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Complements of Grassmann substructures in projective Grassmannians

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Abstract. We prove that a projective Grassmannian can be recovered from the complement of one of its Grassmann substructures. Even more, the underlying projective space with the interval of its distinguished subspaces can be recovered.

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Introduction

The idea of deleting a hyperplane from a geometric structure is not new. It has been applied in various contexts starting from an affine space as a result of deleting a hyperplane from a projective space. In [3] the complement of a *geometric hyperplane*, i.e. a proper subspace that either meets each line in a unique point or contains that line, in a Grassmann space, also known as a space of pencils, is investigated.

Instead of a hyperplane any subspace can be deleted to obtain a more interesting structure, which in the case of projective geometry is called a *slit space* (cf. [5,6]). Just to mention [11] which deals with the so called *partial geometry* that consists of points of a finite projective space $\mathfrak{P}$ not contained in a fixed secundum (a subspace of codimension 2) W of $\mathfrak{P}$ and lines of $\mathfrak{P}$ which do not intersect W. In [7] configurations that arise from finite affine planes by removing a pencil of lines are investigated. The authors generalise the result of [13] that the complement of a line in a finite affine plane can be embedded into a projective plane of the same order. The next example is the *affine space of rectangular matrices*, introduced in [12, Ch. 3], which resembles the structure of subspaces in a projective space that are complementary to a distinguished subspace. Two different approaches to this structure are delivered by [1] and [10].

In both papers it is shown that the structure of complements is embeddable into an affine space and not all lines of that affine space arise as lines of this structure.

Another good reason (frankly the real reason) to play with complements or slit spaces is as follows. Consider a finite dimensional projective space $\mathfrak{P}$ coordinatized by an even characteristic field and endowed with a non-symplectic projective polarity $\perp$, a *pseudopolarity*. Let W be the set of all self-conjugate points of $\mathfrak{P}$ w.r.t. $\perp$. Then W is known to be a proper subspace of $\mathfrak{P}$. If p is a point in $\mathfrak{P}$ we write $p^\perp$ for the set of all points conjugate to p; for a subspace U we have then $U^\perp := \bigcap_{p\in U} p^\perp$. We say that a subspace U is *regular* when its *radical* $U \cap U^\perp$ is void. Note that $U \cap U^\perp \subset \mathsf{W}$ and those subspaces that are *skew* (disjoint) to W are regular. So, our present result moves us closer to solving a similar problem of recovering the ambient projective Grassmannian from the Grassmannian of regular subspaces w.r.t. a pseudopolarity.

In spaces of pencils interval subspaces, i.e. those induced by intervals in the lattice of subspaces of the underlying space, form a very important class. Each of them carries the structure of a space of pencils and any subspace with this property is an interval subspace (see [15]). This is the reason to call such subspaces *Grassmann subspaces*. Grassmannians are similar geometries to spaces of pencils in that they have the same point set while lines are "thicker", they are not pencils of subspaces in the underlying space but bases (upper covers) of these pencils, the incidence being inclusion not the membership relation. Interval substructures in Grassmannians have analogous properties to those of spaces of pencils.

On the other hand a subspace W can be identified with the principal ideal $(\mathsf{W}]$ which is a specific interval $[\mathsf{Z}, \mathsf{W}]$, where Z is the zero (bottom) in the lattice of subspaces of $\mathfrak{P}$. This motivated us to investigate the complement of a Grassmann substructure in a projective Grassmannian. Skew subspaces lose their nice meaning in this general setting, so instead we take *outer* subspaces, i.e. those not contained in the deleted interval. We begin with a general case of a Grassmannian of outer subspaces w.r.t. a fixed interval substructure and show that the ambient projective space can be recovered from this Grassmannian. Later, we discuss a more specific case where an ideal is deleted and we prove that also from a skew Grassmannian the outer Grassmannian and thus, the underlying projective space, can be recovered.

