

COMPETITION INDICES OF STRONGLY CONNECTED DIGRAPHS

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ABSTRACT. Cho and Kim [4] and Kim [6] introduced the concept of the competition index of a digraph. Cho and Kim [4] and Akelbek and Kirkland [1] also studied the upper bound of competition indices of primitive digraphs. In this paper, we study the upper bound of competition indices of strongly connected digraphs. We also study the relation between competition index and ordinary index for a symmetric strongly connected digraph.

1. Preliminaries and notations

Let $D = (V, E)$ denote a *digraph* (directed graph) with vertex set $V = V(D)$, arc set $E = E(D)$. Loops are permitted but multiple arcs are not. A $x \rightarrow y$ *walk* in a digraph D is a sequence of vertices $x, v_1, \dots, v_t, y \in V(D)$ and a sequence of arcs $(x, v_1), (v_1, v_2), \dots, (v_t, y) \in E(D)$, where the vertices and arcs are not necessarily distinct. A *closed walk* is a $x \rightarrow y$ walk where $x = y$. A *cycle* is a closed $x \rightarrow y$ walk in which all vertices except x and y are distinct.

The *length of a walk* W is the number of arcs in W . The notation $x \xrightarrow{k} y$ is used to indicate that there is a $x \rightarrow y$ walk of length k . The *distance* from vertex x to vertex y in D , denoted by $d(x, y)$, is the length of a shortest walk from x to y . An l -cycle is a cycle of length l , and is denoted by C_l . If the digraph D has at least one cycle, the length of a shortest cycle in D is called the *girth* of D , and is denoted by $s(D)$.

A digraph D is called *strongly connected* if for each pair of vertices x and y in $V(D)$ there is a walk from x to y . For a strongly connected digraph D , the *index of imprimitivity* of D is the greatest common divisor of the lengths of the cycles in D , and is denoted by $k(D)$. If D is a trivial digraph of order 1, $k(D)$ is undefined. For a strongly connected digraph D , D is *primitive* if $k(D) = 1$. If D is primitive, there exists some positive integer l such that there is a walk

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of length exactly l from each vertex x to each vertex y . The smallest such l is called the *exponent* of D , denoted by $\text{exp}(D)$. Exponents have been studied by many researchers [2, 7, 8, 9, 10].

Cho et al. [5] introduced the m -step competition graph, a generalization of the competition graph. Let D be a digraph (with or without loops) with the vertex set $\{v_1, v_2, \dots, v_n\}$. Given a positive integer m , we say that a vertex v_k of D is an m -step common prey of v_i and v_j if there are two directed walks of length m , one from v_i to v_k and the other from v_j to v_k . Then, the m -step competition graph of D , denoted by $C^m(D)$, has the same vertex set as D , and there is an edge between vertices v_i and v_j ($v_i \neq v_j$) if and only if v_i and v_j have an m -step common prey in D . The m -step digraph of D , denoted by D^m , has the same vertex set as D and an arc (v_i, v_j) if and only if there is a directed walk of length m from v_i to v_j . Then we have $C^m(D) = C(D^m)$ for each positive integer m , see [5].

Consider the sequence $D, D^2, D^3, \dots, D^m, \dots$. Then there exists a smallest positive integer q such that $D^q = D^{q+r}$ for some positive integer r . Such an integer q is called the *index* of D and is denoted by $\text{index}(D)$. There also exists a smallest positive integer p such that $D^q = D^{q+p}$, and such an integer is called the *period* of D and is denoted by $\text{period}(D)$. Now, consider the competition graph sequence $C(D), C(D^2), C(D^3), \dots, C(D^m), \dots$. There exists a smallest positive integer q such that $C(D^{q+i}) = C(D^{q+r+i})$ for some positive integer r and every nonnegative integer i . Such an integer q is called the *competition index* of D and is denoted by $\text{cindex}(D)$. Let $q = \text{cindex}(D)$. Then there exists a smallest positive integer p such that $C(D^{q+i}) = C(D^{q+p+i})$ for every nonnegative integer i . Such an integer p is called the *competition period* of D and is denoted by $\text{cperiod}(D)$. From the definition of index and competition index, we get the following.

