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COMPETITION OF FIRMS: DISCRIMINATORY PRICING AND LOCATION[1]

BY PHILLIP J. LEDERER AND ARTHUR P. HURTER, JR.

Two costlessly mobile firms are to be located in a market region, a subset of the plane. The firms compete by setting locations and delivered price schedules. To study this competitive stiuation an appropriate extensive form game is defined, along with an appropriate noncooperative solution concept. Existence and general properties of the equilibrium are demonstrated. Among the results are: Each firm increases its profit by locating so as to decrease total cost to both firms of serving the market. Firms will never locate coincidentally if they have identical production costs and transport cost rates, or if these are different and the firms are located in a circular market region having a uniform demand distribution.

INTRODUCTION

THIS PAPER STUDIES competition between two profit maximizing firms in space who are costlessly mobile and may discriminate in price. We will allow the firms to set locations and delivered price schedules and we will be concerned with the existence and properties of equilibria in location and price.

Starting with Hotelling [9], the spatial competition literature has focused on location on bounded linear markets by two or more firms. Hotelling assumed identical firms that produced a single good with constant cost of production and considered consumers to be uniformly distributed and to have inelastic demand. He also assumed that the consumers pay transport cost and purchase the good from the cheapest source. Hotelling claimed that a Nash equilibrium in locations for the two firm market existed and yielded "back-to-back" locations at the center of the market. Many authors, most recently, D'Aspremont, Gabszewicz, and Thisse [2] have noted that an equilbrium in prices and location does not exist for Hotelling's model. However, if the firms employ identical exogeneously specified prices, Hotelling's conclusions hold. Subsequent work by Smithies [14], Hartwick and Hartwick [7], and Eaton [3] claimed to show that a Nash equilibrium in f.o.b. prices and locations can exist in markets with a uniform distribution of consumers each of whom have identical elastic demand functions for the two and three firm problems. The work of D'Aspremont et al. casts doubt on these conclusions without the adoption of restrictive conditions.

Our work contrasts with these works and related research which has dealt with linear markets, with uniform distributions of customers and with identical firms. Our work will involve markets that are subsets of the plane having nonuniform distributions of customers. Our firms will be allowed to be different, that is, have differences in production and transport costs. However, the fundamental difference in our approach is that our firms will set discriminatory prices and not price f.o.b. In many ways our work will represent the discriminatory pricing

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analogue of Hotelling's work. This work is based upon Lederer [**12**] which contains additional results and generalizations. Some of these details, including issues of competitive location, pricing, and production are found in Hurter and Lederer [**10**].

We present our model to study markets in which price discrimination through freight absorption occurs, typically those with goods having low values in relation to transport cost, and having oligopolistic competition, and where producers have transport cost advantages. We will also require that demand elasticity with respect to price be low. Markets having these characteristics include cement, plywood, fertilizer, and sugar.

Hoover [**8**] in his early work analyzed spatial price discrimination for firms with fixed locations. He concluded that a firm serving a market point would have a local price constrained by the marginal cost of service of other firms. In situations where demand elasticity is not too high, this will result in delivered prices at market points equal to the marginal cost of the firm in the market with the second lowest marginal cost. This result is similar to that recently presented by Haddock [**6**] relating to the cause for the existence of base point pricing. We will find that the equilibrium prices in our model, defined and characterized in a rigorous manner, have a similar property.

Our approach to the problem will be to study the location-pricing problem in a sequential manner. We will assume that firms locate first, and then set prices. This last stage of price setting and competition is closely related to the models of price competition first studied by Bertrand [**1**], and more recently by Friedman [**4**] and Grossman [**5**].

We will first state our model, define a location-price game for the firm and an appropriate solution concept, and show that an equilibrium in terms of the solution concept exists and is easily characterized. Then, we will demonstrate general properties of the locations of the firms in equilibrium. Among our results will be that firms in equilibrium do not locate coincidentally. Finally, we will indicate generalizations that follow easily.

THE MODEL

Let two firms, denoted A and B, be located in a compact market region, a subset of $\mathbb{R}^2$ denoted S. Let the locations of the firms be respectively indicated by $z_A = (x_A, y_A)$ and $z_b = (x_B, y_B)$. Assume that both firms may costlessly relocate in S and produce the same, single good. Let C'_A be the constant marginal cost of production for firm A; similarly for C'_B.

Let each firm have a cost of transporting the good from its location to a point in the market. Let the transport cost per unit shipped from the firms be given by the functions $f_A(z_A, z)$, $f_B(z_B, z)$. We require that f_i^2 be Lebesque integrable in z for all z_i and continuous in z_i for all z. Thus distance measures need not be Euclidean.

Let the market be distributed with customers each of whom wish to purchase a single unit of good. We assume that the distribution of customers is given by a Lebesque integrable function $\rho: S \to \mathbb{R}$. We will require ρ^2 to be Lebesque integrable over S to insure integrability of our profit functions.

We will assume that each firm announces a "price policy": a function which specifies the price at which the firm will offer to sell and deliver the good for each market point; it is a "*delivered price schedule.*" Let P be the set of all price policies such that its square is Lebesque integrable over S. A price policy for A will be written p_A. $p_A(z)$ is the price at which A will deliver a unit of good to a customer at $z \in S$.

We will assume that customers buy from the cheapest source. If $p_A(z) = p_B(z)$, it is not clear whether the customer will buy from A or B. Then define:

$$S_A(p_A, p_B) = \{z \in S \mid p_A(z) < p_B(z)\},$$

$$S_B(p_A, p_B) = \{z \in S \mid p_B(z) < p_A(z)\},$$

$$S_C(p_A, p_B) = \{z \in S \mid p_A(z) = p_B(z)\}.$$

S_A is the market controlled by A. S_B is the market controlled by B. S_C is the "shared" market.

We may wish to specify a sharing rule which describes how demand of customers in S_C is allocated between firms.

DEFINITION: For firms A and B, using price policies (p_A, p_B) and locating at (z_A, z_B), a *sharing rule* is a function,

$$r(z_A, p_A, z_B, p_B, z) = [r_A(z_A, p_A, z_A, p_B, z), r_B(z_B, p_A, z_B, p_B, z)]$$

such that for all z

(i) $\quad r_A(z_A, p_A, z_B, p_B, z) + r_B(z_A, p_A, z_B, p_B, z) = 1,$

(ii) $\quad r_A(z_A, p_A, z_B, p_B, z) \geq 0,$

(iii) $\quad r_B(z_A, p_A, z_B, p_B, z) \geq 0,$

(iv) $\quad r_i(z_A, p_A, z_B, p_B, z) = 1 \quad$ if $\quad z \in S_i(p_A, p_B) \quad$ for $\quad i \in \{A, B\}$.

Let the set of all sharing rules be called $\mathcal{R}$.