1. Basic notions

Let us consider a point-line incidence structure $\mathfrak{A} = \langle S, \mathcal{L}, | \rangle$, where S is a non-empty set whose elements are called *points*, $\mathcal{L}$ is a non-empty set whose members are called *lines*, and $| \subseteq S \times \mathcal{L}$ is an *incidence relation*. We say that a point $a \in S$ is incident to (or lies on) a line $l \in \mathcal{L}$ and write $a \mid l$. It is said that $\mathfrak{A}$ is a *partial linear space*, when no two distinct points are incident to two distinct lines (cf. [2]).

Now, let $S' \subseteq S$ and $\mathcal{L}' \subseteq \mathcal{L}$. We call $\mathfrak{A}' = \langle S', \mathcal{L}' \rangle$ a *closed substructure* of $\mathfrak{A}$ (or a *Baire substructure*) iff the following two conditions for $a \in S, l \in \mathcal{L}$ are met:

(A) if two points of $\mathfrak{A}'$ are incident to a line l, then l is a line of $\mathfrak{A}'$, and dually
(B) if two lines of $\mathfrak{A}'$ are incident to a point a, then a is a point of $\mathfrak{A}'$.

This definition can be stated equivalently in this way

Lemma 1.1. *$\langle S', \mathcal{L}' \rangle$ is a closed substructure of $\mathfrak{A}$ iff the following two conditions are met:*

(C) *If $a \in S \setminus S'$, then there is at most one line $l \in \mathcal{L}'$ such that $a \mid l$.*
(D) *If $l \in \mathcal{L} \setminus \mathcal{L}'$, then there is at most one point $a \in S'$ such that $a \mid l$.*

A *triangle* in $\mathfrak{A}$ is a system of three points called *vertices*, and three lines, called *sides*, where every vertex is incident to exactly two sides (or dually, every side is incident to exactly two vertices). If $\mathfrak{A}$ contains a triangle, then we call it *non-trivial*.

We write $\boldsymbol{\Delta}^-_{\mathfrak{A}}(a_1, a_2, a_3)$ when a_1, a_2, a_3 are the vertices of a triangle in $\mathfrak{A}$ and we write $\boldsymbol{\Delta}^+_{\mathfrak{A}}(l_1, l_2, l_3)$ when l_1, l_2, l_3 are the sides of a triangle in $\mathfrak{A}$. To a triangle of vertices a_1, a_2, a_3 we associate two sets of points:

$$\boldsymbol{\lambda}^-_{\mathfrak{A}}(a_1, a_2, a_3) := \{p \in S\colon\ \boldsymbol{\Delta}^-_{\mathfrak{A}}(a_1, a_2, p) \vee \boldsymbol{\Delta}^-_{\mathfrak{A}}(a_2, a_3, p) \vee \boldsymbol{\Delta}^-_{\mathfrak{A}}(a_3, a_1, p)\}, \quad (1)$$

$$\boldsymbol{\Lambda}^-_{\mathfrak{A}}(a_1, a_2, a_3) := \bigcup \big\{\boldsymbol{\lambda}^-_{\mathfrak{A}}(b_1, b_2, b_3)\colon\ \boldsymbol{\Delta}^-_{\mathfrak{A}}(b_1, b_2, b_3) \wedge b_1, b_2, b_3 \in \boldsymbol{\lambda}^-_{\mathfrak{A}}(a_1, a_2, a_3)\big\}. \quad (2)$$

Note that the condition defining $\boldsymbol{\lambda}^-_{\mathfrak{A}}(a_1, a_2, a_3)$ is simply a requirement that two specific lines joining p with two of the vertices a_1, a_2, a_3 exist. Formally we can drop $\boldsymbol{\Delta}^-_{\mathfrak{A}}(\cdot, \cdot, \cdot)$ and write a bit more complex formula:

$$\boldsymbol{\lambda}^-_{\mathfrak{A}}(a_1, a_2, a_3) = \big\{p \in S\colon (\exists m_1, m_2 \in \mathcal{L})[p \mid m_1, m_2 \wedge \\ \vee_{\{i,j,k\}=\{1,2,3\}}(a_i \mid m_1 \wedge a_j \mid m_2 \wedge m_1, m_2 \neq l_k)]\big\}. \quad (3)$$