Proposition 1.1 (Cho and Kim [4]). *For a digraph D , we have*

$$\text{cindex}(D) \leq \text{index}(D).$$

Note that for a primitive digraph D , $\text{exp}(D) = \text{index}(D)$ and $\text{cindex}(D)$ is the smallest integer q such that $C(D^k)$ is a complete graph for any $k \geq q$. It is well known that the index of imprimitivity of D is equal to $\text{period}(D)$ for a strongly connected digraph D (For details on exponent (index) and period, refer to [3, 5, 8, 9, 10]). However, the competition period of a strongly connected digraph is always one, as shown below;

Proposition 1.2 (Cho and Kim [4]). *For a strongly connected digraph D , we have*

$$\text{cperiod}(D) = 1.$$

Proof. Note that each vertex of D has an outgoing arc. Thus, every edge in $C(D^m)$ is an edge in $C(D^{m+i})$ for every positive integer i , since any two vertices having an m -step common prey also have an $(m+i)$ -step common prey. Therefore, we have $\text{cperiod}(D) = 1$. \square

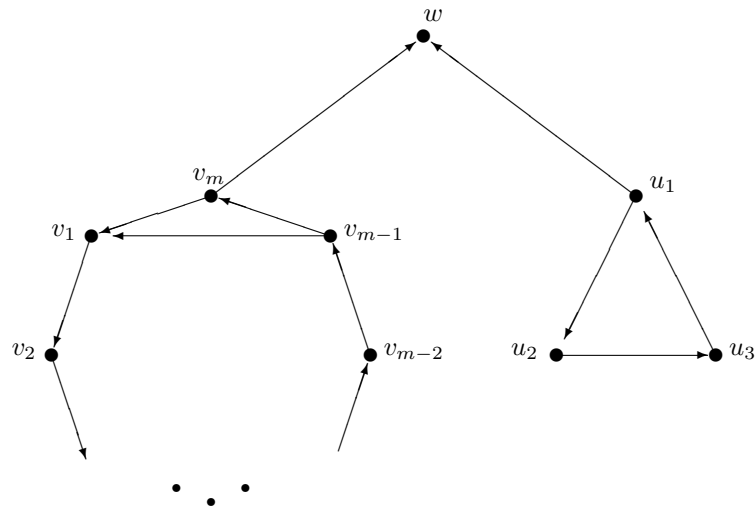


FIGURE 1. U_n where $n = m + 4$

Note that the competition period of a digraph may not be 1 when it is not strongly connected. For example, consider the above digraph U_n . Notice that there exist walks $u_1 \xrightarrow{3k-2} w$, $u_2 \xrightarrow{3k} w$ and $u_3 \xrightarrow{3k-1} w$ for each positive integer k . Therefore we have the competition period of U_n is not 1, but 3.

In [4], the upper bound of competition indices of primitive digraphs were studied as follows.

Lemma 1.3 (Cho and Kim [4]). *Let D be a primitive digraph D of order n (≥ 3) with girth s .*

- (1) *If n is odd, then $\text{cindex}(D) \leq n + \left(\frac{n-3}{2}\right) s$.*
- (2) *If n is even, then $\text{cindex}(D) \leq n - 1 + \left(\frac{n-2}{2}\right) s$.*

Akelbek and Kirkland [1] have introduced the *scrambling index* of a primitive digraph. Scrambling index is the smallest positive integer k such that for every pair of vertices u and v , there exists a vertex w such that $u \xrightarrow{k} w$ and $v \xrightarrow{k} w$ in D . Akelbek and Kirkland’s definition of scrambling index is the same as our definition of the competition index in the case of primitive digraphs. In [1], they presented the following result regarding the scrambling index.

