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Complementary Distance Pattern Uniform Graphs

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Abstract

A graph $G = (V, E)$ is *Complementary Distance Pattern Uniform* if there exists $M \subset V(G)$ such that $f_M(u) = \{d(u, v) : v \in M\}$, for every $u \in V(G) - M$, is independent of the choice of $u \in V(G) - M$ and the set M is called the Complementary Distance Pattern Uniform Set (CDPU set). The least cardinality of CDPU set in G is called the *CDPU number* of G.

Keywords: Complementary Distance Pattern Uniform, CDPU set, CDPU number

1 Introduction

For all terminology and notation in graph theory, not defined specifically in this paper, we refer the reader to Harary [3]. Unless mentioned otherwise, all the graphs considered in this note are simple, self-loop-free and finite.

Let $G = (V, E)$ represent the structure of a chemical molecule. Often, a topological index (TI), derived as an invariant of G, is used to represent a chemical property of the molecule. There are a number of TIs based on distance concepts in graphs [4] and some of them could be designed using distance patterns of vertices in a graph. There are strong indications in the literature [4] that the notion of CDPU sets in G could be used to design a class of TIs that represent certain stereochemical properties of the molecule.

B.D.Acharya define the M - distance pattern of a vertex as follows :

Definition 1.1. [5] Given an arbitrary non-empty subset M of vertices in a graph $G = (V, E)$, each vertex $u \in G$ is associated with the set $f_M(u) = \{d(u, v) : v \in M\}$, where $d(u, v)$ denotes the usual distance between the vertices u and v in G, is called the M- vertex distance pattern of u. If for a subset M of vertices in a graph $G = (V, E)$, f_M is injective, then the set M is called the distance pattern distinguishing set (DPD-set in short).

As another version of *distance-pattern distinguishing set* (or, a 'DPD-set') of G, we define *Complementary Distance Pattern Uniform (CDPU) Graph* as follows:

Definition 1.2. If $f_M(u)$ is independent of the choice of $u \in V - M$, then G is called a Complementary Distance Pattern Uniform (CDPU) Graph and the set M is called the CDPU set.

Theorem 1.3. *Every connected graph has a CDPU set.*

Proof. Let G be a connected (p, q) graph with $p \geq 2$. For $u \in V(G)$, let $M = V(G) - \{u\}$. Then, clearly M is a CDPU set of G. □

Corollary 1.4. *All connected graphs are CDPU.*

Definition 1.5. The least cardinality of CDPU set in G is called the *CDPU number* of G, denoted $\sigma(G)$.

Theorem 1.6. *Every self centered graph of order p has a CDPU set M with $|M| \leq p - 2$.*

Proof. Let G be a self-centered graph. Then $e(v) = r(G)$, for all $v \in G$, where $r(G)$ is the radius of G. Take $M = V(G) - \{u, v\}$, where u, v are any two adjacent vertices of G. Then $f_M(u) = \{1, 2, \ldots, r(G)\}$ and $f_M(v) = \{1, 2, \ldots, r(G)\}$. Therefore, M is a CDPU set in G. Thus, for a self-centered graph G, $\sigma(G) \leq |V(G)| - 2$. □

Corollary 1.7. *For a self centered graph G, max $f_M(v) = rad(G)$, for every $v \in V(G) - M$.*

Theorem 1.8. *If G is a self-median graph of order $n(2n - 13), n \geq 8$, then $\sigma(G) \leq 2n(n - 7)$.*

Proof. Let G be a self-median graph. As well known [**?**], one can construct G with C_n, $n \geq 8$ and two disjoint copies of $K_{n(n-7)}$, by joining each vertex of C_n to $n - 7$ vertices in each copy of $K_{n(n-7)}$ so that each vertex in each copy of $K_{n(n-7)}$ is adjacent to precisely one vertex of the cycle. Choose M as the set of all vertices in both the copy of $K_{n(n-7)}$. Then $f_M(v_i) = \{1, 2\}$ for every $v_i \in C_n$. Hence the theorem. □

Remark 1.9. Let G be a connected graph of order p and let $(e_1, e_2, \ldots, e_k)$ be the non decreasing sequence of eccentricities of its vertices. Let M consists of the vertices with eccentricities $e_1, e_2, \ldots, e_{k-1}$ and let $|V - M| = p - m$ where $|M| = m$. Then $\sigma(G) \leq m$, since all the vertices in $V - M$ have $f_M(v) = \{1, 2, \ldots, e_{k-1}\}$.

