COMPLEMENTARY GRAPHS AND THE CHROMATIC NUMBER

Colin L. Starr and Galen E. Turner III

Abstract. Zykov proved that if G and $\overline{G}$ are complementary graphs having chromatic numbers χ and $\overline{\chi}$, respectively then $\chi \cdot \overline{\chi}$ is at least the number of vertices of G. Nordhaus and Gaddum gave an upper bound for $\chi \cdot \overline{\chi}$ and gave both upper and lower bounds for the analogue $\chi + \overline{\chi}$.

In this paper we characterize those graphs for which $\chi \cdot \overline{\chi}$ and $\chi + \overline{\chi}$ reach the bounds of Nordhaus and Gaddum.

1. Introduction. Unless stated otherwise, the terminology used here will follow Diestel [4]. In particular, a graph G is a finite set of elements $V(G)$, called *vertices*, and a set $E(G)$ of unordered pairs of vertices, called *edges*. The number of vertices of a graph G is denoted $|V(G)|$ and the complete graph on n vertices, denoted K_n, is a graph containing the set of all $n(n-1)/2$ possible edges. Two graphs G and $\overline{G}$ having the same n vertices are called *complementary* if each edge determined by the n vertices is in either G or $\overline{G}$ but not in both. If $X \subseteq V(G)$, the *subgraph induced by* X, denoted $G[X]$, is the subgraph H of G with $V(H) = X$ and $E(H) = \{xy \in E(G) | x, y \in X\}$. The *degree* of a vertex x is the number $d_G(x)$ of edges incident with it, and we denote the maximum and minimum vertex degrees of a graph by Δ and δ, respectively. If every vertex of a graph has degree k then we say that the graph is *k-regular*.

A k-coloring of a graph G is a coloring of the vertices of G so that no two adjacent vertices are colored with the same color and the total number of colors used is at most k. The chromatic number of a graph G is the smallest number of colors required to color the vertices of G. The *chromatic number* of G is denoted $\chi(G)$ or simply χ while the chromatic number of $\overline{G}$ is denoted $\overline{\chi}$; that is, $\overline{\chi} = \chi(\overline{G})$. The *clique number* of a graph G is the largest integer ω such that G contains a subgraph isomorphic to K_ω. The clique number of G is denoted by ω or $\omega(G)$, and $\overline{\omega} = \omega(\overline{G})$. It is easy to see that $\omega(G) \leq \chi(G)$ for any graph G. Moreover, it is easy to show that $\chi(G) \leq \Delta(G) + 1$. The following well-known theorem of Brooks [4] characterizes the graphs that are extremal with this property.

Theorem 1.1 (Brooks' Theorem). If G is a graph such that G is neither a clique nor an odd cycle, then $\chi(G) \leq \Delta(G)$.

A *factor* of a graph G is a subgraph containing all the vertices of G, and a *k-factor* is a factor that is k-regular. If we require every component of a factor to be a complete graph, then the factor is said to be *complete*; thus, a complete k-factor of a graph G is a factor of G all of whose components are isomorphic to K_{k+1}. Note that if G has a complete k-factor, then $k+1$ is a factor of the integer $|V(G)|$.

In 1949, Zykov [7] showed that $|V(G)| \leq \chi(G) \cdot \overline{\chi}(G)$. Nordhaus and Gaddum [6] extended this result by proving the following theorem.

<u>Theorem 1.2</u>. Let G and $\overline{G}$ be complementary graphs on n vertices having chromatic numbers χ and $\overline{\chi}$, respectively. Then

$$2\sqrt{n} \leq \chi + \overline{\chi} \leq n+1 \qquad \text{and} \qquad n \leq \chi \cdot \overline{\chi} \leq \left(\frac{n+1}{2}\right)^2.$$

In this paper, we characterize the graphs in the extreme cases of the inequalities in this theorem. Finck [5] offered a characterization of such graphs from one point of view, which was applied incorrectly in [2]. We find Finck's characterization somewhat unintuitive; our characterization approaches these graphs from a different perspective, which we explain in Sections 2 and 3.

Finally, it is worth noting that it is not difficult to show that every graph has a complete factor. To see this, consider a graph G and its complement $\overline{G}$. Let $T_1, T_2, \ldots, T_{\overline{\chi}}$ be the color classes of a $\overline{\chi}$-coloring of $\overline{G}$. In G, it is clear that the induced subgraph $G[T_i]$ is complete for every T_i. Thus, G has a complete factor, namely

$$\bigcup_{i=1}^{\overline{\chi}} G[T_i].$$

2. Graphs Satisfying the Lower Bounds. We begin with those graphs G for which $\chi \cdot \overline{\chi} = |V(G)|$. Clearly, any complete graph satisfies the equation; moreover, if $\chi(G) \cdot \overline{\chi}(G) = |V(G)|$, then both χ and $\overline{\chi}$ are integer factors of $|V(G)|$. What is surprising is that the graphs G and $\overline{G}$ must contain complete factors as we shall now show.