By (1) it is clear that $a_1, a_2, a_3 \in \boldsymbol{\lambda}^-_{\mathfrak{A}}(a_1, a_2, a_3)$. Hence in view of (2) we get

$$\boldsymbol{\lambda}^-_{\mathfrak{A}}(a_1, a_2, a_3) \subseteq \boldsymbol{\Lambda}^-_{\mathfrak{A}}(a_1, a_2, a_3). \quad (4)$$

Similarly, for a triangle of the sides l_1, l_2, l_3 we define:

$$\boldsymbol{\lambda}^+_{\mathfrak{A}}(l_1, l_2, l_3) := \{m \in \mathcal{L}\colon\ \boldsymbol{\Delta}^+_{\mathfrak{A}}(l_1, l_2, m) \vee \boldsymbol{\Delta}^+_{\mathfrak{A}}(l_2, l_3, m) \vee \boldsymbol{\Delta}^+_{\mathfrak{A}}(l_3, l_1, m)\}, \quad (5)$$

$$\boldsymbol{\Lambda}^+_{\mathfrak{A}}(l_1, l_2, l_3) := \bigcup \big\{\boldsymbol{\lambda}^+_{\mathfrak{A}}(k_1, k_2, k_3)\colon\ \boldsymbol{\Delta}^+_{\mathfrak{A}}(k_1, k_2, k_3) \wedge k_1, k_2, k_3 \in \boldsymbol{\lambda}^+_{\mathfrak{A}}(l_1, l_2, l_3)\big\}, \quad (6)$$

and have

$$\boldsymbol{\lambda}^{+}_{\mathfrak{A}}(l_1, l_2, l_3) \subseteq \boldsymbol{\Lambda}^{+}_{\mathfrak{A}}(l_1, l_2, l_3). \tag{7}$$

The meaning of these notions will become clearer later when we use maximal cliques of adjacencies of points and lines in an incidence structure, which our paper is devoted to.

1.1. Grassmannians

Grassmann spaces frequently appear in the literature, just to mention [2,9]. The most general definition could be probably as follows. Let X be a non-empty set and let $\mathcal{P}$ be a family of subsets of X. Assume that there is a dimension function $\dim : \mathcal{P} \to \{0, \ldots, n\}$ such that $\langle \mathcal{P}, \subset, \dim \rangle$ is an incidence geometry. Write $\mathcal{P}_k$ for the set of all $U \in \mathcal{P}$ with $\dim(U) = k$. For $H \in \mathcal{P}_{k-1}$ and $B \in \mathcal{P}_{k+1}$ with $H \subset B$ the set

$$\mathbf{p}(H, B) := \{U \in \mathcal{P}_k \colon H \subset U \subset B\} \tag{8}$$

is called a *pencil*; $\mathcal{P}_k$ stands for the family of all such pencils. If $0 < k < n$, then the *k-th Grassmann space* is a partial linear space

$$\mathbf{P}_k(\mathcal{P}) := \langle \mathcal{P}_k, \mathcal{P}_k, \in \rangle.$$

This is the most common understanding of a Grassmann space. We used to call it a *space of pencils* for its specific line set and to distinguish it from a closely related point-line geometry consisting of $\mathcal{P}_k$ as points and $\mathcal{P}_{k+1}$ as lines. Namely, if $0 < k < n - 1$, then

$$\mathbf{G}_k(\mathcal{P}) := \langle \mathcal{P}_k, \mathcal{P}_{k+1}, \subset \rangle,$$

is a partial linear space which we call a *k-th Grassmannian* (cf. [8, Section 1.1.2]). Points $U_1, U_2 \in \mathcal{P}_k$ are collinear in $\mathbf{G}_k(\mathcal{P})$ if there is $B \in \mathcal{P}_{k+1}$ such that $U_1, U_2 \subset B$.

When $\mathcal{P}$ is the family of subspaces in a projective (resp. affine) space, then we say that $\mathbf{G}_k(\mathcal{P})$ is a *projective* (or resp. an *affine*) Grassmannian.