An important class of sharing rules, the *cost advantage sharing rules*, will now be defined. A cost advantage sharing rule will be a sharing rule where the firms split the market $S_C(p_A, p_B)$ by specifying that each point in S_C will be served by the firm with the least total marginal cost. Formally, we have the following definition.

DEFINITION: *A cost advantage sharing rule* r^* is a function such that: (i) $r^* \in \mathcal{R}$; (ii) for all (z_A, z_B) and (p_A, p_B) if $z \in S_C(p_A, p_B)$, then for $i \in \{A, B\}$

$$r_i(z_A, p_A, z_B, p_B, z) = \begin{cases} 1 & \text{if } f_i(z_i, z) + C_i' < f_{-i}(z_{-i}, z) + C_{-i}', \\ 0 & \text{if } f_{-i}(z_{-i}, z) + C_{-i}' < f_i(z_i, z) + C_i', \end{cases}$$

with $-i$ the complement of i. The class of all cost advantage sharing rules will be denoted $\mathcal{R}^*$.

The class $\mathscr{R}^*$ will be important in later results. Technically, it will enable us to avoid defining equilibria in terms of ε-equilibrium concepts. In order to explicitly calculate a firm's profit, a sharing rule must be specified.

DEFINITION: Suppose that sharing rule $r \in \mathscr{R}$ is specified. The profit of the firm $i \in \{A, B\}$, is defined to be

$$\Pi_i^r(z_A, p_A, z_B, p_B) = \iint_{S_i(p_A, p_B)} [p_i(z) - f_i(z_i, z) - C_i'] \rho(z)\, dz + \iint_{S_C(p_A, p_B)} [p_i(z) - f_i(z_i, z) - C_i'] r_i(z_A, p_A, z_B, p_B, z) \rho(z)\, dz$$

for all $(z_A, p_A, z_B, p_B) \in S \times P \times S \times P$.

THE GAME

To analyze the behavior of the firms with respect to their choice of locations and price policies, we will consider the following game with an exogeneously chosen r:

1. Let the firms privately and simultaneously select locations.
2. Let each firm become aware of the other's choice.
3. Let the firms privately and simultaneously choose price policies.
4. As a result of these decisions, the firms receive payoffs Π_A^r, Π_B^r, respectively.

We view the game as one of complete information.

A strategy for firm i will be denoted d_i. Firm i's choice of location will be denoted $d_{i(0)}$. If the firms chose locations z_A, z_B, i's price policy choice specified by strategy d_i will be $d_i(z_A z_B)$. The set of all strategies for player i will be denoted D_i. We write the strategy space of the game as $D = D_A \times D_B$. If $d \in D$, $\hat{d}_A \in D_A$, $\hat{d}_B \in D_B$ then $d/\hat{d}_A = (\hat{d}_A, d_B)$ and $d/\hat{d}_B = (d_A, \hat{d}_B)$. We refer to the competitive game as Γ^r, the location-price game using sharing rule r.

EQUILIBRIUM

We next define a meaningful noncooperative solution for the location-price game. The natural solution concept is the Nash equilibrium. It turns out however that the set of Nash equilibria for Γ^r will be too large, will be too hard to characterize, and will contain many strategy pairs that reflect irrational threats. To eliminate these irrational threats and characterize a set of solutions we will use a stronger equilibrium concept which we will call the location-price equilibrium.

DEFINITION: A location-price equilibrium for Γ^r will be a $d \in D$ that is a subgame perfect and undominated.

Subgame Perfectness

The proper subgames for Γ^r include the entire game and a game for each pair of location choices for the firms involving the choice of price policies by the firms assuming they are located at the locations specified.

If d is subgame perfect, Selten [**13**], d is a Nash equilibrium for the entire game and d must specify where the firms will locate. Also, d must specify price policies that are optimal against each other, in the sense of Nash, for every possible location pair for the firms.

Undominated Strategies

DEFINITION: A strategy $d_i \in D_i$ is said to be *dominated* if $\exists \hat{d}_i \in D_i$ such that (i) $\Pi_i^r(d/\hat{d}_i) \geqslant \Pi_i^r(d/d_i)$ $\forall d \in D$ and (ii) for at least one $d \in D$, $\Pi_i^r(d/\hat{d}_i) > \Pi_i^r(d/d_i)$. If d_i is not dominated it is *undominated.*

Our restriction on d is similar to that proposed by Grossman [**5**] and many others in the literature. He requires that firms propose strategic price-quantity proposals that insure the firms earn nonnegative profits, considering fixed costs as well as variable ones.

CHARACTERIZING THE LOCATION-PRICE EQUILIBRIUM

The requirement that a location-price equilibrium be subgame perfect, as opposed to just Nash, eliminates strategies where a firm threatens to price both below its own total marginal cost *and* its competitor's total marginal cost *if* the competitor does not locate at a particular point. Such strategies can be designed to make any pair of locations for the firms a Nash equilibrium. Such strategies are unreasonable. If the competitor does not locate at the desired point the firm will not rationally be expected to make good with the threat. In fact, the firm will not price at any market points where the firm serves customers on any subgame below its own total marginal cost. This result, like many to follow, is qualified by the provision that this rule may not hold on a set of market points where total demand is zero.

DEFINITION: We will say that a property holds on a set $T \subset S$ *except on a set of demand zero, edz,* if the property holds on every point of T except perhaps on a measurable set $T' \subset T$ such that $\iint_{T'} \rho(z)\, dz = 0$. If $T \subset S$ and is measurable and $\iint_T \rho(z)\, dz > 0$, we say that T is a set having *positive demand.*

LEMMA 1: *For Γ^r suppose d is a subgame perfect equilibrium and for (z_A, z_B), $d_i(z_A, z_B) = p_i$, $i \in \{A, B\}$. Then $r_i(z_A, p_A, z_B, p_B, z) > 0$ implies $p_i(z) \geqslant f_i(Z_i, z) + C_i'$ for all $z \in S$, edz.*

This lemma is proved by demonstrating that if the conclusion is not true a firm serving and pricing below marginal cost can alter its price schedule and increase its profit. A formal proof is found in the Appendix.

We next demonstrate that unless the sharing rule utilized is a cost advantage sharing rule, a subgame perfect equilibrium cannot be expected to exist. The insight afforded by the previous lemma is the following: Suppose two firms share a set of customers, and one of the firms is pricing strictly above its total marginal cost on some subset of the shared set that has positive demand; then this firm can strictly increase its profit by undercutting its competitor. A formal proof will be found in the Appendix.

LEMMA 2: *For Γ^r consider $d \in D$ and any (z_A, z_B). Suppose that $d_i(z_A z_B) = p_i$, $i \in \{A, B\}$. Define for $i \in \{A, B\}$:*

$$T_i = \{z \in S_C(p_A, p_B) \mid r_i(z_A, p_A, z_B, p_B, z)\rho(z) \neq r_i^*(z_A, p_A, z_B, p_B, z)\rho(z) \text{ for all } r^* \in \mathscr{R}^*\}.$$

If T_i is a region having positive demand for some $i \in \{A, B\}$, then d cannot be subgame perfect.

