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Complementary tree nil domination number of Cartesian Product of Graphs

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Abstract: *A set D of a graph G = (V, E) is a dominating set, if every vertex in V(G) – D is adjacent to some vertex in D. The domination number* γ *(G) of G is the minimum cardinality of a dominating set. A dominating set D is called a complementary tree nil dominating set, if the induced subgraph < V(G) – D > is a tree and also the set V(G) – D is not a dominating set. The minimum cardinality of a complementary tree nil dominating set is called the complementary tree nil domination number of G and is denoted by* $\gamma_{ctnd}(G)$. *In this paper, complementary tree domination numbers of Cartesian product of some standard graphs are found.*

Key words: *Domination number, Complementary tree nil domination number, Cartesian product.*

1. Introduction

Graphs discussed in this paper are finite, undirected and simple connected graphs. For a graph G, let V(G) and E(G) denote its vertex set and edge set respectively. A graph G with p vertices and q edges is denoted by G(p, q). The concept of domination in graphs was introduced by Ore[5]. A set D $\subseteq$ V(G) is said to be a dominating set of G, if every vertex in V(G) – Dis adjacent to some vertex in D. The cardinality of a minimum dominating set in G is called the domination number of G and is denoted by γ(G). Muthammai, Bhanumathi and Vidhya[5] introduced the concept of complementary tree dominating set. A dominating set D $\subseteq$ V(G) is said to be a complementary tree dominating set (ctd-set), if the induced subgraph < V(G) - D >is a tree. The minimum cardinality of a ctd-set is called the complementary tree domination number of G and is denoted by $\gamma_{ctd}(G)$. Any undefined terms in this paper may be found in Harary[2].

The cartesian product of two graphs G_1 and G_2 is the graph, denoted by G_1 x G_2 with $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ (where x denotes the cartesian product of sets) and two vertices u = (u_1, u_2) and v = (v_1, v_2) in V $(G_1 \times G_2)$ are adjacent in $G_1 \times G_2$ whenever [$u_1 = v_1$ and $(u_2, v_2) \in E(G_2)$] or [$u_2 = v_2$ and $(u_1, v_1) \in E(G_1)$].The corona $G_1 \odot G_2$of two graphs $G_1$and $G_2$are defined as the graph G obtained by taking one copy of $G_1$of order $p_1$and $p_1$copies of $G_2$and then joining the i^{th}vertex of $G_1$to every vertex in the i^{th}copy of G_2. The Corona $G_1 \odot G_2$has $p_1(1 + p_2)$ vertices and $q_1 + p_1q_2 + p_1p_2$edges. The concept of complementary tree nil dominating set is introduced in [4]. A dominating set D $\subseteq$ V(G) is said to be a

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complementary tree nil dominating set (ctnd-set), if the induced subgraph <V(G)−D> is a tree and the set V(G) - D is not a dominating set. The minimum cardinality of a ctnd-set is called the complementary tree nil domination number of G and is denoted by $\gamma_{ctnd}(G)$.

In this paper, we find an upper bound for complementary tree nil domination number of Cartesian product of $P_m \times P_n$ and this number found for $K_m \times K_n$, $K_m \times P_n$, $K_m \times C_n$ and $C_m \times P_n$.

2. Main Results

Theorem 2.1:

If $G \cong K_m \times K_n$ ($m,n \geq 3$ and $m \leq n$), then $\gamma_{ctnd}(G) = \begin{cases} m(n-2)+3, \text{if } m = n \\ m(n-2)+2, \text{if } m < n \end{cases}$

Proof:

Let $G \cong K_m \times K_n$.

Let $V(G) = \bigcup_{i=1}^{m}\{v_{i1}, v_{i2}, \dots, v_{in}\}$ such that $< \{v_{i1}, v_{i2}, \dots, v_{in}\} > \cong K_n^i$, i = 1,2, …, m and $< \{v_{1j}, v_{2j}, \dots, v_{mj}\} > \cong K_m^j$, j=1,2, …, n, where K_n^i is the i[th] copy of K_n and K_m^j is the j[th] copy of K_m in $K_m \times K_n$. $|V(G)| = mn$.

Case 1: m = n.

Let $D' = \left(\bigcup_{i=2}^{m-1}\{v_{ii}, v_{i,i+1}\}\right) \cup \{v_{m,m}\}$ and $D = V(G) - D'$. Then $V(G) - D = D'$ and $|D'| = 2(m-2)+1 = 2m-3$. The vertices $v_{ii}, v_{i,i+1}$ in V(G) – D are adjacent to v_{i1} in D, i = 2,3, …, m-1 and the vertex v_{mm} is adjacent to v_{m1} in D. Therefore D is a dominating set of G. Also $<V(G) - D> \cong P_{2(m-2)+1} = P_{2m-3}$. Therefore D is a ctd-set of G and since $N(v_{11}) \subseteq D$, D is a ctnd-set of G. Therefore $\gamma_{ctnd}(G) \leq |D| = |V(G)| - |D'| = mn - (2m-3) = m(n-2) + 3$.