2. Slit spaces and their Grassmannians

In the paper we shall follow a rather old-fashioned tradition of investigations on a projectively embeddable geometry G where all the considered objects are projective; some of them are referred to as *proper*, and these are, strictly speaking, the objects of G, and the others are referred to as *improper*. Nevertheless, the incidence is *one and fixed*: the projective one.

Let $\mathfrak{P}$ be a finite dimensional projective space coordinatized in a vector space V with $\dim(V) = n$. We write $\mathcal{H}(V)$, or $\mathcal{H}$ in short, for the set of all subspaces of V and $\mathcal{H}_k(V)$, or just $\mathcal{H}_k$, for the set of all k-subspaces (k-dimensional subspaces) of V. The null subspace of V will be denoted by

Θ. For convenience we identify $\mathfrak{P}$ with the lattice of subspaces of V, so that $\mathfrak{P} \cong \langle \mathcal{H}_1, \mathcal{H}_2, \subset \rangle$ and generally, k-subspaces of $\mathfrak{P}$ are $(k+1)$-subspaces of V. Two k-subspaces U_1, U_2 in V are said to be *adjacent* if their meet $U_1 \cap U_2$ is a $(k-1)$-subspace, or equivalently, their join $U_1 + U_2$ is a $(k+1)$-subspace. For $Z, W \in \mathcal{H}$ an *interval* is the set $[Z, W] = \{U \in \mathcal{H} \colon Z \subseteq U \subseteq W\}$.

It is folklore that

Fact 2.1. *The underlying projective incidence geometry, i.e.* $(\mathcal{H}_i \colon 0 \leq i \leq n)$, *can be recovered up to isomorphism and duality in terms of the Grassmann space* $\mathbf{P}_k(\mathcal{H})$ *if* $0 < k < n$ *and it can be recovered in terms of the Grassmannian* $\mathbf{G}_k(\mathcal{H})$ *if* $0 < k < n-1$.

2.1. Triangles in Grassmannians

Let us recall a known but quite essential fact.

Fact 2.2. *Three pairwise adjacent k-subspaces in $\mathfrak{P}$ either cover a $(k-1)$-subspace, or they are covered by a $(k+1)$-subspace.*

Consider a projective Grassmannian $\mathfrak{M} := \mathbf{G}_k(\mathcal{H})$ and let $U_1, U_2, U_3 \in \mathcal{H}_k$ be points of $\mathfrak{M}$ such that $\mathbf{\Delta}^-_{\mathfrak{M}}(U_1, U_2, U_3)$. Then they are pairwise distinct, pairwise adjacent in $\mathfrak{M}$, and not collinear in $\mathfrak{M}$. Hence in view of 2.2 there is $H \in \mathcal{H}_{k-1}$ with $H \subset U_1, U_2, U_3$ and there are lines Y_1, Y_2, $Y_3 \in \mathcal{H}_{k+1}$ of $\mathfrak{M}$ with $Y_i = U_j + U_l$, where $\{i, j, l\} = \{1, 2, 3\}$. This means that $\mathbf{\Delta}^+_{\mathfrak{M}}(Y_1, Y_2, Y_3)$ and again by 2.2 there is $B \in \mathcal{H}_{k+2}$ with $Y_1, Y_2, Y_3 \subset B$. We can say even more: $H = U_1 \cap U_2 \cap U_3 = U_i \cap U_j$ and $B = Y_1 + Y_2 + Y_3 = Y_i + Y_j$ for $1 \leq i < j \leq 3$.

In other words we have an interval $[H, B]$ that contains the vertices and sides of our triangle Δ. This interval is of length 3 hence it can be viewed as a projective plane (more precisely $\mathbf{G}_1(B/H) = \mathbf{G}_1(\mathcal{H}(B/H))$ is a projective plane). Hence Δ is a triangle on some projective plane.