Lemma 1.4 (Akelbek and Kirkland [1]). *Let D be a primitive digraph with n vertices and girth s .*

- (1) *If s is odd, then $\text{cindex}(D) \leq n - s + \left(\frac{s-1}{2}\right) n$.*
- (2) *If s is even, then $\text{cindex}(D) \leq n - s + \left(\frac{n-1}{2}\right) s$.*

Lemma 1.4 gives a better upper bound for the competition indices of primitive digraphs than Lemma 1.3.

For each integer $n (n \geq 3)$, a Wielandt digraph of order n is a digraph isomorphic to W_n , as shown in below. Then W_n is a primitive digraph with girth $n-1$, and $\text{cindex}(W_n) = \omega_n$ where $\omega_n = \left\lceil \frac{(n-1)^2+1}{2} \right\rceil$ for a positive integer n , [1, 4].

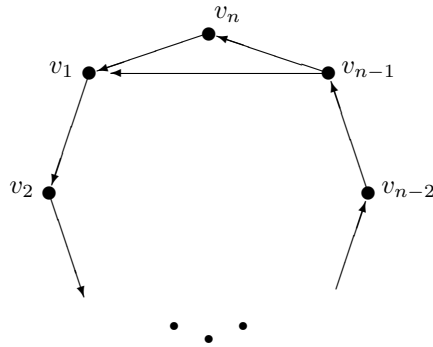


FIGURE 2. the Wielandt digraph W_n

Akelbek and Kirkland [1] and Cho and Kim [4] have considered the upper bound of the competition index of a primitive digraph as follows.

Theorem 1.5 (Akelbek and Kirkland [1], Cho and Kim [4]). *Let D be a primitive digraph of order n . Then,*

$$\text{cindex}(D) \leq \left\lceil \frac{(n-1)^2+1}{2} \right\rceil.$$

Equality holds if and only if D is a Wielandt digraph.

In Section 2, we consider the upper bound of the competition index of a strongly connected digraph. We also study the competition index of a symmetric digraph in Section 3.

2. Competition indices of strongly connected digraphs

We consider competition indices of strongly connected digraphs. Suppose $k = k(D) \geq 1$. Then, there exists a vertex partition $\{V_1, V_2, \dots, V_k\}$ of the vertices V of D such that for each $1 \leq i \leq k$, every arc goes from V_i to V_{i+1} for some i , where $V_{k+1} = V_1$. Let E_i be the subgraph of D^k induced by V_i . Then, E_i is primitive (refer to Theorem 3.4.5 in [2]).

Consider the strongly connected digraph T_{11} shown in Figure 3. Note that $k(T_{11}) = 2$. Let $V_1 = \{v_2, v_4, v_6, v_8, v_{10}\}$ be a set of black vertices and $V_2 = \{v_1, v_3, v_5, v_7, v_9, v_{11}\}$ be a set of white vertices. Then, $\{V_1, V_2\}$ forms a vertex partition of T_{11} .

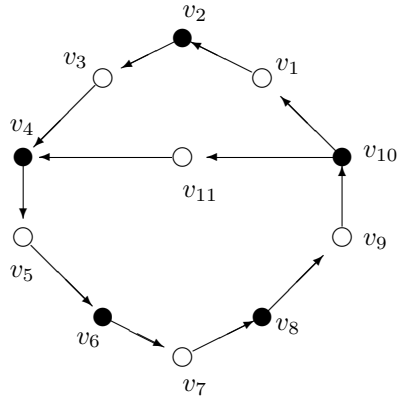


FIGURE 3. Digraph T_{11} with index of imprimitivity 2

Now, consider the induced subgraph E_1 (respectively, E_2) of V_1 (respectively, V_2) in the 2-step digraph T_{11}^2 of T_{11} as shown in Figure 4. Then T_{11}^2 is a disjoint union of E_1 and E_2 . Note that E_1 is a Wielandt digraph of order 5. We also have $\text{cindex}(E_1) = 9$ and $\text{cindex}(E_2) = 10$. Furthermore, we have $\text{cindex}(T_{11}) = 19$.