It is to be noted that, any tree in G is a path and $\delta(G) = m$. Let D' be a γ_{ctnd}-set of G. Then there exists a vertex $u \in D'$ such that $N(u) \subseteq D'$. The longest path that can be obtained from the subgraph of G induced by the vertices of V(G) – N(u) is P_{2m-3}. Therefore $<V(G) - D'> \cong P_{2m-3}$.

Therefore D' contains atleast $mn - (2m - 3) = m(n - 2) + 3$ vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq m(n-2) + 3$.

Hence $\gamma_{ctnd}(G) = m(n-2) + 3$.

Case 2: m < n.

Let $D' = \bigcup_{i=2}^{m}\{v_{ii}, v_{i,i+1}\}$ and $D = V(G) - D'$. Then $V(G) - D = D'$ and $|D'| = 2(m-1)$. The vertices $v_{ii}, v_{i,i+1}$ (i = 2, 3, …, m) are adjacent to v_{i1}, (i = 2, 3, …, m) in D. Therefore D is a dominating set of G. Also $<V(G) - D> \cong P_{2(m-2)} = P_{2m-2}$. Therefore D is a ctd-set of G and since $N(v_{11}) \subseteq D$, D is a ctnd-set of G.

Therefore $\gamma_{ctnd}(G) \leq |V(G)| - |D'| = mn - (2m-2) = m(n-2) + 2$.

As in case 1, any tree in G is a path and $\delta(G) = m$. Let D' be γ_{ctnd}-set of G. Then there exists a vertex $u \in D'$ such that $N(u) \subseteq D'$. The longest path that can be obtained from the subgraph of G induced by the vertices of $V(G) - N(u)$ is P_{2m-2}.

Therefore $<V(G) - D'> \cong P_{2m-2}$. Therefore D' contains atleast $mn - (2m - 2) = m(n - 2) + 2$ vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq m(n - 2) + 2$.

Therefore $\gamma_{ctnd}(G) = m(n - 2) + 2$.

Hence $\gamma_{ctnd}(G) = \begin{cases} m(n-2) + 3, \text{if } m = n \\ m(n-2) + 2, \text{if } m < n \end{cases}$

Example 2.1:

For the graph G given in Figure 1.a and Figure 1.b, the set of vertices within the ⦿ is a minimum ctnd-set of $K_m \times K_n$ and $\gamma_{ctnd}(K_4 \times K_4) = 11$ and $\gamma_{ctnd}(K_4 \times K_5) = 14$.

$G \cong K_4 \times K_4$

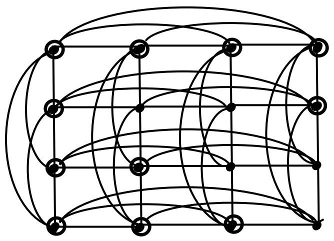

Figure 1.a

$G \cong K_4 \times K_5$

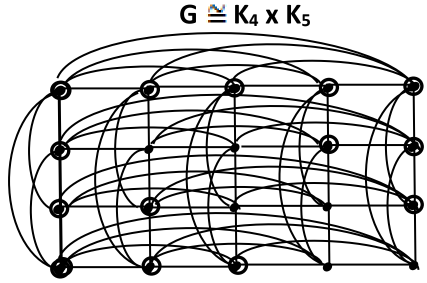

Figure 1.b

Theorem 2.2:

If $G \cong K_m \times P_n$ ($4 \leq m \leq n$), then $\gamma_{ctnd}(G) = n(m - 2) + 2$.

Proof:

Let $G \cong K_m \times P_n$.

Let $V(G) = \bigcup_{i=1}^{m}\{v_{i1}, v_{i2}, \dots, v_{in}\}$ such that $< \{v_{i1}, v_{i2}, \dots, v_{in}\} > \cong K_n^i$, i = 1, 2, …, m and $< \{v_{1j}, v_{2j}, \dots, v_{mj}\} > \cong P_m^j$, j = 1, 2, …, n, where K_n^i is the i^{th} copy of K_n and K_m^j is the j^{th} copy of P_m in $K_m \times P_n$.

Let $D' = \begin{cases} [\bigcup_{i=2}^{n}\{v_{2i}\}] \cup [\bigcup_{i=1}^{\frac{n-1}{2}}\{v_{3,2i}, v_{1,2i+1}\}], \text{if n is odd} \\ [\bigcup_{i=2}^{n}\{v_{2i}\}] \cup [\bigcup_{i=1}^{\frac{n}{2}}\{v_{1,2i-1}, v_{3,2i}\}], \text{if n is even} \end{cases}$.

Then $|D'| = 2(n - 1)$. If $D = V(G) - D'$, then D is a dominating set of G and $N(v_{11}) \subseteq D$. Also $<V(G) - D> = < D' > \cong P_{n-1} \circ K_1$. Therefore D is a ctnd-set of G.