It is intuitively plausable to require that the sharing rule belongs to $\mathscr{R}^*$. There can be no equilibrium without it! We make this requirement and show later that the set of location-price equilibria for Γ^{r^*} is invariant over $\mathscr{R}^*$.

It is to be noted that subgame perfectness does not eliminate the threat of predatory pricing of the following type: "If you do not locate where I wish you to, I will price on sets having positive demand at or above your marginal cost but below my marginal cost of supplying these points." In effect, I will allow you positive profits on these sets but will hold your profits down. Such threats, though strategically important, are dangerous for the initiator. If there is the slightest chance that the competitor would not match the firm's price on the described sets (this would be his optimal price choice) the firm would suffer a loss. If this consideration occurs to the firm, and it considers the chance of the competitor mispricing to be small but positive, then the firm should never price as described: a firm in equilibrium should never price below its marginal cost. In spirit, this analysis is that of Selten's perfect (or trembling hand) equilibrium. The analytic details involved in dealing with this solution concept for strategy spaces comprising, in part, the choice of a price policy from a Lebesgue integrable set of functions are considerable. (A probability space of such functions must be described along with convergence concepts.) Instead, identical results can be obtained by requiring each firm to utilize undominated strategies.

LEMMA 3: *For Γ^{r^*}, d is undominated iff for (z_A, z_B), $d_i(z_A, z_B) = p_i$ for $i \in \{A, B\}$ implies*

$$(1) \qquad p_i(z) \geq f_i(z_i, z) + C_i' \quad \forall z, \qquad edz.$$

This result is proved by showing that if (1) does not hold for p_i, a $\hat{p}_i$ can be constructed dominating p_i. Conversely, if (1) holds it can be shown that p_i is undominated. A formal proof is found in the Appendix.

EQUILIBRIUM PRICE POLICIES

THEOREM 1: *If d is a location-price equilibrium for* Γ^{r^*} *and* $(z_A, z_B) \in S \times S$ *then* $d_i(z_A z_B) = p^*(z_A, z_B, \cdot)$ *for* $i \in \{A, B\}$ *where*

$$p^*(z_A, z_B, z) = \max\,[f_A(z_A, z) + C'_A, f_B(z_B, z) + C'_B]$$

for all $z \in S$, *edz.*

The proof is demonstrated by showing that if the low cost firm does not serve or share service of demand on a set having positive demand it could undercut the low price firm. The current low price firm must be pricing above its marginal cost by Lemma 3; thus by cutting its prices the low cost firm can raise its profits. Further, in equilibrium the low cost firm must price at the marginal cost of the next most efficient firm at each market point and that firm must price at its marginal cost. If the next most efficient firm priced above this amount, the low cost firm would price at this price and serve all demand. This would induce the next most efficient firm to cut its price. A formal proof of this result is found in the Appendix.

In equilibrium, prices are the maximum of the firms' marginal cost, both firms price identically, and each customer is served by the cost advantaged firm. The cost advantage sharing rule helped us to avoid defining an equilibrium in terms of an ε-equilibrium where the advantaged firm slightly undercuts the other's marginal cost.

We will refer to p^* as an equilibrium price policy. If equilibrium price policies are used by both firms and the firms locate at (z_A, z_B), then the profit for firm i under sharing rule $r^* \in \mathscr{R}^*$ will be $\Pi_i^{r^*}(z_A, p^*(z_A, z_B, \cdot), z_B, p^*(z_A, z_B, \cdot))$. We will abbreviate this without confusion as $\Pi_i^{r^*}(z_A, p^*, z_B, p^*)$.

We can see that if $r_1^*, r_2^* \in \mathscr{R}^*$ then $\Pi_i^{r_1^*}(z_A, p^*, z_B, p^*) = \Pi_i^{r_2^*}(z_A, p^*, z_B, p^*)$ for all $i \in \{A, B\}$, $(z_A, z_B) \in S \times S$. This follows because r_1^*, r_2^* may disagree only on the set $\{z \in S \,|\, f_A(z_A, z) + C'_A = f_B(z_B, z) + C'_B\}$, and on this set both firms earn zero profits. We may conclude that the set of location-price equilibria for $\Gamma_1^{r^*}$ and $\Gamma_2^{r^*}$ are identical.

Presuming the firms employ equilibrium price policies, the market regions controlled by the firms may be categorized. If the firms are located at (z_A, z_B) then the market points served by A, B and jointly shared are

$$S_A(z_A, z_B) = \{z \in S \,|\, f_A(z_A, z) + C'_A < f_B(z_B, z) + C_B\},$$

$$S_B(z_A, z_B) = \{z \in S \,|\, f_B(z_B, z) + C'_B < f_A(z_a, z) + C_A\},$$

$$S_C(z_A, z_B) = \{z \in S \,|\, f_A(z_A, z) + C'_A = f_B(z_B, z) + C_B\}.$$

We may summarize these remarks:

THEOREM 2: *If* Γ^{r^*} *is a location-price game,* r^* *is a cost advantage sharing rule, and equilibrium price policies are being used, then for* $i \in \{A, B\}$

$$\Pi_i^{r^*}(z_A, p^*, z_B, p^*) = \iint_{S_i(z_A, z_B)} [(f_{-i}(z_{-i}, z) + C'_{-i}) - (f_i(z_i, z) + C'_i)]\, \rho(z)\, dz \quad \textit{for all } (z_A, z_B).$$

We will assume a cost advantage sharing rule is being used in the remainder and drop r^* from our notation.

SOCIAL COST AND LOCATION EQUILIBRIA

In order to show the existence of a location-price equilibrium for Γ, we must demonstrate the existence of location choices (z_A^*, z_B^*) that are Nash, i.e., such that

$$\Pi_A(z_A^*, p^*, z_B^*, p^*) \geq \Pi_A(z_A, p^*, z_B^*, p^*) \quad \forall z_A \in S,$$
$$\Pi_B(z_A^*, p^*, z_B^*, p^*) \geq \Pi_B(z_A^*, p^*, \dot{z}_B, p^*) \quad \forall z_B \in S.$$

Such a pair will be called a location equilibrium. Interesting economic results and the decomposition of the firms' profit functions may be realized with the following definition.

DEFINITION: The *social cost* is the total cost incurred by the firms to supply demand to customers in S in a cooperative, cost minimizing manner. If the firms are located at (z_A, z_B), then if the firms are cooperating to supply demand in a cost minimizing manner, social cost is

$$K(z_A, z_B) = \iint_S \min\,[f_A(z_A, z) + C'_A, f_B(z_B, z) + C'_B]\rho(z)\, dz.$$

Social cost is a continuous function of (z_A, z_B). There is an interesting relationship between social cost and a firm's profit under equiibrium price policies.