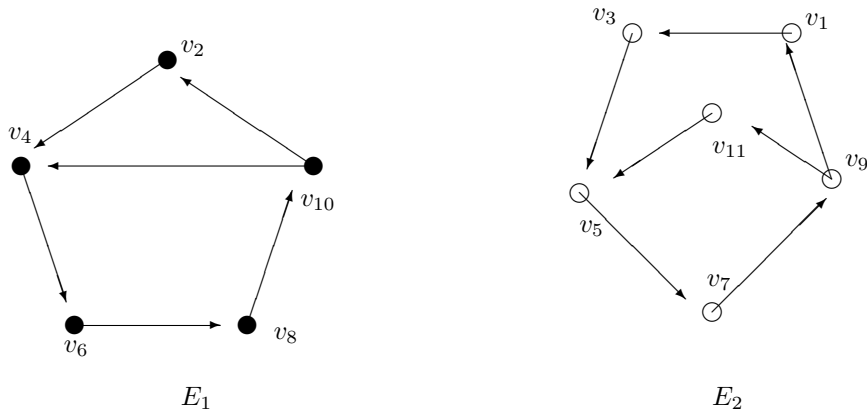


FIGURE 4. Its induced subgraphs E_1, E_2

Let D be a strongly connected digraph. For a pair of vertices u and v having a k -step common prey, let $\text{cindex}(D : u, v)$ denote the smallest positive integer k such that u and v have a k -step common prey. If u and v do not have any k -step common prey, we let $\text{cindex}(D : u, v) = 1$. We also let $\text{cindex}(D : u, u) = 1$. Let $\text{cindex}(D : u)$ denote $\max\{\text{cindex}(D : u, v) | v \in V(D)\}$. Then we have $\text{cindex}(D) = \max\{\text{cindex}(D : u) | u \in V(D)\}$ and $\text{cindex}(D) = \max\{\text{cindex}(D : u, v) | u, v \in V(D)\}$.

Now, let D be a primitive digraph. For a pair of vertices u and v in $V(D)$, let $\text{exp}(D : u, v)$ denote the smallest integer k such that there exists a directed walk of length t from u to v in D for all $t \geq k$, and let $\text{exp}(D : u) = \max\{\text{exp}(D : u, v) | v \in V(D)\}$. Then, we have $\text{exp}(D) = \max\{\text{exp}(D : u, v) | u, v \in V(D)\}$.

Theorem 2.1. *Suppose $k(D) = k$ for a strongly connected digraph D . Let V_1, V_2, \dots, V_k be k nonempty sets, with $V_{k+1} = V_1$, where each arc of D issues from V_i and enters V_{i+1} for some i with $1 \leq i \leq k$. Let E_i be the subgraph of D^k induced by V_i , where $1 \leq i \leq k$, and $m = \min\{\text{cindex}(E_i) | i = 1, 2, \dots, k\}$ in D^k . Then, we have*

$$\text{cindex}(D) \leq k(m + 1) - 1.$$

Proof. We show that $\text{cindex}(D : u, v) \leq k(m + 1) - 1$ for any two vertices u and v . If $u \in V_i, v \in V_j$, and $i \neq j$, then u and v do not have an l -step common prey for any positive integer l . Thus, $\text{cindex}(D : u, v) = 1$. Suppose $u, v \in V_j$ and $m = \text{cindex}(E_r)$ for some $1 \leq j, r \leq k$. Let u', v' be vertices in V_r such that there exist walks $u \xrightarrow{f} u'$ and $v \xrightarrow{f} v'$, where $0 \leq f \leq k - 1$. Since $\text{cindex}(D^k : u', v') \leq \text{cindex}(E_r)$, we have

$$\begin{aligned} \text{cindex}(D : u, v) &\leq f + \text{cindex}(D : u', v') \\ &\leq k - 1 + k \cdot \text{cindex}(D^k : u', v') \\ &\leq k - 1 + k \cdot \text{cindex}(E_r). \end{aligned}$$

We have $\text{cindex}(D : u, v) \leq k(m + 1) - 1$. Thus, we have $\text{cindex}(D) = \max\{\text{cindex}(D : u, v) | u, v \in V(D)\} \leq k(m + 1) - 1$. This establishes the result. \square

Note that the inequality in Theorem 2.1 is sharp. For the digraph $D = T_{11}$ shown in Figure 3, we have that $k = 2, m = 9 = \text{cindex}(E_1)$, and $\text{cindex}(T_{11} : v_1, v_5) = 19$. Thus, we have $\text{cindex}(T_{11}) = k(m + 1) - 1$.