Theorem 1.10. *Let G be a connected non-self centered graph on n vertices and k distinct eccentricities. Then there are exactly k distinct CDPU sets for G.*

Proof. Let G be a connected non-self centered graph with $V(G) = \{v_1, v_2, \ldots, v_n\}$. Suppose that G has exactly k different eccentricities. Let the vertices corresponding to e_i be $\{v_{i1}, v_{i2}, \ldots, v_{im}\}$. Take
$M_i = V(G) - \{vertices\ corresponding\ to\ eccentricity\ e_i\}$. Then $f_M(v_i) = \{1, 2, \ldots, e_i\}$, for every $v_i \in V - M$. Since G has k different eccentricities, we get k distinct CDPU sets for G.

To show that there are exactly k CDPU sets for G, take $V - M = \{v_{i1}, v_{i2}, \ldots, v_{im}, v_j\}$, where v_j is a vertex corresponding to eccentricity e_j. Then $f_M(v_i) = \{1, 2, \ldots, j-1, j+1, \ldots, e_i\}$, for every $v_i \in V - \{M, v_j\}$ and $f_M(v_j) = \{1, 2, \ldots, j-1, j+1, \ldots, e_j\}$. Hence the distance pattern is different showing that there are exactly k CDPU sets. □

Corollary 1.11. *Let ε denotes the set of all different eccentricities of a non-self centered graph G and ζ denotes the set of all possible CDPU sets of G. Then $|\varepsilon| = |\zeta|$.*

Remark 1.12. If G is a non-self centered graph, then all the vertices in the complement of the CDPU set should have the same eccentricity.

Theorem 1.13. *Let G be a graph with n vertices. If G is a self centered graph, then $1 \leq \sigma(G) \leq n - 2$. If G is not a self centered graph, then $1 \leq \sigma(G) \leq n - r$, where r is the number of vertices with maximum eccentricity.*

Proof. The proof follows from Theorem 1.6 and Remark 1.9 □

Theorem 1.14. *Let G be a non self centered graph with exactly two different eccentricities. Then $diam(G) \leq 3$.*

Proof. When $diam(G) = 1$, G is a complete graph, which is not the case.

Then it is enough to prove that $diam(G)$ is either two or three. The smallest graph with exactly two different eccentricities are P_3 and $K_{1,2}$. Let $V(P_3) = \{v_1, v_2, v_3\}$, whose diameter is two. When we add any edge to the vertex v_2, say, (v_2, v_i), then it becomes a star whose diameter is also two. Also if $e_i = (v_i, v_3)$ is any edge, then the diameter is also two.

The next possibility is to add edges to the antipodal vertex, say, (v_3, v_4). Then the diameter of the graph increases by one. Hence $e(v_1) = e(v_4) = 3$ and $e(v_2) = e(v_3) = 2$. Thus $diam(G) = 3$. □

Corollary 1.15. *If G is a non-self centered graph having no full degree vertex, then $diam(G) = 3$.*

Theorem 1.16. *Let G be a non-self centered graph having no full degree vertex. Then $\sigma(G) = 2$ if G has exactly two different eccentricities, with the number of vertices corresponding to atleast one of the eccentricities should be two.*

Proof. Suppose that G has exactly two different eccentricities, say, e_i and e_j. The by Corollary 1.15, $diam(G) = 3$. Hence G atleast four vertices with $e_i = 3$ and $e_j = 2$. Therefore atleast two vertices should have eccentricity two and three. since the number of vertices corresponding to atleast one of the eccentricity should be two, we get $\sigma(G) = 2$. $\square$

Conjecture 1. Let G be a non-self centered graph having no full degree vertex. Then $\sigma(G) = 2$ if and only if G has exactly two different eccentricities with the number of vertices corresponding to atleast one of the eccentricity should be two.

Remark 1.17. For a graph G which is not self centered, $\max.f_M(v) \leq diam(G) - 1$.

Theorem 1.18. *A graph G has $\sigma(G) = 1$ if and only if G has atleast one vertex of full degree.*

Proof. Suppose that G has one vertex v_i with full degree. Take $M = \{v_i\}$. Then $f_M(u) = \{1\}$, for every $u \in V - M$. Hence $\sigma(G) = 1$.
Conversely, suppose that G is a graph with $\sigma(G) = 1$. That is, there exists an M which contains only one vertex v_i which is a CDPU set of G. Also $\sigma(G) = 1$ implies that v_i is adjacent to all other vertices. Hence v_i is a vertex with full degree. $\square$

Corollary 1.19. *For any graph G, if $\sigma(G) = 1$, then $r(G) = 1$.*

Proof. Let G be a graph with $\sigma(G) = 1$. Then from Theorem 1.18, G has a vertex, say, v_i of full degree. Then $e(v_i) = 1$. Hence $r(G) = 1$. $\square$

Theorem 1.20. *For any integer n, $\sigma(P_n) = n - 2$.*

Proof. Let P_n be the path on n vertices and $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. Choose M as the set of all cut vertices, $\{v_2, v_3, \ldots, v_{n-1}\}$. Then $f_M(v_1) = \{1, 2, \ldots, n - 2\}$ and $f_M(v_n) = \{1, 2, \ldots, n - 2\}$. Therefore $\sigma(P_n) \leq n - 2$.