$\gamma_{ctnd}(G) \leq |D| = mn - 2(n - 1) = mn - 2n + 2 = n(m - 2) + 2$.

Hence $\gamma_{ctnd}(G) \leq n(m - 2) + 2$.

Let D' be a γ_{ctnd}-set of G. Since D' is a ctd-set of G, D' contains atleast (m – 2) vertices in each of (n – 1)K_m's and since, $V(G) - D'$ is not a dominating set, D' contains all the vertices of the remaining K_m. Hence D' contains atleast (m – 2)(n -1) + m = mn – m -2n +2 + m = n(m – 2) + 2 vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq n(m-2) + 3$.

Hence $\gamma_{ctnd}(K_m \times P_n) = n(m-2) + 2$.

Example 2.2:

For the graph G given in Figure 2, the set of vertices within the ⦿ is a minimum ctnd-set of $K_m \times K_n$ and $\gamma_{ctnd}(K_4 \times K_9) = 20$.

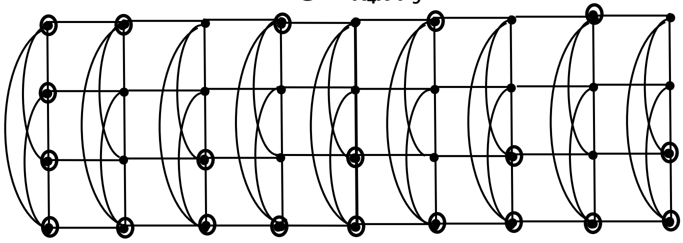

Figure 2

Remark 2.1:

In view of Theorem 2.2,

$\gamma_{ctnd}(K_m \times C_n) = n(m-2) + 3$.

Theorem 2.3:

If $G \cong P_m \times P_n$ (m, n $\geq$ 2), then $\gamma_{ctnd}(G) \leq \gamma_{ctd}(G) + 2$.

Proof:

Let $G \cong P_m \times P_n$. Then $\delta(G) = 2$.

Let D be a γ_{ctd} - set of G. Let $u \in D$ be a vertex of minimum degree in G and $\deg(u) = \delta(G)$. Then $D' = D \cup N(u)$ is a ctnd –set of G, since $N(u) \subseteq D'$. Therefore $\gamma_{ctnd}(G) \leq |D'| = |D| + |N(u)| = \gamma_{ctd}(G) + \delta(G) = \gamma_{ctd}(G) + 2$.

Hence $\gamma_{ctnd}(G) \leq \gamma_{ctd}(G)+2$.

Equality holds, if $G \cong P_2 \times P_n$, n $\geq$ 3.

Theorem 2.4:

If $G \cong C_3 \times P_n$, then $\gamma_{ctnd}(G) = n + 2$, $n \geq 3$.

Proof:

Let $G \cong C_3 \times P_n$.

Let $V(G) = \bigcup_{i=1}^{n}\{v_{1i}, v_{2i}, v_{3i}\}$ such that $< \{v_{i1}, v_{i2}, \ldots, v_{in}\} > \cong P_n^i$, i =1,2,3 and $< \{v_{1j}, v_{2j}, v_{3j}\} > \cong C_3^j$, j=1,2, …, n,where P_n^i is the i[th] copy of P_n and C_3^j is the j [th] copy of C_3 in $C_3 \times P_n$.

Let $D = \begin{cases} \{v_{11}, v_{21}\} \cup [\cup_{i=1}^{\frac{n}{2}} \{v_{2,2i}, v_{3,2i-1}\}], & \text{if n is even} \\ \{v_{11}, v_{21}, v_{31}\} \cup [\cup_{i=1}^{\frac{n-1}{2}} \{v_{2,2i}, v_{3,2i+1}\}], & \text{if n is odd.} \end{cases}$

Then D is a dominating set of G and $N(v_{11}) \subseteq D$. Also $<V(G) - D> \cong P_n \circ K_1$. Therefore D is a ctnd-set of G.

$$\gamma_{ctnd}(G) \leq |D| = \begin{cases} 2\left(\frac{n}{2}\right) + 2 = n + 2, & \text{if n is even} \\ 2\left(\frac{n-1}{2}\right) + 3 = n + 2, & \text{if n is odd.} \end{cases}$$

Let D' be a γ_{ctnd}-set of G. Then D' contains atleast one vertex from each cycle. Since $C_3 \times P_n$ contains n copies of C_3, D' contains atleast n vertices. Also, since $V(G) - D'$ is not a dominating set, the remaining vertices of first cycle C_3 in $C_3 \times P_n$ must be included in D'.

Therefore D' contains atleast n + 2 vertices and $\gamma_{ctnd}(G) = |D'| \geq n + 2$.

Hence $\gamma_{ctnd}(C_3 \times P_n) = n + 2$, $n \geq 3$.

Theorem 2.5:

If $G \cong C_4 \times P_n$, then $\gamma_{ctnd}(G) = \left\lfloor \frac{3n+4}{2} \right\rfloor$, $n \geq 2$.