Lemma 2.2. *Let D be a strongly connected digraph of order $n (\geq 3)$. If $k(D) > \frac{n}{2}$, then we have*

$$\text{cindex}(D) \leq \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Proof. Let the girth of D be s . If $k(D) > \frac{n}{2}$, then every cycle in D has the same length $s (= k(D))$. Let V_1, V_2, \dots, V_s be s nonempty sets, with $V_{s+1} = V_1$, where each arc of D issues from V_i and enters V_{i+1} for some i with $1 \leq i \leq s$.

Consider two vertices $u \in V_i$ and $y \in V_j$. If $i \neq j$, or $i = j$ and $u = v$, then $\text{cindex}(D : u, v) = 1$. If u and v do not have l -step common prey for any positive number l , then $\text{cindex}(D : u, v) = 1$. Without loss of generality, we may assume that $\text{cindex}(D : u, v) = m > 1$ and $u, v \in V_s$. Then we have walks such that $u \rightarrow u_1 \rightarrow \dots \rightarrow u_m = w$ and $v \rightarrow v_1 \rightarrow \dots \rightarrow v_m = w$, where $u_i \neq v_i$ and $u_i, v_i \in V_i$ ($i = 1, \dots, m - 1$). Therefore we have $2m + 1 \leq n$ and $\text{cindex}(D : u, v) = m \leq \lfloor \frac{n-1}{2} \rfloor$. This establishes the result. \square

From Theorem 2.1 and Lemma 2.2, we have the following theorem.

Theorem 2.3. *Suppose D is a strongly connected digraph of order n (≥ 3). Then, we have*

$$\text{cindex}(D) \leq \omega_n = \left\lceil \frac{(n-1)^2 + 1}{2} \right\rceil.$$

This equality holds if and only if D is isomorphic to W_n .

Proof. Let $k(D) = k$. If $k = 1$, the above equality holds by Theorem 1.5. If $n = 3$ and $k \geq 2$, we can check that $\text{cindex}(D) \leq \omega_n$. If $k(D) > \frac{n}{2}$, this equality holds by Lemma 2.2. Assume that $2 \leq k \leq \frac{n}{2}$ and $n \geq 4$. Let V_1, V_2, \dots, V_k be k nonempty sets, with $V_{k+1} = V_1$, where each arc of D issues from V_i and enters V_{i+1} for some i with $1 \leq i \leq k$. Let E_i be the subgraph of D^k induced by V_i , where $1 \leq i \leq k$. Then, there exists a positive integer r such that the order of E_r is less than or equal to $\lfloor \frac{n}{k} \rfloor$. From Theorem 1.5, we have $\text{cindex}(E_i) \leq \omega_{|V_i|}$ for each $1 \leq i \leq k$ since each E_i is a primitive digraph. Therefore we have $\text{cindex}(E_r) \leq \omega_{\lfloor \frac{n}{k} \rfloor}$. By Theorem 2.1, we have

$$\begin{aligned} \text{cindex}(D) &\leq k - 1 + k \cdot \text{cindex}(E_r) \\ &\leq k - 1 + k \cdot \omega_{\lfloor \frac{n}{k} \rfloor} \\ &\leq k - 1 + k \cdot \frac{\left(\frac{n}{k}\right)^2 - 2\left(\frac{n}{k}\right) + 3}{2} \\ &= \frac{n^2}{2k} + \frac{5}{2}k - n - 1. \end{aligned}$$

Let $g(k) = \frac{n^2}{2k} + \frac{5}{2}k - n - 1$ ($2 \leq k \leq \frac{n}{2}$). Then $g(k)$ attains the maximum value when $k = 2$. $g(2) = \frac{n^2 - 4n + 16}{4} < \omega_n$ since $n \geq 4$. Therefore, we have $\text{cindex}(D) < \omega_n$ for a nonprimitive strongly connected digraph D . This establishes the result. \square

Next we study the competition indices of symmetric strongly connected digraphs.