2.2. Complement of a Grassmann substructure

Let us fix two subspaces Z and W in V so that $[\mathsf{Z}, \mathsf{W}] \neq \emptyset$ (in other words $\mathsf{Z} \subseteq \mathsf{W}$). We will analyze the following class of *outer* subspaces:

$$\mathcal{D} := \big\{U \in \mathcal{H} \colon U \notin [\mathsf{Z}, \mathsf{W}]\big\} = \{U \in \mathcal{H} \colon \mathsf{Z} \nsubseteq U \text{ or } U \nsubseteq \mathsf{W}\}.$$

Set $\mathcal{D}_k := \mathcal{D} \cap \mathcal{H}_k$. Note that $\mathcal{D}_k = \emptyset$ iff $[\mathsf{Z}, \mathsf{W}] = \mathcal{H}$, that is iff $\mathsf{Z} = \Theta$ and $\mathsf{W} = V$. So, to avoid an empty $\mathcal{D}_k$ we assume in the sequel that

$$\mathsf{Z} \neq \Theta \quad \text{or} \quad \mathsf{W} \neq V. \tag{9}$$

On the other hand we have:

$$\mathcal{D}_k = \begin{cases} \mathcal{H}_k, & \text{if } k < \dim(\mathsf{Z}) \text{ or } \dim(\mathsf{W}) < k, \\ \mathcal{H}_k \setminus \{\mathsf{Z}\}, & \text{if } k = \dim(\mathsf{Z}), \\ \mathcal{H}_k \setminus \{\mathsf{W}\}, & \text{if } k = \dim(\mathsf{W}). \end{cases} \tag{10}$$

Consider the k-th Grassmannian

$$\mathfrak{D}_k := \mathbf{G}_k(\mathcal{D}) = \langle \mathcal{D}_k, \mathcal{D}_{k+1}, \subset \rangle.$$

It is a non-trivial partial linear space for $1 \leq k \leq n-2$. Now let us have a look at the case where $k = 1$. If $\dim(\mathsf{Z}) = 0$, then $\mathfrak{D}_1$ is a *slit space*, i.e. a projective space with one of its subspaces deleted (cf. [5,6]) and if additionally $\operatorname{codim}(\mathsf{W}) = 1$, then $\mathfrak{D}_1$ simply becomes an affine space. If $\dim(\mathsf{Z}) = 1$, then we delete from $\mathfrak{P}$ the point Z and some bundle of lines through Z, so $\mathfrak{D}_1$ is a variant of a punctured projective space. If $\dim(\mathsf{Z}) = 2$, then $\mathfrak{D}_1$ is $\mathfrak{P}$ with the line Z deleted. So, further we assume that

$$2 \leq k \leq n-2. \tag{11}$$

Generally, $\mathfrak{D}_k$ is not a gamma space in the sense of [2] as it does not need to satisfy *none-one-or-all* of the axioms. To verify this take a triangle of vertices U, U_1, U_2 so that $U \subset \mathsf{W}, \mathsf{Z} \not\subset U$ (which implies $\mathsf{Z} \neq \Theta$), and $\mathsf{W} \cap \overline{U_1, U_2}$ contains some point U_0 of $\mathfrak{D}_k$. Then the potential line $\overline{U_0, U}$ may lie in $[\mathsf{Z}, \mathsf{W}]$. So, instead $\mathfrak{D}_k$ satisfies *none-one-all_except_one-or-all* of the axioms. If $\mathsf{Z} = \Theta$, or dually when $\mathsf{W} = V$, the Grassmannian $\mathfrak{D}_k$ is a gamma space.

We can think of $\mathfrak{D}_k$ in a bit different way. The set $[\mathsf{Z}, \mathsf{W}]_k$ of k-subspaces in the interval $[\mathsf{Z}, \mathsf{W}]$ will be of interest for us. Note that the elements of $[\mathsf{Z}, \mathsf{W}]$ uniquely determine both Z and W, contrariwise to the elements of $[\mathsf{Z}, \mathsf{W}]_k$. The problem is when $\dim(\mathsf{Z}) = k$ or $\dim(\mathsf{W}) = k$ (comp. (10)). Then $[\mathsf{Z}, \mathsf{W}]_k$ becomes $\{\mathsf{Z}\}$ or $\{\mathsf{W}\}$ respectively and the other end of the interval is meaningless. For this reason we introduce the maximal $\mathsf{Z}_{\max}$ and minimal $\mathsf{W}_{\min}$ such that $[\mathsf{Z}, \mathsf{W}]_i = [\mathsf{Z}_{\max}, \mathsf{W}_{\min}]_i$ for $i = k, k+1$, formally,