Proof:

Let $G \cong C_4 \times P_n$ and $V(G) = \cup_{i=1}^{n} \{v_{1i}, v_{2i}, v_{3i}, v_{4i}\}$ such that $< \{v_{i1}, v_{i2}, \ldots, v_{in}\} > \cong P_n^i$, i =1,2,3,4 and $< \{v_{1j}, v_{2j}, v_{3j}, v_{4j}\} > \cong C_4^j$, j=1,2, …, n, where P_n^i is the i^{th} copy of P_n and C_4^j is the j^{th} copy of C_4 in $C_4 \times P_n$ and $|V(G)| = 4n$.

Case 1: n is even.

Let $D' = \{v_{31}, v_{3n}\} \cup [\cup_{i=1}^{\frac{n-2}{2}} \{v_{1,2i+1}, v_{4,2i+1}, v_{3,2i}\}] \cup [\cup_{i=2}^{n} \{v_{2i}\}]$ and $D = V(G) - D'$. Then $|D'| = 2 + 3\left(\frac{n-2}{2}\right) + n - 1 = \frac{5n-4}{2}$. Then D is a dominating set of G and $N(v_{11}) \subseteq D$. Also $<V(G) - D> = <D'>$ is a tree obtained from a path $P_{n-1} = <\{v_{2,i}, i = 2,3,\ldots,n\}>$, $(n \geq 2)$ by attaching P_3 at each of the vertices $v_{22}, v_{23}, v_{25}, \ldots, v_{2,n-1}$ and attaching a pendant edge at each of the vertices $v_{24}, v_{26} \ldots, v_{2,n}$. Therefore D is a ctnd-set of G.

$$\gamma_{ctnd}(G) \leq |D| = |V(G) - D'| = 4n - \left(\frac{5n-4}{2}\right) = \frac{3n+4}{2}.$$

Hence $\gamma_{ctnd}(G) \leq \frac{3n+4}{2}$.

Let D' be a γ_{ctnd}-set of G. Since $<V(G) - D'>$ is not a dominating set, D' contains a vertex u such that $N(u) \subseteq D$. u is taken to be a vertex of minimum degree $\delta(G) = 3$ in G. The blocks A, B, C are constructed as given below.

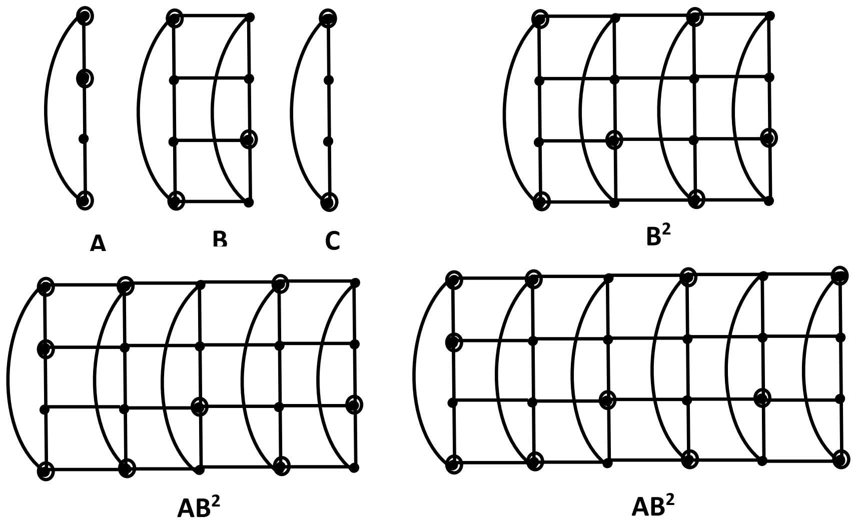

Figure 3

G is obtained by concatenating the blocks A, $B^{\frac{n-2}{2}}$ andC. That is, G $\cong$A $B^{\frac{n-2}{2}}$ C. The vertices with the symbol ⦿ in each of the blocks represent the vertices that are to be included in D'.

Therefore D' contains 3 vertices from block A and atleast 3 vertices from each block B of $B^{\frac{n-2}{2}}$ and 2 vertices from block C.Therefore$\gamma_{ctnd}(G) = |D'| \geq 3 + 3\left(\frac{n-2}{2}\right) + 2 = \frac{3n+4}{2}$.

and hence $\gamma_{ctnd}(G) = \frac{3n+4}{2}$.

Case 2: n is odd.

Let $D' = \{v_{31}\} \cup [\cup_{i=1}^{\frac{n-2}{2}}\{v_{1,2i+1}, v_{4,2i+1}, v_{3,2i}\}] \cup [\cup_{i=2}^{n}\{v_{2i}\}]$.