<u>Theorem 2.1</u>. Let G be a graph on n vertices. Then the following are equivalent.

(1) $\chi \cdot \overline{\chi} = n$
(2) G has a complete $(\chi - 1)$-factor
(3) $\overline{G}$ has a complete $(\overline{\chi} - 1)$-factor.

<u>Proof</u>. We begin by assuming (1) that $\chi \cdot \overline{\chi} = n$, and we will prove that this implies (2) G has a complete $(\chi - 1)$-factor and (3) $\overline{G}$ has a complete $(\overline{\chi} - 1)$-factor. Let $S_1, S_2, \ldots, S_\chi$ be the color classes of a χ-coloring of G. Clearly, each component of $G[S_i]$ has no edge and $\cup_{i=1}^{\chi} S_i = V(G)$. We require the following lemma.

<u>Lemma 2.2</u>. $|S_i| = |S_j|$ for any $i, j \in \{1, \ldots, \chi\}$.

Proof. Suppose $|S_i| > |S_j|$ for some $i, j \in \{1, \ldots, \chi\}$, and let S_t be a set of maximum size among all sets $S_1, S_2, \ldots S_\chi$. Since G has exactly $\chi \cdot \overline{\chi}$ vertices, $|S_t| > \overline{\chi}$. However, because $\overline{G}$ must have a subgraph isomorphic to $K_{|S_t|}$ we obtain $\overline{\chi} \geq |S_t| > \overline{\chi}$; a contradiction.

By Lemma 2.2, each of $S_1, S_2, \ldots, S_\chi$ has the same size. Moreover, as $\chi \cdot \overline{\chi} = n$, each S_i has size $\overline{\chi}$. Thus, $\overline{G}$ has a complete $(\overline{\chi}-1)$-factor, namely

$$\bigcup_{i=1}^{\chi} \overline{G}[S_i].$$

Now, if $T_1, T_2, \ldots, T_{\overline{\chi}}$ are the color classes of a $\overline{\chi}$-coloring of $\overline{G}$, then using Lemma 2.2 with $\overline{G}$ and $\overline{\chi}$ instead of G and χ, it is clear that each T_i has size χ. Therefore, G has a complete $(\chi - 1)$-factor, namely

$$\bigcup_{i=1}^{\overline{\chi}} G[T_i].$$

We will now show that (2) is equivalent to (3). First, suppose that (2) holds; that is, G has a complete $(\chi - 1)$-factor. This means that G has a spanning subgraph H in which each component is isomorphic to K_χ. Let k be the number of components of H, and observe that $\chi \cdot k = n$. Since each component C of H is complete, every vertex of C can be colored the same color in $\overline{G}$. Thus, $\overline{\chi} \leq k$.

Now, since each component of H is isomorphic to K_χ, each of the χ colors of G is used exactly once in each component of H. Thus, the color classes of G form χ independent sets of size k each, so $\overline{G}$ has a spanning subgraph in which each component is isomorphic to K_k; it follows that $\overline{\chi} \geq k$. Thus, $\overline{\chi} = k$ and $\overline{G}$ has a complete $(\overline{\chi} - 1)$-factor, so (2) implies (3). The argument that (3) implies (2) is symmetric. Moreover, since $n = \chi \cdot k = \chi \cdot \overline{\chi}$, we have shown that (2) implies (1), and the theorem is established.

The next theorem characterizes those graphs for which $\chi + \overline{\chi} = 2\sqrt{|V(G)|}$. Its proof relies on the arithmetic-geometric mean inequality; namely, $\sqrt{\chi \cdot \overline{\chi}} \leq \frac{\chi + \overline{\chi}}{2}$ with equality holding if and only if $\chi = \overline{\chi}$.

Theorem 2.3. Let G and $\overline{G}$ be complementary graphs on n vertices. Then $\chi + \overline{\chi} = 2\sqrt{n}$ if and only if $\chi = \overline{\chi}$ and G has a complete $(\chi - 1)$-factor.

Proof. Suppose that $\chi + \overline{\chi} = 2\sqrt{n}$.