$$\mathsf{Z}_{\max} = \bigwedge\big([\mathsf{Z}, \mathsf{W}]_k \cup [\mathsf{Z}, \mathsf{W}]_{k+1}\big) \quad \text{and} \quad \mathsf{W}_{\min} = \bigvee\big([\mathsf{Z}, \mathsf{W}]_k \cup [\mathsf{Z}, \mathsf{W}]_{k+1}\big),$$

where $\bigwedge$ is a meet operation and $\bigvee$ is a join operation in the lattice of subspaces of V. Most of the time they are not needed but sometimes they are indispensable. For projective Grassmannians the following is known or could be easily obtained (cf. [4,14]).

Fact 2.3. *For any* $\mathsf{Z}, \mathsf{W} \in \mathcal{H}$ *with* $\mathsf{Z} \subset \mathsf{W}$ *we have*

$$\mathbf{G}_k(\mathcal{H})\big|[\mathsf{Z}, \mathsf{W}] = \big\langle [\mathsf{Z}, \mathsf{W}]_k, [\mathsf{Z}, \mathsf{W}]_{k+1} \big\rangle \tag{12}$$

and under a lax embedding the image of any other projective Grassmannian in $\mathbf{G}_k(\mathcal{H})$ *is of the form* (12).

It is clear that (12) is a closed substructure of $\mathbf{G}_k(\mathcal{H})$. We call such a substructure a *Grassmann substructure* (cf. [15]). Now, take the complement of the Grassmann substructure (12) in $\mathbf{G}_k(\mathcal{H})$ to get our Grassmannian $\mathbf{G}_k(\mathcal{D})$.

Fact 2.4. *The Grassmannian $\mathbf{G}_k(\mathcal{H})$ with the substructure (12) distinguished, that is the Grassmannian $\mathfrak{D}_k$, satisfies (C) and (D) formulated in 1.1.*

2.3. Stars and tops

Let $H \in \mathcal{H}_{k-1}$. The set

$$\mathrm{S}(H) := \{U \in \mathcal{D}_k \colon H \subset U\} \tag{13}$$

will be called a *star*. It is a set of points of $\mathfrak{D}_k$. Dually, with each $B \in \mathcal{H}_{k+2}$ we associate the set

$$\mathrm{T}(B) := \{Y \in \mathcal{D}_{k+1} \colon U \subset B\} \tag{14}$$

called a *top*. It is a set of lines of $\mathfrak{D}_k$. One observation is trivial:

Fact 2.5. *Let $H \in \mathcal{H}_{k-1}, B \in \mathcal{H}_{k+2}$. If $\mathsf{Z} \subseteq H$ and $\mathsf{W} = V$, then $\mathrm{S}(H) = \emptyset$, and if $\mathsf{Z} = \Theta$ and $B \subseteq \mathsf{W}$, then $\mathrm{T}(B) = \emptyset$.*

These are the only cases when a star or a top can be void. Otherwise, every star and every top contains some triangle in $\mathfrak{D}_k$. Before we prove that in 2.7, a key observation is in order.

Fact 2.6. *For $H \in \mathcal{H}_{k-1}$, if $\mathsf{Z} \nsubseteq H$ or $\mathsf{W} \neq V$, i.e. if $\mathrm{S}(H) \neq \emptyset$, then*

$$\mathfrak{D}_H := \mathbf{G}_k(\mathcal{D})|\mathrm{S}(H) = \big\langle [H)_k \setminus [H+\mathsf{Z},\mathsf{W}]_k, [H)_{k+1} \setminus [H+\mathsf{Z},\mathsf{W}]_{k+1} \big\rangle, \tag{15}$$

which means that a star induces a projective space with an interval of its subspaces deleted. More precisely,