Then $|D'| = 1 + 3\left(\frac{n-1}{2}\right) + n - 1 = \frac{5n-3}{2}$ and D = V(G) - D'. Then D is a dominating set of G and $N(v_{11}) \subseteq D$. Also <V (G) – D > = < D'> is a tree obtained from a path $P_{n-1} = <\{v_{2,i}, i = 2,3,\ldots,n\}>$, ($n \geq 2$) by attaching P_3 at each of the vertices $v_{22}, v_{23}, v_{25}, \ldots, v_{2,n}$ and attaching a pendant edge at each of the vertices $v_{24}, v_{26} \ldots, v_{2,n-1}$. Therefore D is a ctnd-set of G.

$\gamma_{ctnd}(G) \leq |D| = |V(G) - D'| = 4n - \left(\frac{5n-3}{2}\right) = \frac{3n+3}{2}$.

Hence $\gamma_{ctnd}(G) \leq \frac{3n+3}{2} = \left\lfloor \frac{3n+4}{2} \right\rfloor$.

Let D' be a γ_{ctnd}-set of G. Since <V(G) – D'> is not a dominating set, D' contains a vertex u such that $N(u) \subseteq D$. u is taken to be a vertex of minimum degree $\delta(G) = 3$ in G. The blocks A, B are constructed as in case 1.

G is obtained by concatenating the blocks A and $B^{\frac{n-1}{2}}$ as in case 1. That is, $G \cong AB^{\frac{n-1}{2}}$. The vertices with the symbol ⊙ in each of the blocks represent the vertices that are to be included in D'.

Therefore D' contains 3 vertices from block A and atleast 3 vertices from each block B of $B^{\frac{n-1}{2}}$.

Therefore $\gamma_{ctnd}(G) = |D'| \geq 3 + 3\left(\frac{n-1}{2}\right) = \frac{3n+3}{2} = \left\lfloor \frac{3n+4}{2} \right\rfloor$.

Hence $\gamma_{ctnd}(C_4 \times P_n) = \left\lfloor \frac{3n+4}{2} \right\rfloor$, $n \geq 2$.

Theorem 2.6:

If $G \cong C_5 \times P_n$, then $\gamma_{ctnd}(G) = 2n + 1$, $n \geq 3$.

Proof:

Let $G \cong C_5 \times P_n$ and $V(G) = \cup_{i=1}^{n}\{v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i}\}$ such that $< \{v_{i1}, v_{i2}, \ldots, v_{in}\} > \cong P_n^i$, i =1, 2, 3, 4, 5 and $< \{v_{1j}, v_{2j}, v_{3j}, v_{4j}, v_{5j}\} > \cong C_5^j$, j = 1,2, …, n, where P_n^i is the i^{th} copy of P_n and C_5^j is the j^{th} copy of C_5 in $C_5 \times P_n$. $|V(G)| = 5n$.

Case 1: n is odd

Let $D = \{v_{21}, v_{12}, v_{32}\} \cup [\cup_{i=1}^{\frac{n+1}{2}}\{v_{1,2i-1}, v_{5,2i-1}\}] \cup [\cup_{i=2}^{\frac{n-1}{2}}\{v_{3,2i}, v_{4,2i}\}]$.

Then $|D| = 3 + 2\left(\frac{n+1}{2}\right) + 2\left(\frac{n-3}{2}\right) = 2n + 1$.

Consider the blocks

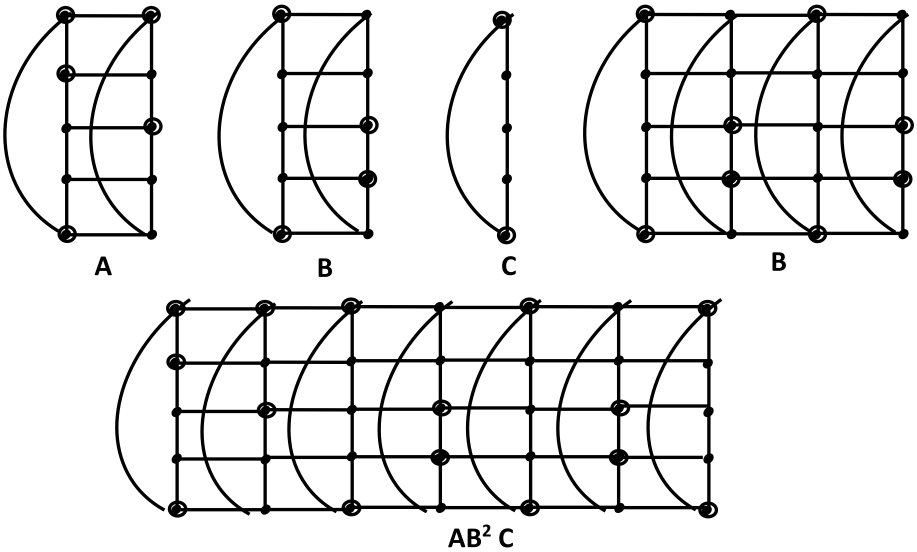

Figure 4

Then G $\cong AB^{\frac{n-3}{2}}C$. Let D be the set of vertices with the symbol ⦿ in each of the blocks A, $B^{\frac{n-3}{2}}$ and C. D contains 5 vertices from block A, and 4 vertices from each block B of $B^{\frac{n-3}{2}}$ and 2 vertices from block C. Then D is a dominating set of G and the vertex v_{11} is such that $N(v_{11}) \subseteq D$ and $<V(G) - D> \cong T$, where T is a tree constructed as below.