3. Competition indices of symmetric strongly connected digraphs

In this section, we assume that D is a symmetric strongly connected digraph of order n (≥ 2). Observe that in symmetric graphs, strongly connected is the same as connected. Since there is a 2-cycle in D , we have $k(D) \leq 2$. For

a connected symmetric primitive digraph D , $k(D) = 2$ if D is bipartite, and otherwise $k(D) = 1$.

Lemma 3.1 (Shao [8]). *Let u and v be vertices of a symmetric primitive digraph D such that there exist directed walks of lengths k_1 and k_2 , respectively, with different parity between u and v . Then, we have*

$$\exp(D : u, v) \leq \max\{k_1, k_2\} - 1.$$

Proposition 3.2 (Shao [8]). *Let $E_n = \{\exp(D) : D \text{ is a symmetric primitive digraph of order } n \geq 4\}$. Then we have*

$$E_n = \{1, 2, \dots, 2n - 2\} \setminus S,$$

where S is the set of all odd numbers in $\{n, n + 1, \dots, 2n - 3\}$.

Lemma 3.3. *Let D be a symmetric primitive digraph of order n . Then, for each vertex u , we have*

$$\text{cindex}(D : u) = \left\lceil \frac{\exp(D : u)}{2} \right\rceil.$$

Proof. Consider two vertices u and v in $V(D)$.

First, we have two directed walks $u \xrightarrow{\exp(D:u,v)} v$ and $u \xrightarrow{\exp(D:u,v)+1} v$. Since D is symmetric, u and v have a $\left\lceil \frac{\exp(D:u,v)}{2} \right\rceil$ -step common prey. Therefore, we have

$$\text{cindex}(D : u, v) \leq \left\lceil \frac{\exp(D : u, v)}{2} \right\rceil.$$

Suppose $u = v$. Then there exists a vertex $w (\neq u)$ such that $u \xrightarrow{1} w$, and there exists a directed walk $u \xrightarrow{2 \cdot \text{cindex}(D:u,w)} w$. From Lemma 3.1, we have $\exp(D : u, w) \leq 2 \cdot \text{cindex}(D : u, w) - 1$. Therefore, we have

$$\exp(D : u, v) \leq \exp(D : u, w) + 1 \leq 2 \cdot \text{cindex}(D : u, w) \leq 2 \cdot \text{cindex}(D : u).$$

Suppose $u \neq v$. If $d(u, v)$ is odd, then we have two walks $u \xrightarrow{d(u,v)} v$ and $u \xrightarrow{2 \cdot \text{cindex}(D:u,v)} v$. From Lemma 3.1, we have $\exp(D : u, v) \leq 2 \cdot \text{cindex}(D : u) - 1$ since $2 \cdot \text{cindex}(D : u) > d(u, v)$. If $d(u, v)$ is even, then we have a vertex w such that $u \xrightarrow{d(u,v)-1} w (\neq u) \rightarrow v$. We also have $u \xrightarrow{2 \cdot \text{cindex}(D:u,w)} w$. Since $d(u, v) - 1$ is odd, we have $\exp(D : u, w) \leq 2 \cdot \text{cindex}(D : u, w) - 1$. Therefore, $\exp(D : u, v) \leq \exp(D : u, w) + 1 \leq 2 \cdot \text{cindex}(D : u, w) \leq 2 \cdot \text{cindex}(D : u)$. $\exp(D : u, v) \leq 2 \cdot \text{cindex}(D : u)$. Therefore, we always have