Next we have to show that $\sigma(G) \not< n - 2$. For a path P_n, there are atmost two vertices with same eccentricity. If three vertices are outside M, then atleast one of the vertices should have different eccentricity and the distance pattern of that vertex is different. Hence $\sigma(P_n) = n - 2$. $\square$

Theorem 1.21. *For all integers $a_1 \geq a_2 \geq \cdots \geq a_n \geq 2$, $\sigma(K_{a_1,a_2,\ldots,a_n}) = min\{min.\{a_1, a_2, \ldots, a_n\}, n\}$.*

Proof. Let $G = K_{a_1,a_2,\ldots,a_n}$ be a complete n - partite graph. Then $V(G)$ can be partitioned into n subsets $V_1, V_2, \ldots, V_n$ where $|V_1| = a_1, |V_2| = a_2, \ldots, |V_n| = a_n$.
Case1: Take one vertex from each partite set of $K_{a_1,a_2,\ldots,a_n}$ to constitute the set M. Since each element of a partite set is non-adjacent to the other vertices in it and is adjacent to all other partite sets, we get, $f_M(v) = \{1,2\}, \forall v \in V(K_{a_1,a_2,\ldots,a_n}) - M$. Hence $\sigma(K_{a_1,a_2,\ldots,a_n}) \leq n$.

Next suppose that no vertex from the partite set V_i belong to M. Let $v_i \in V_i$. Then $f_M(v_i) = \{1\}$, for every $v_i \in V_i$ and $f_M(u) = \{1,2\}$ for every $u \in V(G) - M$, u does not belong to V_i. Hence M is not a CDPU set.
Case 2: Let M_i corresponds to the partite set V_i. Then $f_{M_i}(u) = \{1\}$, for every $u \in V(K_{a_1,a_2,\ldots,a_n}) - M_i$. Hence all M_i's form CDPU sets. Thus $\sigma(K_{a_1,a_2,\ldots,a_n}) \leq min.\{a_1, a_2, \ldots, a_n\}$.

Next suppose that $v_i \in V_i$ does not belong to M_i. Then $f_M(v_i) = \{1,2\}$ and $f_M(u) = \{1\}$, for every $u \in V(G) - M$. Hence M_i is not CDPU.

Thus $\sigma(K_{a_1,a_2,\ldots,a_n}) = min\{min.\{a_1, a_2, \ldots, a_n\}, n\}$. □

Corollary 1.22. $\sigma(K_{a_1,a_2}) = 2$.

Proposition 1. $\sigma(C_n) \leq \begin{cases} n-2, & \text{if n is odd;} \\ \frac{n}{2}, & \text{if } n \geq 8 \text{ is even.} \end{cases}$

Proof. Let C_n be a cycle on n vertices and $V(C_n) = \{v_1, v_2, \ldots, v_n\}$.
Case 1 : n is even.
Choose M as the set of alternate vertices on C_n, say, $\{v_2, v_4, \ldots, v_n\}$. Then for $i = 1, 3, \ldots, n-1$, $f_M(v_i) = \begin{cases} \{1,3,5,\ldots,m-1\}, & \text{if } C_n = 2m \text{ and m is even;} \\ \{1,3,5,\ldots,m\}, & \text{if } C_n = 2m \text{ and m odd.} \end{cases}$
Therefore $f_M(v_i)$ is identical depending on whether m is odd or even. Hence $\sigma(C_n) \leq \frac{n}{2}$.
Case 2 : n is odd.
Choose $n-2$ adjacent vertices, say, $\{v_3, v_4, \ldots, v_n\}$. Then $f_M(v_1) = \{1, 2, 3, \ldots, \frac{n-1}{2}\}$, and $f_M(v_2) = \{1, 2, 3, \ldots, \frac{n-1}{2}\}$. Hence $\sigma(C_n) \leq n-2$. □

Remark 1.23. $\sigma(C_4) = \sigma(C_6) = 2$

Theorem 1.24. *$\sigma(G + \overline{K_m}) \leq m$ if G has no vertex of full degree.*

Proof. Let G be a graph with no vertices of G has full degree. In $G + \overline{K_m}$, every vertex of G is joined to every vertex of $\overline{K_m}$.