Since $2\sqrt{n} = \chi + \overline{\chi}$, using the arithmetic-geometric mean inequality and Theorem 1.2 we obtain

$$\chi + \overline{\chi} \geq 2 \cdot \sqrt{\chi \cdot \overline{\chi}} \geq 2\sqrt{n} = \chi + \overline{\chi}.$$

This means that
$$\chi + \overline{\chi} = 2 \cdot \sqrt{\chi \cdot \overline{\chi}} = 2\sqrt{n}.$$

Now, the first equality shows that $\chi = \overline{\chi}$ by the arithmetic-geometric mean inequality. The second equality implies that $\chi \cdot \overline{\chi} = n$, and the theorem follows from Theorem 2.1.

Conversely, suppose that $\chi = \overline{\chi}$ and that G has a complete $(\chi - 1)$-factor. We need only show that $\chi + \overline{\chi} = 2\sqrt{n}$. Since $\chi = \overline{\chi}$,

$$0 = (\chi - \overline{\chi})^2 = \chi^2 - 2\chi \cdot \overline{\chi} + \overline{\chi}^2,$$

$$\chi^2 + 2\chi \cdot \overline{\chi} + \overline{\chi}^2 = 4 \cdot \chi \cdot \overline{\chi}.$$

Now, by Theorem 2.1, $\chi \cdot \overline{\chi} = n$, and thus,

$$(\chi + \overline{\chi})^2 = \chi^2 + 2\chi \cdot \overline{\chi} + \overline{\chi}^2 = 4n.$$

This implies that $\chi + \overline{\chi} = 2\sqrt{n}$, and the theorem is established.

3. Graphs Satisfying the Upper Bounds.

Theorem 3.1. Let G and $\overline{G}$ be complementary graphs on n vertices. Then $\chi + \overline{\chi} = n + 1$ if and only if $V(G)$ can be partitioned into three sets S, T, and $\{x\}$ such that $G[S]$ is isomorphic to $K_{\chi-1}$ and $\overline{G}[T]$ is isomorphic to $K_{\overline{\chi}-1}$.

Before proving this theorem, we establish several lemmas, the first of which is easy and is stated without proof.

Lemma 3.2. Let G be a graph on n vertices. Then G is Δ-regular if and only if $\Delta + \overline{\Delta} = n - 1$.

Lemma 3.3. If G is a Δ-regular graph and $\chi + \overline{\chi} = n + 1$, then G is isomorphic to K_1 or C_5.

Proof. If G or $\overline{G}$ is neither complete nor an odd cycle, then $\chi \leq \Delta$ or $\overline{\chi} \leq \overline{\Delta}$ by Theorem 1.1. In either case, we have the following chain of inequalities with the last equality being from Lemma 3.2.

$$n + 1 = \chi + \overline{\chi} \leq \Delta + \overline{\Delta} + 1 = n.$$

Since this is a contradiction, it is clear that both $\chi = \Delta + 1$ and $\overline{\chi} = \overline{\Delta} + 1$. Thus, by Theorem 1.1, G must be complete or an odd cycle, and $\overline{G}$ must be complete or an odd cycle. Clearly either both G and $\overline{G}$ must be complete or both be odd cycles. If G is complete and $\overline{G}$ is complete, then G is isomorphic to K_1. If G is an odd cycle and $\overline{G}$ is an odd cycle, then it is easy to see that G is isomorphic to C_5.

Lemma 3.4. Let G be a graph on n vertices and $\omega = \omega(G)$ the clique number of G. If $\omega = \chi$ and $\chi + \overline{\chi} = n + 1$, then $V(G)$ can be partitioned into two sets S and T such that $G[S]$ is isomorphic to K_χ and $\overline{G}[T]$ is isomorphic to $K_{\overline{\chi}-1}$.

Proof. Let S be a subset of $V(G)$ such that $G[S]$ is a χ-clique in G, and let $T = V(G) - V(S)$, so that $|T| = n - \chi = \overline{\chi} - 1$. Since S is independent in $\overline{G}$, each member of S can be colored 1 in $\overline{G}$. Since $\overline{G}$ has chromatic number $\overline{\chi}$, the graph $\overline{G}[T]$ requires at least $\overline{\chi} - 1$ colors. But $\overline{G}[T]$ has exactly $\overline{\chi}$ vertices, so $\overline{G}[T]$ must be complete.

Lemma 3.5. Let G be a graph on n vertices. If $\omega = \chi - 1$ and $\chi + \overline{\chi} = n + 1$, then $V(G)$ can be partitioned into two sets S and T such that $G[S]$ is isomorphic to $K_{\chi-1}$ and one of the following holds:

(1) $\overline{G}[T]$ is isomorphic to $K_{\overline{\chi}}$, or
(2) there exists $x \in T$ such that $\overline{G}[T - x]$ is isomorphic to $K_{\overline{\chi}-1}$.