(i) *if $\dim(H+\mathsf{Z}) = k-1$, i.e. $\mathsf{Z} \subseteq H$, then $\mathfrak{D}_H$ is a slit space,*
(ii) *if $\dim(H+\mathsf{Z}) = k$, then $\mathfrak{D}_H$ is a projective space with the point $H+\mathsf{Z}$ and a bundle of lines contained in W through the deleted point,*
(iii) *if $\dim(H+\mathsf{Z}) = k+1$, then $\mathfrak{D}_H$ is a projective space with the line $H+\mathsf{Z}$ deleted,*
(iv) *if $\dim(H+\mathsf{Z}) \geq k+2$, then $\mathfrak{D}_H$ is a projective space.*

Thanks to duality as well, a top is either void or a projective space with an interval of its subspaces deleted.

Proposition 2.7. *Let $H \in \mathcal{H}_{k-1}, B \in \mathcal{H}_{k+2}$.*

(i) *If $\mathsf{Z} \nsubseteq H$ or $\mathsf{W} \neq V$, then there is a triangle of $\mathfrak{D}_k$ with its vertices in $\mathrm{S}(H)$.*
(ii) *If $\mathsf{Z} \neq \Theta$ or $B \nsubseteq \mathsf{W}$, then there is a triangle of $\mathfrak{D}_k$ with its sides in $\mathrm{T}(B)$.*

Proof. (i) is immediate by 2.6 and (ii) requires noting that it is dual to (i). $\square$

In view of Sect. 2.1 three points U_1, U_2, U_3 that are vertices of a triangle in $\mathfrak{D}_k$, i.e. $\boldsymbol{\Delta}^-_{\mathfrak{D}_k}(U_1, U_2, U_3)$ holds, span a unique star $\mathrm{S}(H)$, where $H = U_1 \cap U_2 \cap U_3 \in \mathcal{H}_{k-1}$. This justifies writing $\mathrm{S}(U_1, U_2, U_3) := \mathrm{S}(H)$.

The key observation is that stars can be defined purely in terms of the incidence structure $\mathfrak{D}_k$. To prove this it is convenient to prove some auxiliary lemma first.

Lemma 2.8. *If* $\boldsymbol{\Delta}^-_{\mathfrak{D}_k}(U_1, U_2, U_3)$*, then*

(i) $\boldsymbol{\lambda}^-_{\mathfrak{D}_k}(U_1, U_2, U_3) \subseteq \mathrm{S}(U_1, U_2, U_3)$,

(ii) $\boldsymbol{\Lambda}^-_{\mathfrak{D}_k}(U_1, U_2, U_3) \subseteq \mathrm{S}(U_1, U_2, U_3)$.

Proof. (i) Let $U \in \boldsymbol{\lambda}^-_{\mathfrak{D}_k}(U_1, U_2, U_3)$. According to (5) we have $\boldsymbol{\Delta}^-_{\mathfrak{D}_k}(U_i, U_j, U)$ for some $1 \leq i < j \leq 3$ but it means that $U_1 \cap U_2 \cap U_3 = U_i \cap U_j \subset U$ which is sufficient for an argument.

(ii) Set $H := U_1 \cap U_2 \cap U_3$. If $W_1, W_2, W_3 \in \boldsymbol{\lambda}^-_{\mathfrak{D}_k}(U_1, U_2, U_3)$ are such that $\boldsymbol{\Delta}^-_{\mathfrak{D}_k}(W_1, W_2, W_3)$, then by (i) we get $\mathrm{S}(H) = \mathrm{S}(W_1, W_2, W_3)$. Now for $U \in \boldsymbol{\Lambda}^-_{\mathfrak{D}_k}(U_1, U_2, U_3)$, in view of (6) and (i) we have

$$U \in \boldsymbol{\lambda}^-_{\mathfrak{D}_k}(W_1, W_2, W_3) \subseteq \mathrm{S}(W_1, W_2, W_3) = \mathrm{S}(H).$$

□

Proposition 2.9. *Let* $\boldsymbol{\Delta}^-_{\mathfrak{D}_k}(U_1, U_2, U_3)$ *and distinguish a special case where*

($*$) *the ground field is* $\mathbb{Z}_2, n = k + 2,$ *and* $\dim(U_1 \cap U_2 \cap U_3 + \mathsf{Z}) = k$.