LEMMA 4: *For* $i \in \{A, B\}$, $(z_A, z_B) \in S \times S$

$$\Pi_i(z_A, p^*, z_B, p^*) = \iint_S [f_{-i}(z_{-i}, z) + C'_{-i}]\rho(z)\, dz - K(z_A, z_B).$$

PROOF: By Theorem 2,

$$\begin{aligned} \Pi_A(z_A, p^*, z_B, p^*) &= \iint_{S_A(z_A, z_B)} [f_B(z_B, z) + C'_B - f_A(z_A, z) - C'_A)]\rho(z)\, dz \\ &= \iint_S (f_B(z_B, z) + C'_B)\rho(z)\, dz \\ &\quad - \iint_S \min\,[f_A(z_A, z) + C'_A, f_B(z_B, z) + C'_B]\rho(z)\, dz \\ &= \iint_S (f_B(z_B, z) + C'_B)\rho(z)\, dz - K(z_A, z_B). \end{aligned}$$

Q.E.D.

This interesting relationship leads to the following:

THEOREM 3: *A location-price equilibrium exists. Further, d^* is a location-price equilibrium with equilibrium locations (z_A^*, z_B^*) iff*

(2) $$K(z_A^*, z_B^*) \leq K(z_A, z_B^*) \quad \forall z_A \in S,$$

(3) $$K(z_A^*, z_B^*) \leq K(z_A^*, z_B) \quad \forall z_B \in S,$$

and equilibrium price policies are being used by the firms.

PROOF: Assume d^* is a location-price equilibrium. If (z_A^*, z_B^*) are equilibrium locations, then $\Pi_A(z_A^*, p^*, z_B^*, p^*) \geq \Pi_A(z_A, p^*, z_B^*, p^*)\ \forall z_A \in S$. Referring to Lemma 4, this is equivalent to

$$\iint_S (f_B(z_B^*, z) + C'_B)\rho(z)\,dz - K(z_A^*, z_B^*)$$

$$\geq \iint_S (f_B(z_B^*, z) + C'_B)\rho(z)\,dz - K(z_A, z_B^*) \quad \forall z_A \in S.$$

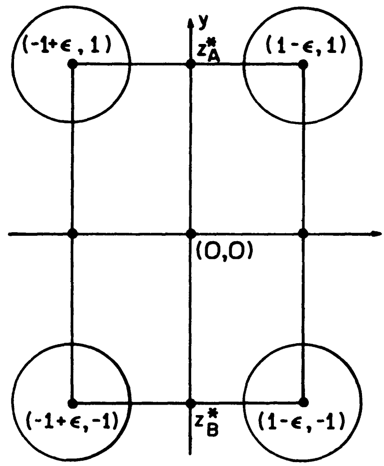
Expression (2) follows immediately. The second is shown similarly. We have previously shown that equilibrium price policies must be employed.

Conversely, if $(z_A^*, z_B^*) \in S \times S$ satisfy (2) and (3) and equilibrium price policies are employed, then it's clear by Lemma 4 that (z_A^*, z_B^*) form a location equilibrium and by Theorem 1 that d^* is subgame perfect. By Lemma 3, d^* is undominated. Thus d^* is a location-price equilibrium.

Finally we note that since K is a continuous function on the compact set $S \times S$, it has a minimum on this set at some (z_A^*, z_B^*). Such a pair satisfies (2) and (3). Equilibrium locations exist. *Q.E.D.*

We have shown that the existence of a location-price equilibrium depends on the existence of locations such that each location minimizes social cost with respect to the other's location. Such locations do exist and one equilibrium pair may be found by minimizing $K(z_A, z_B)$ on the compact set $S \times S$. If a firm anticipates that equilibrium prices will be employed by the other firm and the other firm's location will be fixed, *the firm will strive to minimize social cost, not its own production cost to maximize its profit. However, equilibrium locations need not minimize social cost globally as the following example illustrates.*

EXAMPLE: Consider a market consisting of circular submarkets of radius ε, $\varepsilon < \frac{1}{2}$, having constant unit demand, whose centers are located at the points $(1-\varepsilon, 1)$, $(1-\varepsilon, -1)$, $(-1+\varepsilon, -1)$, $(-1+\varepsilon, 1)$. See Figure 1. We will assume that transport cost is proportional to Euclidean distance.



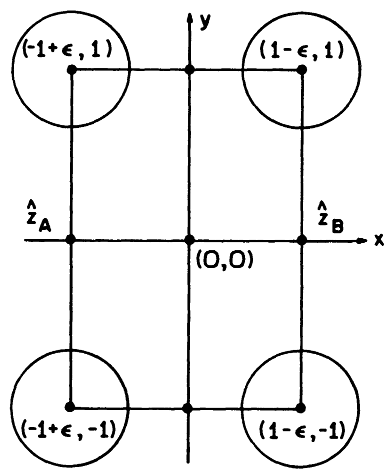


FIGURE 1

Locations $z_A^* = (0, 1)$, $z_B^* = (0, -1)$ may be seen to be equilibrium locations; each minimizes K with the other variable held fixed (and actually globally minimizes K).

Locations $\hat{z}_A = (-1+\varepsilon, 0)$, $\hat{z}_B = (1-\varepsilon, 0)$ may also be seen to be equilibrium locations. With $\hat{z}_B$ held fixed, $K(z_A, \hat{z}_B)$ is minimized at $\hat{z}_A$. Likewise for $K(\hat{z}_A, z_B)$.

If a monopolist controlled both plants, it would choose locations to globally minimize K. The set of monopolist's locations are a subset of the set of equilibrium locations.

PROPERTIES OF EQUILIBRIUM LOCATIONS

We next discuss various properties of equilibrium locations. In this section we will generally assume that transport cost is proportional to Euclidean distance, $f_A(z_A, z) = \alpha_A \| z - z_A \|_2$, $\alpha_A > 0$. Similarly for B. Differential properties of a firm's profit with respect to location are described in the following lemma supplied without proof. We will understand $\iint_{S_A} \nabla_{z_A} f_A(z_A, z)\rho(z)\, dz$ to be a vector valued integral.

LEMMA 5: *For $i \in \{A, B\}$,*

$$\nabla_{z_i} \Pi_i(z_A, p^*, z_B, p^*) = -\iint_{S_i(z_A, z_B)} \nabla_{z_i} f_i(z_i, z)\rho(z)\, dz$$

and is continuous for: (a) *all $(z_A, z_B) \in S \times S$ if $\alpha_A \neq \alpha_B$ and/or $C'_A \neq C'_B$*; (b) *all $(z_A, z_B) \in S \times S$ such that $z_A \neq z_B$ if $\alpha_A = \alpha_B$ and $C'_A = C'_B$.*

We note that the derivative of Π_A with respect to z_A is just the negative of the derivative of A's transport and production costs. We begin our survey of properties of equilibrium locations with the following result about location of identical firms.