Proof. Let H be an ω-clique in G, and let $S = V(H)$ and $T = V(G) - V(H)$. Notice that $|T| = n - \omega = n - (\chi - 1) = \overline{\chi}$. Therefore, $\overline{G}[T]$ has exactly $\overline{\chi}$ vertices. If $\overline{G}[T]$ is complete, then the conclusion of the lemma is satisfied.

Thus, we assume that $\overline{G}[T]$ is not complete. Now, if $\overline{G}[T]$ could be colored with fewer than $\overline{\chi} - 1$ colors, then, since $V(H)$ is independent in $\overline{G}$, the entire graph $\overline{G}$ could be colored with fewer than $\overline{\chi}$ colors. Therefore, $\chi(\overline{G}[T]) = \overline{\chi} - 1$.

Since $\overline{G}[T]$ is a graph on $\overline{\chi}$ vertices with chromatic number $\overline{\chi} - 1$, we can show that $\overline{G}[T]$ must contain a subgraph isomorphic to $K_{\overline{\chi}-1}$. To see this, color $\overline{G}[T]$ with $\overline{\chi} - 1$ colors and observe that there are exactly two vertices, say x and y, colored the same. Since the chromatic number of $\overline{G}[T]$ is $\overline{\chi} - 1$, it is clear that the vertices of $\overline{G}[T] - \{x, y\}$ form a complete graph. (Otherwise, a color could be duplicated on those vertices, reducing the chromatic number of $\overline{G}[T]$ to at most $\overline{\chi} - 2$.) Now, if x is not adjacent to some vertex z of $\overline{G}[T] - \{x, y\}$, then z can be re-colored with the color used on x. But this would require that y be re-colored with the original color on z, for otherwise we can color $\overline{G}[T]$ with fewer colors; thus, y is adjacent to z.

We now have a $\overline{\chi} - 1$ coloring of $\overline{G}[T]$ where x and z are colored the same. As before (when x and y were colored the same), $\overline{G}[T] - \{x, z\}$ forms a complete graph. In particular, y is adjacent to every vertex in $T - \{x, z\}$. But y is also adjacent to z, as we have seen, and z is adjacent to all vertices of $G[T] - \{x, y\}$. This implies that $G[T] - \{x\}$ is complete, so $\overline{G}[T]$ contains a copy of $K_{\overline{\chi}-1}$.

Corollary 3.6. Let G be a graph on n vertices. If $\omega = \chi - 1$ and $\chi + \overline{\chi} = n + 1$, then $V(G)$ can be partitioned into three sets S, T, and $\{x\}$, such that $G[S]$ is isomorphic to $K_{\chi-1}$ and $\overline{G}[T]$ is isomorphic to $K_{\overline{\chi}-1}$.

Lemma 3.7. Let G be a graph on n vertices. If $\chi + \overline{\chi} = n + 1$, then $\omega \geq \chi - 1$ and $\overline{\omega} \geq \overline{\chi} - 1$.

Proof. Let G be a minimal counterexample to the lemma with respect to $|V(G)|$, and let $H = G - x$, where $x \in V(G)$. Note that $\chi(H) \leq \chi(G)$ and $\overline{\chi}(H) \leq \overline{\chi}(G)$. By the theorem of Nordhaus and Gaddum, $\chi(H) + \overline{\chi}(H) \leq n$, so in fact at least one of the inequalities $\chi(H) \leq \chi(G)$ and $\overline{\chi}(H) \leq \overline{\chi}(G)$ must be strict. Thus, there are two cases.

Case 1. If only one of $\chi(H) < \chi(G)$ and $\overline{\chi}(H) < \overline{\chi}(G)$ is true, then without loss of generality, suppose that $\chi(H) = \chi(G)$ and $\overline{\chi}(H) = \overline{\chi}(G) - 1$. (Note that deleting a vertex will decrease the chromatic number by at most one.) Therefore, $\chi(H) + \overline{\chi}(H) = n = |V(H)| + 1$ and so $\omega(H) \geq \chi(H) - 1$ by the minimality of G. Thus, since $\chi(H) = \chi(G)$, it is clear that $\omega(G) \geq \omega(H) \geq \chi(H) - 1 = \chi(G) - 1$. By Lemmas 3.4 and 3.5, $\overline{G}$ must contain a clique of size at least $\overline{\chi} - 1$, a contradiction to the choice of G.