Let H be the graph obtained by subdividing each of the pendant edges of P_{n-2}^{+} exactly once and T be the tree obtained from H by attaching a pendant edge at one pendant vertex say v of P_{n-2} and then joining a vertex of degree 2 of P_4 by an edge to a pendant vertex at a distance 2 from v.

Therefore D is a ctnd-set of G.

$\gamma_{ctnd}(G) \leq |D'| = 2n + 1$.

Let D' be a γ_{ctnd}-set of G. Since $\gamma(C_5) = 2$, D' contains 2 vertices from each of n cycles and D' contains one more vertex from a cycle C_5 and hence D' contains atleast 2n+1 vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq 2n + 1$.

Hence $\gamma_{ctnd}(G) = 2n + 1, n \geq 2$

Case 2: n is even

Let $D = \{v_{11}, v_{12}, v_{21}, v_{32}, v_{51}\} \cup [\cup_{i=2}^{\frac{n}{2}} \{v_{1,2i-1}, v_{3,2i}, v_{4,2i}, v_{5,2i-1}\}]$. Then $|D| = 5 + 4\left(\frac{n-2}{2}\right) = 2n + 1$.

G is obtained by concatenating the blocks A, $B^{\frac{n-2}{2}}$. That is G $\cong AB^{\frac{n-2}{2}}$. Let D be the set of vertices with the symbol ⦿ in each of the blocks A and $B^{\frac{n-2}{2}}$. D contains 5 vertices from block A, and 4 vertices from each block B of $B^{\frac{n-2}{2}}$. Then D is a dominating set of G and the vertex v_{11} is such that $N(v_{11}) \subseteq D$ and $<V(G) - D> \cong T$, where T is a tree constructed as in case 1.

Therefore D is a ctnd-set of G and $\gamma_{ctnd}(G) \leq |D| = 2n + 1$.

Let D' be a γ_{ctnd}-set of G. Since $\gamma(C_5) = 2$, D' contains 2 vertices from each of n cycles and since V(G) – D is not a dominating set of G, D' contains one more vertex from a cycle C_5 and hence D' contains atleast 2n+1 vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq 2n + 1$.

Hence $\gamma_{ctnd}(G) = 2n + 1, n \geq 2$.

Theorem 2.7:

If G $\cong C_5 \times P_2$, then $\gamma_{ctnd}(G) = 5$.

Proof:

Let $G \cong C_5 \times P_2$ and $V(G) = \bigcup_{i=1}^{n}\{v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i}\}$ such that $<\{v_{i1}, v_{i2}\}> \cong P_n^i$, i =1, 2, 3, 4, 5 and $<\{v_{1j}, v_{2j}, v_{3j}, v_{4j}, v_{5j}\}> \cong C_5^j$, j = 1, 2, where P_n^i is the i^{th} copy of P_n and C_5^j is the j^{th} copy of C_5 in $C_5 \times P_2$.

Let $D = \{v_{11}, v_{21}, v_{31}, v_{41}, v_{12}\}$. Then $N(v_{11}) \subseteq D$ and D is a dominating set of G. Also $V(G) - D = \{v_{31}, v_{22}, v_{33}, v_{44}, v_{52}\}$ and <V(G) – D> is a graph obtained from P_3 by attaching 2 pendant edges at a pendant vertex of P_3. Therefore D is a ctnd-set of G.

$\gamma_{ctnd}(G) \leq |D| = 5$.

Let D' be a γ_{ctnd}-set of G. D' contains 4 vertices from C_5^1 and atleast one vertex from C_5^2.

Therefore D' contains atleast 5 vertices. $\gamma_{ctnd}(G) = |D'| \geq 5$.

Hence $\gamma_{ctnd}(G) = 5$.

Theorem 2.8:

If $G \cong C_6 \times P_n$, then $\gamma_{ctnd}(G) = \left\lceil \frac{5n+1}{2} \right\rceil$, $n \geq 3$.

Proof:

Let $G \cong C_6 \times P_n$ and $V(G) = \bigcup_{i=1}^{n}\{v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i}, v_{6i}\}$ such that $<\{v_{i1}, v_{i2}, \dots, v_{in}\}> \cong P_n^i$, i =1,2,3,4,5,6 and $<\{v_{1j}, v_{2j}, v_{3j}, v_{4j}, v_{5j}, v_{6j}\}> \cong C_6^j$, j=1,2, …, n, where P_n^i is the i^{th} copy of P_n and C_6^j is the j^{th} copy of C_6 in $C_6 \times P_n$ and $|V(G)| = 6n$.

Case 1: n is odd.