Case 1: G has more number of vertices than $\overline{K_m}$.
Take M as the set of all m vertices in $\overline{K_m}$. Then $f_M(v) = \{1\}, \forall v \in V - M$.Hence $\sigma(G + \overline{K_m} \leq m$. If we remove any vertex u from the above M, then $f_M(v) = \{1\}, \forall v \in G$ and $f_M(u) = \{1,2\}$. Hence it is not possible. Thus $\sigma(G + \overline{K_m}) = m$.

Case 2: G has lesser number of vertices than $\overline{K_m}$.
Then take M as the set of all vertices in G. Hence in this case $\sigma(G + \overline{K_m}) < m$. □

Theorem 1.25. *If $\sigma(G_1) = k_1$ and $\sigma(G_2) = k_2$, then $\sigma(G_1+G_2) = min\{k_1, k_2\}$.*

Proof. Let $\sigma(G_i) = k_i$, and M_i be a $\sigma(G)$-set (since it is a CDPU-set with $\sigma(G_i)$ vertices). In $G_1 + G_2$, every vertex of G_1 is joined to every vertex of G_2. Therefore, both M_1 and M_2 are CDPU sets of $G_1 + G_2$. If we remove any vertex from M_i, then it does not form a CDPU set for G_i, since $\sigma G_i = k_i$. Hence $\sigma(G_1 + G_2) = min(k_1, k_2)$. □

Corollary 1.26. *For any positive integer n, $\sigma(G + K_m) = 1$.*

Theorem 1.27. *Let G be a bipartite CDPU graph. Then $\sigma(G) = 1$ if and only if G is isomorphic to a star.*

Proof. Suppose $\sigma(G) = 1$. Then, from Theorem 1.18 there is atleast one vertex of full degree in G. Also G is bipartite. Thus G is isomorphic to a star. Conversely, if G is a star, then $\sigma(G) = 1$. □

Theorem 1.28. *Let T be a CDPU tree. Then $\sigma(T) = 1$ if and only if T is isomorphic to P_2, P_3 or $K_{1,n}$.*

Proof. When $T \cong P_2, P_3$ or $K_{1,n}$, clearly $\sigma(T) = 1$. Conversely, suppose that $\sigma(T) = 1$. Since $\sigma(T) = 1$, from Theorem 1.18, there is atleast one vertex of full degree in T. Also T is a tree. Thus the only trees which contains atleast one full degree vertices are P_2, P_3 and $K_{1,n}$. □

Theorem 1.29. *The shadow graph of a complete graph K_n has exactly two $\sigma(K_n)$ disjoint CDPU sets.*

Proof. Let $V(K_n) = \{u_1, u_2, \ldots, u_n\}$ and the shadow vertices be $\{u'_1, u'_2, \ldots, u'_n\}$. Let $M = V(K_n)$ and $M' = \{u'_1, u'_2, \ldots, u'_n\}$. We shall show that $\sigma(K_n) = n$. For M, we have $f_M(u_i) = \{1, 2\}, \forall 1 \leq i \leq n$. Also $f_M(u'_i) = \{1, 2\}, \forall 1 \leq i \leq n$. Also any $n-1$ vertices from M or M' will not form a CDPU set as the distance pattern of the vertex, say u_j (respectively, u'_j) $f_M(u_j) = \{1, 2, 3\}$ (respectively, $f_M(u'_j) = \{1, 2, 3\}$), a contradiction. A similar contradiction occur when we allow the CDPU set to be vertices from both M nd M'. Hence the proof □

More, generally we have the following theorem

Theorem 1.30. *If G is a self-centered graph of order p and $S(G)$ is its shadow graph then $\sigma(S(G)) = p$ and there are exactly two $\sigma(S(G))$-sets M_1 and M_2; further, $M_1 \cap M_2 = \emptyset$.*

Scope and Conclusion

As already stated in the introduction, the concept under study has important applications in the field of Chemistry. The study is interesting due to its applications in Computer Networks and Engineering, especially in Control System. In a closed loop control system, signal flow graph representation is used for gain analysis. So in certain control systems specified by certain characteristics, we can find out M of vertices consisting of two vertices such that one vertex will be the take off point and other vertex will be the summing point.

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References

[1] B.D. Acharya, Set-valuations And Their Applications, MRI Lecture notes in Applied Mathematics, No. 2, The Mehta Research Institute of Mathematics and Mathematical Physics, Allahabad, 1983.

[2] F. Buckley and F. Harary, Distance in Graphs, Addison-Wesley Publishing Company, 1990.

[3] F. Harary, Graph Theory, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1969.

[4] R. Natarajan and S. C. Basak, Numerical Characterization of Molecular Chiralty of Organic Compounds, Current Computer Aided Drug Design, 2009, 5, 13-22.

[5] Technical Report, DST grant-in-aid project No.SR/S4/277/06, funded by the Department of Science & Technology (DST), April 2009.

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