COMPLEMENTS AND SUBSTITUTES IN THE OPTIMAL ASSIGNMENT PROBLEM

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1. INTRODUCTION

When studying a process that depends intricately on several independent input activities, it is sometimes of interest to know whether a given pair of activities are complements or substitutes — that is, whether they reinforce or interfere with each other's influence on the process as a whole. Mathematically, the distinction can often be described in terms of a partial derivative:

$$\frac{\partial^2 F(x, y, \ldots)}{\partial x \, \partial y},$$

which is typically positive if the activities x and y are complements, negative if they are substitutes. Here F is some measure of the value of the over-all process — its output, utility, negative cost, etc. An analogous statement in terms of finite differences would apply to processes that are not sufficiently differentiable.

In this note we consider the "optimal assignment problem" as such a process, and establish the complementarity-substitutability relationships between its elements. In one of the standard interpretations of this problem, a heterogeneous collection of men and machines is available, with each man-machine pair capable of producing at a certain rate, and the object is to match men and machines so as to maximize the total product.[1] In our present approach it is the availabilities of the various men and machines that are the independent inputs, and our function F is the maximized total product. Since the activities are discrete variables, our results must be expressed in difference rather than differential form. The results are not particularly startling: they say that man-machine pairs are complementary, while man-man and machine-machine pairs are substitutes. However, the proofs were elusive enough to deserve being recorded; hence this note.

In a companion note[2] we shall prove some similar relationships for the "maximal network flow" process.

2. NOTATION

Let $a_{ij} \geq 0$ be the "output" resulting from the assignment of "man" i to "machine" j, and let A denote the matrix (a_{ij}). Wellet the ranges of i and j be arbitrary finite index sets M and N, respectively, since the use of specific integers turns out to be notationally inconvenient. Let $m = |M|$ and $n = |N|$, the numbers of men and machines, respectively.

Define the <u>score</u> $S = S(A)$ as

$$S = \max_{\mathcal{P}} \sum_{h=1}^{r} a_{i_h j_h}, \qquad (1)$$

where r denotes min (m, n) and the maximum is taken over all

[1] Another interpretation is the pairing-off of traders in a market so as to maximize total profit. F then becomes the characteristic function of the associated n-person game.

[2] L. S. Shapley, "On Network Flow Functions," The RAND Corporation, Paper P-2185.

<u>assignments</u> (i.e., collections of disjoint pairs) $\mathcal{P} = \{(i_1, j_1), \ldots, (i_r, j_r)\}$. If $k \in M \cup N$ is the index of a row or column of A, define $S_k = S(A_k)$ as the score of the <u>reduced matrix</u> A_k obtained by deleting that row or column from A. Again, if k is the index of a row or column not present in A, then define S^k as the score of the <u>augmented matrix</u> A^k obtained by adjoining that new row or column to A (assuming that the relevant numbers a_{ik} or a_{kj} have already been defined). It is clear that the order in which successive reductions or augmentations are applied is immaterial; thus $S_{k\ell} = S_{\ell k}$, etc. If either index set is empty, so that no matrix is defined, the score is arbitrarily taken to be zero.

3. COMPLEMENTARITY—SUBSTITUTABILITY RELATIONSHIPS

We shall establish two results, as follows.

THEOREM 1. <u>If</u> k <u>and</u> ℓ <u>both refer to rows, or both to columns, then</u>

$$(2) \qquad (S^k - S) + (S^\ell - S) \geq S^{k\ell} - S;$$

<u>i.e., similar elements may be substitutes, but are never complements.</u>

<u>Proof</u>. Without loss of generality we consider just the "row" version: $k, \ell \in M$. The proof proceeds by induction on the number $\lambda = \lambda(m, n) = \min(m, n-1)$. The cases with $\lambda = 0$ are easy, though not wholly trivial. If $\lambda > 0$, then let

$\mathscr{P}^{k\ell}$ be a maximizing assignment for the doubly augmented matrix $A^{k\ell}$. We can assume that both k and ℓ appear in $\mathscr{P}^{k\ell}$, since if k (say) is absent, then $S^{k\ell} = S^{\ell}$ and (2) is trivial. Thus, let u, v ϵ N be the column indices paired with k and ℓ, respectively, in $\mathscr{P}^{k\ell}$. (See Fig. 1. The matrix A is indicated by the heavy outline.)