Let $D' = \{v_{31}, v_{41}, v_{51}, v_{32}, v_{62}\} \cup [\bigcup_{i=1}^{\frac{n-1}{2}}\{v_{1,2i+1}, v_{5,2i+1}, v_{6,2i+1}\}] \cup [\bigcup_{i=2}^{n}\{v_{2i}\}] \cup [\bigcup_{i=2}^{\frac{n-1}{2}}\{v_{3,2i}, v_{4,2i}\}]$.

Then $|D'| = 5 + 3\left(\frac{n-1}{2}\right) + n - 1 + 2\left(\frac{n-3}{2}\right) = \frac{7n-1}{2}$ and $D = V(G) - D'$. Then D is a dominating set of G and $N(v_{11}) \subseteq D$. Also <V (G) – D > = < D'> is a tree obtained from a path $P_{n-1} = <\{v_{2,i},\ i = 2,3,\dots,n\}>$, ($n \geq 2$) by attaching P_4 at each of the vertices $v_{23}, v_{25}, v_{27}, \dots, v_{2,n}$ and attaching P_3 at each of the vertices $v_{24}, v_{26} \dots, v_{2,n-1}$. Therefore D is a ctnd-set of G.

$\gamma_{ctnd}(G) \leq |D| = |V(G) - D'| = 6n - \left(\frac{7n-1}{2}\right) = \frac{5n+1}{2}$.

Hence $\gamma_{ctnd}(G) \leq \frac{5n+1}{2}$.

Let D' be a γ_{ctnd}-set of G. Since <V (G) – D'> is not a dominating set. Therefore D' contains a vertex of u such that $N(u) \subseteq D$. u is taken to be a vertex of minimum degree $\delta(G) = 3$ in G. The blocks A, B, C are constructed as given below.

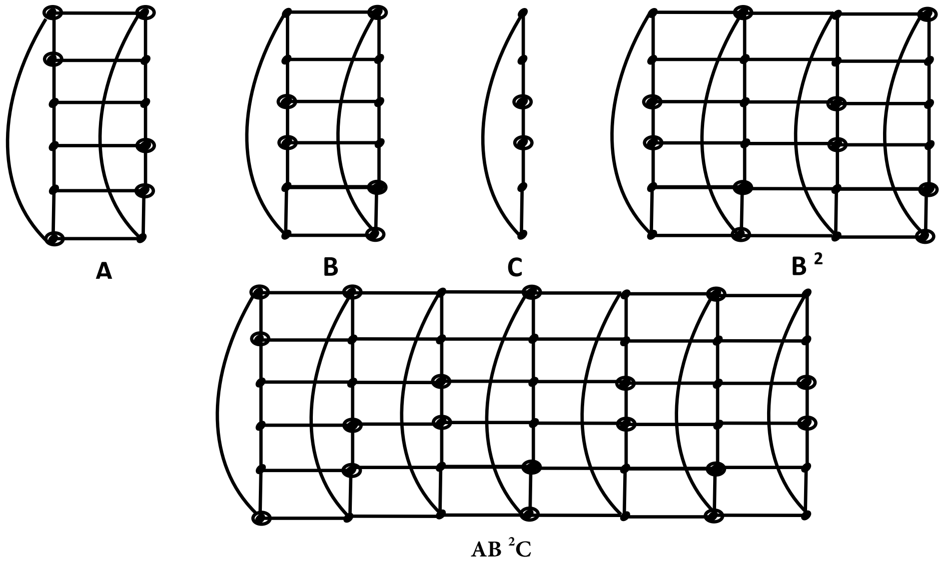

Figure 5

G is obtained by concatenating the blocks A, $B^{\frac{n-3}{2}}$ and C. That is, $G \cong A\, B^{\frac{n-3}{2}}$ C. The vertices with the symbol ⦿ in each of the blocks represent the vertices that are to be included in D'. Therefore D' contains 6 vertices from block A and atleast 5 vertices from each block B of $B^{\frac{n-3}{2}}$ and 2 vertices from block C. Therefore $\gamma_{ctnd}(G) = |D'| \geq 6 + 5\left(\frac{n-3}{2}\right) + 2 = \frac{5n+1}{2}$ and hence $\gamma_{ctnd}(G) = \frac{5n+1}{2}$.

Case 2: n is even.

Let $D' = \{v_{31}, v_{41}, v_{51}, v_{32}, v_{62}\} \cup [\cup_{i=1}^{\frac{n-2}{2}}\{v_{1,2i+1}, v_{5,2i+1}, v_{6,2i+1}\}] \cup [\cup_{i=2}^{n}\{v_{2i,}\}] \cup [\cup_{i=2}^{\frac{n}{2}}\{v_{3,2i}, v_{4,2i}\}]$.