THEOREM 4: *If (z_A^*, z_B^*) are equilibrium locations for the location-price game with $f_A = f_B$ and $C'_A = C'_B$, then $z_A^* \neq z_B^*$.*

This result can be simply proved by remarking that if there is a subset of the market enjoying positive demand, social cost cannot be minimized with coincident location. This result is general to non-Euclidean transport functions as well as Euclidean ones.

Thus, two firms that have the same rate of transport and marginal cost of production cannot locate coincidently in equilibrium. This result is to be contrasted with Hotelling's [**9**] result with fixed identical firm prices which predicted equilibrium location at a central coincident point of a linear market region. We may use this fact and Lemma 5 to state the following lemma.

LEMMA 6: *If (z_A^*, z_B^*) are equilibrium locations, then $\nabla_{z_i} \Pi_i(z_A, p^*, z_B, p^*)$ is continuous at (z_A^*, z_B^*) for $i \in \{A, B\}$.*

Location of Nonidentical Firms

We can establish a similar result about noncoincident location for the case of *two nonidentical firms.* A few preliminary results will prove helpful and add insight to properties that equilibrium locations of nonidentical firms must obey.

THEOREM 5: *Let S be a convex compact market region with positive demand. If (z_A^*, z_B^*) are equilibrium locations and for some $i \in \{A, B\}$, $S_i(z_A^*, z_B^*)$ is a region having positive demand, then $z_i \in$ interior S.*

The result follows from Lemma 5 and Lemma 6: the gradient of the profit of a boundary located firm points into the interior. By Taylor's theorem, a firm must be located in the interior in equilibrium. This leads us to the following conclusion.

THEOREM 6: *If S is a convex, compact market region, then if (z_A^*, z_B^*) are equilibrium locations,*

$$\nabla_{z_i} \Pi_i(z_A^*, p^*, z_B^*, p^*) = 0 \quad \textit{for } i \in \{A, B\}.$$

Also, z_i^* minimizes transport and production cost to customers in $S_i(z_A^*, z_B^*)$.

PROOF: If $\iint_{S_i(z_A^*, z_B^*)} \rho(z)\, dz > 0$ then, by the first order necessary conditions of unconstrained optimization $\nabla_{z_i} \Pi_i(z_A^*, p^*, z_B^*, p^*) = 0$ because of the interior location of z_i^*. If $\iint_{S_i(z_A^*, z_B^*)} \rho(z)\, dz = 0$ we get the same result. $\nabla_{z_A} \Pi_A(z_A^*, p^*, z_B^*, p^*) = 0$ implies that the concave function $-\iint_{S_A(z_A^*, z_B^*)} (f_i(z_A, z_B^*) + C_i'(\rho(z)\, dz$ is at a global maximum, by the sufficient conditions for optimization of a concave function. Thus, transport and production cost associated with customers in $S_A(z_A^*, z_B^*)$ is minimized at z_A^*. Likewise for z_B^* by a similar argument.

Q.E.D.

Next, we will require the market region served to be a "circular" market region: the market is a disc with the center at the origin having radius 1. Also we will assume that ρ is uniform. Our goal is to demonstrate in this simple situation that coincident location for *nonidentical* duopolists will never be an equilibrium. Developing this result in steps, we have the following theorem.

THEOREM 7: *If (z_A^*, z_B^*) is a location equilibrium for two nonidentical firms in a circular market S with an uniform distribution of customers, then z_A^*, z_B^* lie on a diameter of S.*

This result is demonstrated by a symmetry argument detailed in the Appendix. Next, we show that the firms do not lie on the same side of the origin on the diameter.

THEOREM 8: *For the circular market having an uniform demand distribution, if (z_A^*, z_B^*) is a location equilibrium for nonidentical firms with $z_A^* \neq z_B^*$, and $S_A(z_A^*, z_B^*)$, $S_B(z_A^*, z_B^*)$ are markets having positive demand, then z_A^*, z_B^* must lie on opposite sides of the origin on a common diameter.*

PROOF: (z_A^*, z_B^*) may be assumed to lie on the x axis. We will show that:

$$\frac{\partial \Pi_i}{\partial x_i}(z_A^*, p^*, z_B^*, p^*) = 0 \quad \text{implies that}$$

$$x_A^* < 0 \text{ and } x_B^* > 0 \quad \text{(or vice versa).}$$

From Lemma 5:

$$\frac{\partial \Pi_A}{\partial x_A}(z_A, p^*, z_B^*, p^*) = \iint_{S_Z(z_A, z_B^*)} \alpha_A \frac{x - x_A}{\|z - z_A\|_2} \rho(z)\, dz.$$

Define the function:

$$F(x_A) = -\iint_{S_A(z_A^*, z_B^*)} f_A((x_A, 0), z)\rho(z)\, dz.$$

$F(x_A)$ is strictly concave if $S_A(z_A^*, z_B^*)$ is a set having positive demand, which it is by hypothesis. Also,

$$\frac{\partial F(x_A)}{\partial x_A} = -\iint_{S_A(z_A^*, z_B^*)} \alpha_A \frac{x - x_A}{\|z - z_A\|_2} \rho(z)\, dz.$$

The function $F(x_A)$ has a unique maximum at the point $\hat{x}_A$ where $\partial F_A(\hat{x}_A)/\partial x_A = 0$. Therefore, if $\iint_{S_i(z_A^*, z_B^*)} \alpha_i(x/\|z\|_2)\rho(z)\, dz = 0$ for $i \in \{A, B\}$, then $(z_A^*, z_B^*) = (\bar{0}, \bar{0})$, by the uniqueness of the maxima of F and a similarly defined function of z_B, and the first order conditions described in Theorem 6. By assumption $z_A^* \neq z_B^*$, therefore,

$$\iint_{S_A(z_A^*, z_B^*)} \frac{x}{\|z\|_2} \rho(z)\, dz \neq 0.$$

If

$$-\iint_{S_A(z_A^*, z_B^*)} \frac{x}{\|z\|_2} \rho(z)\, dz < 0, \quad \text{then}$$

$$-\iint_{S_B(z_A^*, z_B^*)} \frac{x}{\|z\|_2} \rho(z)\, dz > 0,$$

since by symmetry of ρ and S

$$\iint_S \frac{x}{\|z\|_2} \rho(z)\, dz = 0.$$

If

$$-\iint_{S_A(z_a^*, z_B^*)} \frac{x}{\|z\|_2} \rho(z)\, dz < 0,$$

then $\partial F(0)/\partial x_A < 0$, and x_A^* such that $\partial F(x_A^*)/\partial x_A = 0$ must lie to the left of 0: $x_A^* < 0$. Likewise for x_B^*: $x_B^* > 0$. *Q.E.D.*

Finally, we have Theorem 9.

