hence by the above $T_1 \cap T_2$ satisfies condition (iii) of Theorem 2.

NOTE. The fact that $\{R_x, \subseteq\}$ is directed downwards will be used in [6] paper to define a convergence (in terms of the order) on any abelian partially ordered group, which agrees with relative uniform convergence on directed integrally closed partially ordered vector spaces, and with order convergence on fully ordered abelian groups.

LEMMA 4. *Let (G, \leq) be divisible and archimedean, then it has a smallest CTRO if and only if (G, \leq) has a strong unit. In fact the smallest CTRO consists precisely of the strong units.*

PROOF. Suppose that a belongs to the smallest CTRO. Then by Theorem 3 Corollary 2 $a^0 \succcurlyeq x^0$ for all $x \succ 0$. Hence a is a strong unit and the rest follows easily.

The l -group $R \times Z$ (pointwise order) is archimedean and has exactly one CTRO, containing elements other than strong units. The group of real-valued sequences with finitely many non-zero terms can be fully ordered, by letting its strictly positive cone consist of sequences whose last non-zero term is positive. By Theorem 4 Corollary this group has exactly one CTRO, but does not contain any strong units.

LEMMA 5. *The poset $\{R_\alpha, \subseteq\}$ has a maximal element.*

PROOF. The standard Zorn's Lemma argument proves the existence of a maximal element.

Acknowledgments

The author would like to thank Professor J. B. Miller and Dr B. F. Sherman for suggesting improvements to the paper. This research was carried out while the author held a Commonwealth Postgraduate Award.

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