$$\frac{\exp(D : u)}{2} \leq \text{cindex}(D : u).$$

From the above results, we have $\frac{\exp(D:u)}{2} \leq \text{cindex}(D : u) \leq \left\lceil \frac{\exp(D:u)}{2} \right\rceil$. For each vertex u in D , we have

$$\text{cindex}(D : u) = \left\lceil \frac{\exp(D : u)}{2} \right\rceil.$$

This establishes the result. □

Theorem 3.4. *If D is a symmetric strongly connected digraph, then we have*

$$\text{cindex}(D) = \left\lceil \frac{\text{index}(D)}{2} \right\rceil.$$

Proof. Let the order of D be n . If $n = 1$, this equality holds. Suppose $n \geq 2$. Then, we have $k(D) \leq 2$.

Case 1. $k(D) = 1$

Note that $\text{index}(D) = \max\{\text{exp}(D : u)\}$ and $\text{cindex}(D) = \max\{\text{cindex}(D : u)\}$. From Lemma 3.3, the result is established.

Case 2. $k(D) = 2$

There are bipartition V_1 and V_2 of the vertex set $V(D)$. Consider two vertices $u \in V_i$ and $v \in V_j$.

If $i \neq j$, we have $\text{cindex}(D : u, v) = 1$. If $i = j$ and $\text{index}(D)$ is even, there is a directed walk such that $u \xrightarrow{\text{index}(D)} v$. Then, we have $\text{cindex}(D : u, v) \leq \frac{\text{index}(D)}{2}$. If $i = j$ and $\text{index}(D)$ is odd, there is a directed walk such that $u \xrightarrow{\text{index}(D)+1} v$. In all cases, we have $\text{cindex}(D : u, v) \leq \frac{\text{index}(D)+1}{2}$. Therefore, we have $\text{cindex}(D) \leq \left\lceil \frac{\text{index}(D)}{2} \right\rceil$.

If $i = j$, there exists $u \xrightarrow{2 \cdot \text{cindex}(D)} v$. If $i \neq j$, there is no $u \xrightarrow{2 \cdot \text{cindex}(D)} v$ and there exist a vertex $u' \in V_j$ and a directed walk such that $u \xrightarrow{1} u' \xrightarrow{2 \cdot \text{cindex}(D)} v$. Then, we have $D^{2 \cdot \text{cindex}(D)} = D^{2 \cdot \text{cindex}(D)+2}$. Therefore, we have $\text{index}(D) \leq 2 \cdot \text{cindex}(D)$.

This establishes the result. □

Corollary 3.5. *If D is a symmetric primitive digraph of order $n \geq 4$, then $\text{cindex}(D) \leq n - 1$, and equality holds if and only if D is a path with a loop on an end vertex.*

Proof. From Proposition 3.2 and Theorem 3.4, we have $\text{cindex}(D) = \left\lceil \frac{\text{index}(D)}{2} \right\rceil \leq n - 1$. Since there are no symmetric primitive digraphs whose index is $2n - 3$ from Proposition 3.2, this corollary is proved. □

4. Closing remark

In this paper, we have introduced the concept of the competition index of a digraph using the concept of the m -step competition graph introduced in [5]. For digraphs whose competition periods are one, there are some similarities between the property of the competition indices and that of ordinary indices or exponents. However, for digraphs whose competition periods are not one, there are significant differences between their competition indices and ordinary indices. In the case of the ordinary index, it is well known that $\text{index}(D) \leq (n - 2)^2 + 2$ for a reducible digraph D of order n , [7, 9]. This implies that the

index of a Wielandt digraph is the maximum possible index among indices of digraphs of order n . However, in the case of a competition index, the similarity between the properties does not hold. For example, consider the competition index of U_n shown in Figure 1. Then, we have $\text{cindex}(U_n) = m^2 - 2m + 3$ and $\text{cperiod}(U_n) = 3$. Thus $\text{cindex}(U_n) > \omega_n$ when $n \geq 15$. That is, there is a reducible digraph of order n whose competition index is greater than that of any strongly connected digraph of order n . As shown in the example U_n , the competition indices of reducible digraphs are very interesting, and there is much work to be done.

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