THEOREM 9: *Let two nonidentical firms be located in a circular market region having uniform demand distribution and let $z_A^* = z_B^* = \bar{0}$. Assume that $\alpha_A > \alpha_B$ and $C'_A < C'_B$. Suppose that $S_A(\bar{0}, \bar{0})$, $S_B(\bar{0}, \bar{0})$ are market regions having positive demand.*

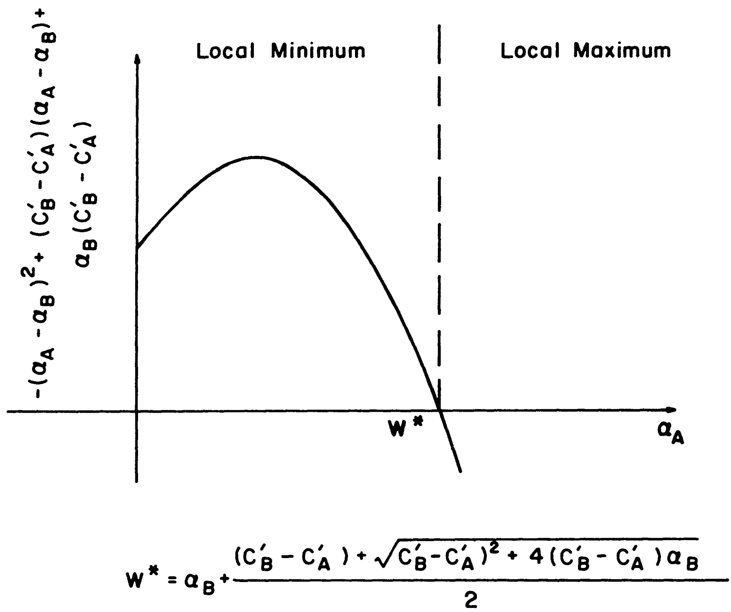


FIGURE 2

Then:

$$\frac{\partial^2 \Pi}{\partial x_A^2}(\bar{0}, p^*, \bar{0}, p^*) > 0,$$

$$\frac{\partial^2 \Pi_B}{\partial x_B^2}(\bar{0}, p^*, \bar{0}, p^*) = -(\alpha_A - \alpha_B)^2 + (C'_B - C'_A)(\alpha_A - \alpha_B) + \alpha_B(C'_B - C'_A),$$

and (z_A^*, z_B^*) *cannot be equilibrium locations.*

The proof is a computation found in the Appendix.

With coincident location at the origin, the firm with the higher transport rate and lower marginal cost of production will be located at a local minimum of its profit as a function of its location choice. The firm with the higher marginal cost of production will be at a local minimum of its profit with respect to its location if the difference between the firm's transport rates is small. If the difference is large, it will be at a local maximum. Holding $C'_B - C'_A$ fixed, we may explicitly state what "small" and "large" mean. See Figure 2. We can summarize the last three theorems and state the following:

THEOREM 10: *Let two nonidentical firms be located in a circular market region having a uniform demand distribution. If* (z_A^*, z_B^*) *are equilibrium locations and both firms serve markets having positive demand, then* $z_A^* \neq z_B^*$ *and the firms must be located on a common diameter on opposite sides of the origin.*

EXTENSIONS

Our model of two firm competition is readily generalizable. In subsequent works we will supply the details; here we point out obvious directions.

The model may be extended to study n-firm competition, and it may be seen that a location-price equilibrium will exist. In addition, we may allow each firm to control a number of mobile plants or warehouses which may or may not be identical. The coordination problem of the firm in the context of competition may be studied and again a location-price equilibrium may be shown to exist.

We can also allow the firms to have location dependent fixed costs and marginal costs of production. Specifically, we may allow the firms to have linear homogeneous production functions, and assume there to be fixed factor sites in the market region. Then the firm's problem will be to compete in price and location, all the while worrying about how location affects the optimal mix of factors and the marginal cost of production. The existence of a location-price equilibrium and its properties are presented in Hurter and Lederer [**10**].

The results of this work are general in compact spaces other than of location in a subset of $\mathbb{R}^2$ because no special properties of $\mathbb{R}^2$ were used in establishing a location-price equilibrium. We need only require that whatever the compact space along with an appropriately defined measure in which we are working, f_i^2 and ρ^2 are both integrable. This means a location-price equilibrium exists in problems of competitive location in discrete point markets, and on networks, and in spaces of dimension higher than two. Application of our model to problems of competitive network design, i.e., in the transport and telecommunications industries is found in Lederer [**11**].

The results also apply in location problems involving product differentiation, i.e., product attributes. In this case, firms are locating their technology or product and are customizing the product to customers' specifications and setting a "delivered" price for this service. Thus, competitive product differentiation under discriminatory pricing can be studied with our model.

This model is also applicable to many firm competitors with multiple facilities. The development of that model, found in Lederer [**12**], is similar to that presented here and a location-price equilibrium is shown to exist.

It may be asked does a location-price equilibrium exist in markets with elastic demand? Unfortunately our methods which rely upon the decomposition of the firm's profit function, willl not be useful. It can be shown that for specific examples a location-price equilibrium will exist. However, general existence has yet to be demonstrated. Some results can be found in Lederer [**12**].

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APPENDIX

PROOF OF LEMMA 1: Suppose for $i=\{A\}$ the implication does not hold on some $T\subset S$ having positive demand. Then, there is a $T'\subset T$ having positive demand such that $z\in T'$ implies $f_A(z_A, z)+C'_A-p_A(z)>\delta>0$. Defining the price policy $p^{\varepsilon}_A(z)=\max[p_A(z)-\varepsilon, f_A(z_A, z)+C'_A]$, for sufficiently small $\varepsilon>0$, A's profit on this subgame will rise. Therefore, d is not subgame perfect, a contradiction. *Q.E.D.*

PROOF OF LEMMA 2: Suppose T_A is a set having positive demand. Then some firm, say A, is serving customers in T_A even though it does not have the cost advantage, and there exists a subset $T\subset T_A$ having positive demand such that $f_A(z_A, z)+C'_A>f_B(z_B, z)+C'_B$ on T. By Lemma 1, for some $\delta>0$, there is a subset $T'\subset T$ having positive demand on which $p_A(z)-f_B(z_B, z)-C'_B>\delta$. Defining, $p^{\varepsilon}_B=\{p_B(z)-\varepsilon$ if $z\in S_C(p_A, p_B)$, $p_B(z)$ otherwise$\}$, for sufficiently small $\varepsilon>0$, B can increase his profit by serving customers on T'. Therefore, d is not subgame perfect, a contradiction. *Q.E.D.*

PROOF OF LEMMA 3: If $p_A(z)<f_A(z_A, z)+C'_A$ on some set S_A having positive demand, p_A is dominated on the subgame commencing at (z_Az_B). Conversely, suppose for all $(z_A, z_B)\in S\times S$, p_A obeys (1). Fix $(z_A, z_B)\in S\times S$. There exists a p_B such that p_A is an optimal response to p_B. For any $\hat{p}_A$ such that A's profits are identical using either p_A or $\hat{p}_A$ against p_B, and $\{z\in S|p_A(\ )\neq\hat{p}_A(\ )\}$ is a set having positive demand, there exists a $\hat{p}_B$ such that p_A is superior to $\hat{p}_A$ against $\hat{p}_B$. Therefore, there does not exist a $\hat{d}_A$ dominating d_A on the subgame commencing at (z_Az_B) and this is true for all such pairs in $S\times S$. Similarly d_A can be show to be undominated on the whole game. *Q.E.D.*

PROOF OF THEOREM 1: Let d be a location-price equilibrium and let $(z_A, z_B)\in S\times S$. Suppose $d_i(z_Az_B)=p^*_i(z_A, z_B)$ for $i\in(A, B)$. We will first show $p^*_A(z_A, z_B, \cdot)=p^*_B(z_A, z_B, \cdot)$, edz.