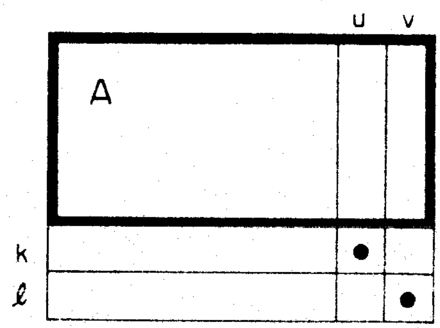

Fig. 1

The following statements are then more or less immediate from (1):

(3) $$S^{k\ell} = S_{uv} + a_{ku} + a_{\ell v},$$

(4) $$S_u + a_{ku} \leq S^k,$$

(5) $$S_v + a_{\ell v} \leq S^{\ell}.$$

If (2) is false, we have

(6) $$S^k + S^{\ell} < S^{k\ell} + S.$$

Adding (3), (4), (5), and (6) yields

$$S_u + S_v < S + S_{uv}.$$

This implies that the "column" version of (2) is false for the reduced matrix A_{uv}, and hence that the "row" theorem fails for the transpose of A_{uv}, which is a matrix of $n-2$ rows and m columns. But this contradicts the induction hypothesis, since $\lambda(n-2, m) = \min(n-2, m-1) = \lambda - 1$. Therefore (2) is true. Q.E.D.

THEOREM 2. If k refers to a row and ℓ to a column, then

$$(S^k - S) + (S^\ell - S) \leq S^{k\ell} - S; \tag{7}$$

i.e., dissimilar elements may be complementary, but are never substitutes.

Proof. This time let the induction be on $\lambda = \min(m, n)$. The cases with $\lambda = 0$ are trivial. If $\lambda > 0$, then let $\mathcal{P}^k$ be a maximizing assignment for A^k. We can assume that k appears in $\mathcal{P}^k$, since otherwise $S^k = S$ and (7) is trivial. Thus, let u be the column paired with k in $\mathcal{P}^k$, and similarly let v be the row paired with ℓ in a $\mathcal{P}^\ell$ that maximizes for A^ℓ. (See Fig. 2.)

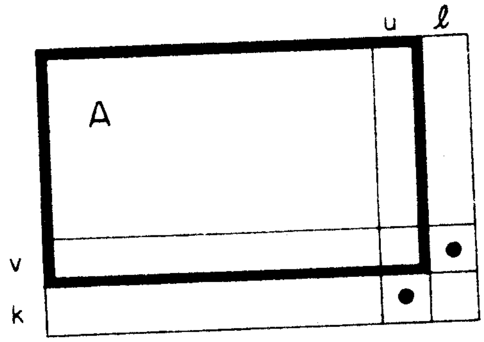

Fig. 2

The following statements are then more or less immediate from (1):

(8) $S^{k\ell} \geq S_{uv} + a_{ku} + a_{v\ell}$,

(9) $S_u + a_{ku} = S^k$,

(10) $S_v + a_{v\ell} = S^{\ell}$.

If (7) is false we have

(11) $S^k + S^{\ell} > S^{k\ell} + S$.

Adding (8), (9), (10), and (11) yields

$$S_u + S_v > S + S_{uv}.$$

This implies that (7) is false for the matrix A_{uv} with induction number $\min(m-1, n-1) = \lambda - 1$, contradicting the induction hypothesis. Therefore (7) is true. Q.E.D.

4. COMMENT

The above results can be extended in a straight forward manner to the Hitchcock–Koopmans transportation model. A typical result is that in the presence of an oversupply the requirements at two ports are never complementary in their effect on the (negative) total shipping cost.