Case 2. Assume that $\chi(G - x) = \chi - 1$ and $\overline{\chi}(G - x) = \overline{\chi} - 1$ for every vertex x of G. This implies that $d_G(x) \geq \chi - 1$ and $d_{\overline{G}}(x) \geq \overline{\chi} - 1$; otherwise, x could be restored to $G - x$ and colored the same as some nonadjacent vertex, giving G a chromatic number less than χ, a contradiction.

Therefore, for each vertex x of G, $n - 1 = d_G(x) + d_{\overline{G}}(x) \geq \chi + \overline{\chi} - 2 = n - 1$. In order to achieve this, we must have $d_G(x) = \chi - 1$ and $d_{\overline{G}}(x) = \overline{\chi} - 1$ for every vertex x of G, which means that both G and $\overline{G}$ are regular. By Lemma 3.3, G must be isomorphic to K_1 or C_5, a contradiction as these both satisfy the statement of the lemma.

We now prove Theorem 3.1, the characterization of the case $\chi + \overline{\chi} = |V(G)| + 1$, which we restate here for convenience.

Theorem 3.8. Let G and $\overline{G}$ be complementary graphs on n vertices. Then $\chi + \overline{\chi} = n + 1$ if and only if $V(G)$ can be partitioned into three sets S, T, and $\{x\}$ such that $G[S]$ is isomorphic to $K_{\chi-1}$ and $\overline{G}[T]$ is isomorphic to $K_{\overline{\chi}-1}$.

Proof. First assume that $V(G)$ can be partitioned into three sets S, T, and $\{x\}$ such that $G[S]$ is isomorphic to $K_{\chi-1}$ and $\overline{G}[T]$ is isomorphic to $K_{\overline{\chi}-1}$. We merely need to count vertices: $|V(G)| = |S \cup T \cup \{x\}| = (\chi - 1) + (\overline{\chi} - 1) + 1 = \chi + \overline{\chi} - 1$. Thus, $\chi + \overline{\chi} = n + 1$.

Now assume that $\chi + \overline{\chi} = n + 1$. By Lemma 3.7, we have $\omega(G) \geq \chi - 1$ and $\overline{\omega}(G) \geq \overline{\chi} - 1$. First, if $\omega(G) = \chi - 1$, then the theorem holds by Corollary 3.6. Second, if $\omega(G) = \chi$, then by Lemma 3.4, G can be partitioned into a χ-clique S' and an independent set T of size $\overline{\chi} - 1$. Let $x \in V(S')$, and put $S = V(S') - \{x\}$. Then $G[S]$ is isomorphic to $K_{\chi-1}$ and $\overline{G}[T]$ is isomorphic to $K_{\overline{\chi}-1}$.

Capobianco [2] incorrectly states, "The only graphs that attain the upper bound $[\chi + \overline{\chi} = n + 1]$ are K_n, $\overline{K_n}$, and C_n." Theorem 3.1 gives the means to construct counterexamples; one simple counterexample is $G = K_4 \bigcup \overline{K_3}$.

The following theorem completes the characterizations of the graphs in the extreme cases of Theorem 1.2.

Theorem 3.9. Let G be a graph on n vertices. Then $\chi \cdot \overline{\chi} = \left(\frac{n+1}{2}\right)^2$ if and only if $\chi + \overline{\chi} = n+1$ and $\chi = \overline{\chi}$.

Proof. Since $4\chi \cdot \overline{\chi} = (n+1)^2 \geq (\chi + \overline{\chi})^2$, we must have $(\chi - \overline{\chi})^2 \leq 0$. Therefore, $\chi = \overline{\chi}$, and the inequality is actually an equality. Thus, $\chi + \overline{\chi} = n+1$.

4. Conclusion. A natural extension of Theorem 2.1 would be to characterize those graphs H where $\chi(H)$ and $\chi(\overline{H})$ are factors of $|V(H)|$, but $\chi(H) \cdot \chi(\overline{H}) > |V(H)|$.

Alavi and Behzad [1] proved bounds for edge chromatic numbers and Cook [3] proved bounds for total chromatic numbers similar to the bounds of Nordhaus and Gaddum. It is hoped that the approach presented here will shed light on a characterization of the graphs that attain those bounds.

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Colin L. Starr
Department of Mathematics
Willamette University
Salem, OR 97301
email: cstarr@willamette.edu

Galen E. Turner III
Mathematics and Statistics Program
Louisiana Tech University
Ruston, LA 71272
email: gturner@coes.LaTech.edu