Suppose that there is some $T\subset S_A(z_A, z_B)=\{z\in S|f_A(z_A, z)+C'_A<f_B(z_B, z)+C'_B\}$ having positive demand on which $p^*_A\neq p^*_B$. Then for one of the firms, say A, there exists a $T'\subset T$, having positive demand, on which $p^*_A(z_A, z_B, z)-p^*_B(z_A, z_B, z)>\delta>0$. A can price at p^*_B on T' and lower its current price by ε outside T', and for sufficiently small $\varepsilon>0$, increase its profit. Therefore, d is not subgame perfect. Instead, if $p^*_B(z_A, z_B, z)-p^*_A(z_A, z_B, z)>\delta>0$ on T', then A can raise its price by at least δ on T' and lower its current price by ε outside T' and for sufficiently small $\varepsilon>0$, increase its profit, again a contradiction. Therefore, $p^*_A=p^*_B$ on S_A, edz. Similarly, it can be shown $p^*_A=p^*_B$ on S_B, edz.

Now consider $S_C(z_A, z_B)=\{z\in S|f_A(z_A, z)+C'_A=f_B(z_B, z)+C'_B\}$. Suppose there is a $T\subset S_C$ having positive demand on which $p^*_A\neq p^*_B$. Then for some firm, say A, there is a $T'\subset T$ having positive demand where $p^*_B-p^*_A>\delta>0$. Firm A can reset its price by undercutting B's price on T' by ε and reducing its current price outside of T' by ε, and for sufficiently small $\varepsilon>0$ raise its profit. This again contradicts subgame perfectness and we conclude, $p^*_A=p^*_B$, edz.

Next, we show that $p^*(z_A, z_B, z)=p^*_A(z_A, z_B, z)=p^*_B(z_A, z_B, z)=\max[f_A(z_A, z)+C'_A, f_B(z_B, z)+C'_B]$. Suppose that on some subset $T\subset S$ having positive demand, $p^*(z_A, z_B, z)-\max[f_A(z_A, z)+C'_A, f_B(z_B, z)+C'_B]>\delta>0$. There are three cases.

1. Suppose there is a subset $T'\subset T$ having positive demand such that $T'\subset S_A(z_A, z_B)\cap T$. Then, firm B can adopt the price policy $p^{\varepsilon}_B(z_A, z_B, z)=p^*(z_A, z_B, z)-\varepsilon$ for all $z\in S$. For sufficiently small $\varepsilon>0$, B's profits will increase. d is not subgame perfect.

2. The same as above, except $T'\subset S_B(z_A, z_B)\cap T$. The argument is the same. Now, A can readjust its price schedule and increase its profit. Again we see d is not subgame perfect.

3. Suppose there is a subset T' of T, having positive demand such that $T'=S_C(z_A, z_B)\cap T$. Then, firm A can increase its profit by matching B's price on T and reducing its current price by $\varepsilon>0$ outside of T', for sufficiently small $\varepsilon>0$. We see that d is not subgame perfect. *Q.E.D.*

PROOF OF THEOREM 7: Suppose that z^*_A, and z^*_B are not on a common diameter and each firm serves a market with positive demand. Assume, without loss of generality, that $y^*_A=y^*_B>0$. Let $w=(0, 1)$, set $\alpha=z^*_A\cdot w=y^*_A$. Define $S_1=S_A(z^*_A, z^*_B)\cap\{z\in S|z\cdot w\geq\alpha\}$, $S_2=S_A(z^*_A, z^*_B)\cap\{z\in S|z\cdot w\leq\alpha\}$. S_2 can be partitioned into two sets S_{21}, S_{22} both having positive demand such that $S_{21}\cup S_{22}=S_2$, and such that S_{21} is the reflection of S_1 in the line $y=y^*_A$. For almost all $z\in S_{22}$, $w\cdot(z-z_A)>0$ and by the symmetry of S_1 and S_{21}, $\iint_{S_1}(w\cdot z)\rho(z)\,dz=-\iint_{S_{21}}(w\cdot z)\rho(z)\,dz$. Also $\iint_{S_{22}}w\cdot(z-z_A)\rho(z)\,dz>0$.

By Lemma 5

$$\nabla_{z_A}\Pi_A(z_A^*, p^*, z_B^*, p^*)\cdot w = -\iint_{S_A(z_A^*, z_B^*)} (\nabla_{z_A} f_A(z_A^*, \tilde{z}))\cdot w\rho(z)\,dz$$
$$= \iint_{S_1} \alpha_A \frac{(z-z_A)}{\|z-z_A\|_2}\cdot w\rho(z)\,dz$$
$$+ \iint_{S_{21}} \alpha_A \frac{(z-z_A)}{\|z-z_A\|_2}\cdot w\rho(z)\,dz$$
$$+ \iint_{S_{22}} \alpha_A \frac{(z-z_A)}{\|z-z_A\|_2}\cdot w\rho(z)\,dz > 0.$$

Because $\nabla_{z_A}\Pi_A(z_A, p^*, z_B^*, p^*)$ is continuous in z_A at z_A^*, by Taylor's theorem $\Pi_A(z_A^* + \lambda w, p^*, z_B^*, p^*) > \Pi_A(z_A^*, p^*, z_B^*, p^*)$ for some $\lambda > 0$ sufficiently small. Therefore (z_A^*, z_B^*) are not equilibrium locations. *Q.E.D.*

PROOF OF THEOREM 9: Suppose $C'_A < C'_B$, $\alpha_A > \alpha_B$, $z_A = z_B = \bar{0}$, and $\rho(\cdot) = 1$. Then $S_A(\bar{0}, \bar{0}) = \{z \in S \mid \|z\|_2 \leq r\}$ where $r = (C'_B - C'_A)/(\alpha_B - \alpha_A)$ and $r < 1$. We may calculate

$$\frac{\partial \Pi_A}{\partial x_A}(\bar{0}, p^*, \bar{0}, p^*) = 0.$$

In some neighborhood of $x_A = 0$ we may parameterize the region $S_A((x_A, 0), \bar{0})$ and write

$$\frac{\partial \Pi_A}{\partial x_A}((x_A, 0), p^*, \bar{0}, p^*) = \int_{-y(x_A)}^{y(x_A)} \left(\int_{\alpha(x_A, y)}^{\beta(x_A, y)} \frac{\partial f_A}{\partial x_A}((x_A, 0), z)\,dx\right) dy$$

with $y(0) = r$ and $\beta(0, y) = \sqrt{r^2 - y^2}$ and $\alpha(0, y) = -\sqrt{r^2 - y^2}$.