Then $|D'| = 5 + 3\left(\frac{n-2}{2}\right) + n - 1 + 2\left(\frac{n-2}{2}\right) = \frac{7n-2}{2}$ and $D = V(G) - D'$. Then D is a dominating set of G and $N(v_{11}) \subseteq D$. Also $<V(G) - D> = <D'>$ is a tree obtained from a path $P_{n-1} = <\{v_{2,i}, i = 2,3,\dots,n\}>$, $(n \geq 2)$ by attaching P_4 at each of the vertices $v_{23}, v_{25}, v_{27}, \dots, v_{2,n-1}$ and attaching P_3 at each of the vertices $v_{24}, v_{26}, \dots, v_{2,n}$. Therefore D is a ctnd-set of G.

$\gamma_{ctnd}(G) \leq |D| = |V(G) - D'| = 6n - \left(\frac{7n-2}{2}\right) = \frac{5n+2}{2}$.

Hence $\gamma_{ctnd}(G) \leq \frac{5n+2}{2}$.

Let D' be a γ_{ctnd}-set of G. Since $<V(G) - D'>$ is not a dominating set, D' contains a vertex of u such that $N(u) \subseteq D$. u is taken to be a vertex of minimum degree $\delta(G) = 3$ in G. The blocks A, B are constructed as in case 1.

G is obtained by concatenating the blocks A and $B^{\frac{n-2}{2}}$. That is, $G \cong AB^{\frac{n-2}{2}}$. The vertices with the symbol ⊙ in each of the blocks represent the vertices that are to be included in D'.

Therefore D' contains 6 vertices from block A and atleast 5 vertices from each block B of $B^{\frac{n-2}{2}}$. Therefore $\gamma_{ctnd}(G) = |D'| \geq 6 + 5\left(\frac{n-2}{2}\right) = \frac{5n+2}{2}$ and hence $\gamma_{ctnd}(G) = \frac{5n+2}{2} = \left\lceil\frac{5n+1}{2}\right\rceil$.

Hence $\gamma_{ctnd}(C_6 x P_n) = \left\lceil\frac{5n+1}{2}\right\rceil$, $n \geq 2$.

Theorem 2.9:

If $G \cong C_6$ x P_2, then $\gamma_{ctnd}(G) = 5$.

Proof:

$G \cong C_6$ x P_n and $V(G) = \cup_{i=1}^{n}\{v_{1i}, v_{2i}, v_{3i}, v_{4i}, v_{5i}, v_{6i}\}$ such that $< \{v_{i1}, v_{i2}, \dots, v_{in}\} > \cong P_n^i$, i=1, 2, 3, 4,5,6 and $< \{v_{1j}, v_{2j}, v_{3j}, v_{4j}, v_{5j}, v_{6j}\} > \cong C_6^j$, j = 1, 2, where P_n^i is the i[th] copy of P_n and C_6^j is the j [th] copy of C_6 in C_6 x P_2.

Let $D = \{v_{11}, v_{21}, v_{61}, v_{12}, v_{42}\}$. Then $N(v_{11}) \subseteq D$ and D is a dominating set of G. Also $V(G) - D = \{v_{31}, v_{41}, v_{51}, v_{22}, v_{32}, v_{44}, v_{52}, v_{62}\}$and $<V(G) - D> \cong P_7$. Therefore D is a ctnd-set of G.$\gamma_{ctnd}(G) \leq |D| = 5$.

Let D' be a γ_{ctnd}-set of G. D' contains 3 vertices from C_6^1 and atleast 2 vertices from C_6^2.

Therefore D' contains atleast 5 vertices. Therefore $\gamma_{ctnd}(G) = |D'| \geq 5$.

Hence $\gamma_{ctnd}(G) = 5$.

Remark 2.2:

In view of Theorem 2.4,Theorem 2.5, Theorem 2.6, and Theorem 2.8,

1. $\gamma_{ctnd}(C_3 x C_n) = n+3$, $n \geq 3$.
2. $\gamma_{ctnd}(C_4 x C_n) = \left\lceil\frac{3n+6}{2}\right\rceil$, $n \geq 3$.
3. $\gamma_{ctnd}(C_5 x C_n) = 2n + 3$, $n \geq 3$.
4. $\gamma_{ctnd}(C_6 x C_n) = 3n$, $n \geq 3$.

Remark 2.3:

1. If $G_1 \cong K_m$ and $G_2 \cong K_n$, then $\gamma_{ctnd}(G_1 + G_2) = m + n$.
2. If G_1 and G_2 are any two non-complete connected graphs of order m and n respectively, with minimum degree atleast two, then $\gamma_{ctnd}(G_1 + G_2) \leq m + n - 1$. Equality holds, if $G_1 \cong K_m - e, G_2 \cong K_n - e$.

3. For any two connected graphs G_1 and G_2 of order m and n respectively, $\gamma_{ctnd}(G_1 \circ G_2) \leq m+n-1$. Equality holds, if $G_1 \cong P_2$ and $G_2 \cong nK_1$.
4. For any two nontrivial connected graphs G_1 and G_2 with the of order m and n respectively, $\gamma_{ctnd}(G_1 \circ G_2) \leq m + n - 2$. Equality holds, if $G_1 \cong P_2$ and $G_2 \cong C_3$.

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