Differentiating,

$$\frac{\partial \Pi_A^2}{\partial x_A^2}(\bar{0}, p^*, \bar{0}, p^*) = -\int_{-y(0)}^{y(0)} \frac{\partial}{\partial x_A}\left(\int_{\alpha(x_A, y)}^{\beta(x_A, y)} \frac{\partial f_A}{\partial x_A}((x_A, 0), z)\,dx\right)\Bigg|_{x_A = 0} dy$$
$$-\frac{\partial y}{\partial x_A}(0)\left(\int_{\alpha(0,r)}^{\beta(0,r)} \frac{\partial f_A}{\partial x_A}(\bar{0}, z)\,dx\right) + \frac{\partial y}{\partial x_A}(0)\left(\int_{\alpha(0,-r)}^{\beta(0,-r)} \frac{\partial f_A}{\partial x_A}(\bar{0}, z)\,dx\right)$$
$$= -\int_{-r}^{r}\left(\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{\partial^2 f_A}{\partial x_A^2}(\bar{0}, (x, y))\,dx\right) dy - \int_{-r}^{r}\left(\frac{\partial f_A}{\partial x_A}(0, (x, y))\frac{\partial \beta}{\partial x_A}(0, y)\right.$$
$$\left. - \frac{\partial f_A}{\partial x_A}(0, (x, y))\frac{\partial \alpha}{\partial x_A}(0, y)\right) dy.$$

But

$$\frac{\partial \beta}{\partial x_A}(0, y) = -\frac{\alpha_A}{\alpha_B - \alpha_A}, \qquad \frac{\partial \alpha}{\partial x_A}(0, y) = -\frac{\alpha_A}{\alpha_B - \alpha_A}, \qquad \frac{\partial^2 f_A}{\partial x_A^2} = -\frac{\partial^2 f_A}{\partial x_A\,\partial x}$$

and

$$\frac{\partial f_A}{\partial x_A}(z_A, z) = -\alpha_A \frac{z - z_A}{\|z - z_A\|_2}.$$

Therefore, substituting and integrating

$$\frac{\partial^2 \Pi_A}{\partial x_A^2}(\bar{0}, p^*, \bar{0}, p^*) = 2\alpha_A\left(\int_{-r}^{r} \frac{\sqrt{r^2 - y^2}}{r}\,dy\right)\left(-1 - \frac{\alpha_A}{\alpha_B - \alpha_A}\right) > 0.$$

We may similarly compute for B:

$$\frac{\partial \Pi_B}{\partial x_B}(\bar{0}, p^*, \bar{0}, p^*) = 0;$$

$$\frac{\partial^2 \Pi_B}{\partial x_B^2}(\bar{0}, p^*, \bar{0}, p^*) = -\int_{-1}^{1}\left(\int_{-\sqrt{1-r^2}}^{\sqrt{1-y^2}} \frac{\partial^2 f_B}{\partial x_B^2}(\bar{0}, z)\, dx\right) dy + \int_{-r}^{r}\left(\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{\partial^2 f_B}{\partial x_B^2}(\bar{0}, z)\, dx\right) dy$$

$$-\int_{-r}^{r} \frac{\partial f_B}{\partial x_B}(\bar{0}, z)\bigg|_{(\sqrt{r^2-y^2}, y)} \frac{\alpha_B}{\alpha_B - \alpha_A}\, dy$$

$$+\int_{-r}^{r} \frac{\partial f_B}{\partial x_B}(\bar{0}, z)\bigg|_{(\sqrt{r^2-y^2}, y)} \frac{\alpha_B}{\alpha_B - \alpha_A}\, dy$$

$$= -2\alpha_B \int_{-1}^{1} \sqrt{1-y^2}\, dy + 2\alpha_B\left(1 - \frac{\alpha_B}{\alpha_B - \alpha_A}\right)\int_{-r}^{r} \frac{\sqrt{r^2-y^2}}{r}\, dy$$

$$= -(\alpha_A - \alpha_B)^2 + (C'_B - C'_A)(\alpha_A - \alpha_B) + \alpha_B(C'_B - C'_A). \qquad Q.E.D.$$

REFERENCES

[1] BERTRAND, J.: "Theorie Mathematique de la Richesse Social," *Journal des Savants*, 1883, 499-503.

[2] D'ASPREMONT, C., J. GABSZEWICZ, AND J.-F. THISSE: "On Hotelling's Stability in Competition," *Econometrica*, 47(1979), 1145-1150.

[3] EATON, B. C.: "Spatial Competition Revisited," *Canadian Journal of Economics*, 5(1972), 268-278.

[4] FRIEDMAN, J. W.: *Oligopoly and Theory of Games.* Amsterdam: North-Holland, 1977.

[5] GROSSMAN, SANDFORD J.: "Nash Equilibrium and the Industrial Organization of Markets with Large Fixed Costs," *Econometrica*, 49(1981), 1149-1172.

[6] HADDOCK, D. D.: "Basing Point Pricing: Competitive vs. Collusive Theories," *American Economic Review*, 82(1982), 289-306.

[7] HARTWICK, J. M., AND P. G. HARTWICK: "Duopoly in Space," *Canadian Journal of Economics*, 4(1971), 483-505.

[8] HOOVER, E.: "Spatial Price Discrimination," *Review of Economic Studies*, 4(1936), 182-191.

[9] HOTELLING, HAROLD: "Stability in Competition," *Economic Journal*, 39 (1929), 4-57.

[10] HURTER, A. P., JR., AND P. J. LEDERER: "Spatial Duopoly With Discriminatory Pricing," *Regional Science and Urban Economics*, 15(1985), 541-553.

[11] LEDERER, P. J.: "Duopoly Competition in Networks," in *Location Decisions: Methodology and Applications*, ed. by J. P. Osleeb. Basel: J. C. Baltzer AC, Scientific Publishing Company, 1986.

[12] ———: "Location-Price Games," Ph.D. Dissertation, Northwestern University, Evanston, Illinois, August, 1981.

[13] SELTEN, R.: "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Form Games," *International Journal of Game Theory*, 4(1976), 25-55.

[14] SMITHIES, ARTHUR: "Optimum Location in Spatial Competition," *Journal of Political Economy*, 41(1941), 423-439.