(i) *If* ($*$) *does not hold, then*

$$\mathrm{S}(U_1, U_2, U_3) = \boldsymbol{\Lambda}^-_{\mathfrak{D}_k}(U_1, U_2, U_3).$$

(ii) *If* ($*$) *holds and* Y_1, Y_2, Y_3 *are the sides of the triangle* U_1, U_2, U_3*, then*

$$\mathrm{S}(U_1, U_2, U_3) = \boldsymbol{\Lambda}^-_{\mathfrak{D}_k}(U_1, U_2, U_3) \cup \Big\{U \in \mathcal{D}_k \colon (\exists W \in \mathcal{D}_k)(\exists Y \in \mathcal{D}_{k+1}) \\ \big[U, W \subset Y \wedge \big(\vee_{1 \leq i < j \leq 3}(U \subset Y_i \wedge W \subset Y_j \wedge W \not\subset Y_i)\big)\big]\Big\}.$$

Consequently, stars are definable in terms of $\mathfrak{D}_k$.

Proof. Let us write Δ for the triangle of vertices U_1, U_2, U_3 and sides Y_1, Y_2, Y_3. Set $H := U_1 \cap U_2 \cap U_3$. We begin with (i).

$\supseteq$: By 2.8(ii).

$\subseteq$: Let $U \in \mathrm{S}(H)$. By (4) it suffices to show that $U \in \boldsymbol{\lambda}^-_{\mathfrak{D}_k}(U_1, U_2, U_3)$.

First, consider the case where U is on no side of Δ. Suppose on the contrary that $U \notin \boldsymbol{\lambda}^-_{\mathfrak{D}_k}(U_1, U_2, U_3)$, i.e. none of the possible triples U_i, U_j, U forms a triangle in $\mathfrak{D}_k$. In $\mathbf{G}_k(\mathcal{H})$ however all of these three triangles exist. So, some side of each of those three triangles must be in the deleted interval. It means that there are distinct $i, j \in \{1, 2, 3\}$ such that $U + U_i, U + U_j \notin \mathcal{D}_{k+1}$. A contradiction to 2.4.

Now, let us consider the case where U is on some side of Δ. We can assume that U is not a vertex of Δ as otherwise $U \in \boldsymbol{\lambda}^-_{\mathfrak{D}_k}(U_1, U_2, U_3)$. With

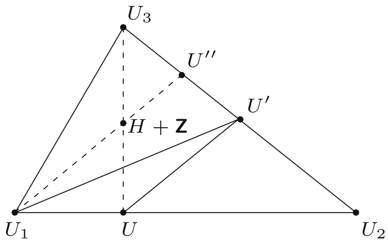

FIGURE 1. The case where the ground field is not $\mathbb{Z}_2$

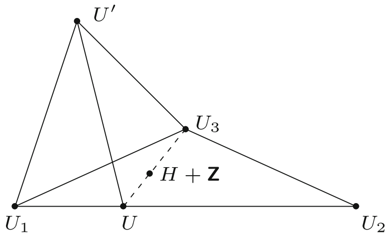

FIGURE 2. The case where the projective space $\mathfrak{D}_H$ is not a plane

no loss of generality, assume that $U \subset U_1 + U_2$. If $U + U_3 \in \mathcal{D}_{k+1}$, then $U \in \boldsymbol{\lambda}^-_{\mathfrak{D}_k}(U_1, U_2, U_3)$. So, assume that $U + U_3 \notin \mathcal{D}_{k+1}$. In view of 2.6, four possibilities need to be examined.

In case 2.6(i) the line $U + U_3$ is deleted together with its points U and U_3, which is impossible.

In case 2.6(ii) the sole deleted point $H + \mathsf{Z}$ is on the deleted line $U + U_3$. If the ground field is not $\mathbb{Z}_2$, then the size of lines in our projective space $\mathfrak{D}_H$ is at least 4. So, there are two additional points U', U'' on the side $U_2 + U_3$ (see Fig. 1). The line $U_1 + U''$ could be in the deleted bundle through $H + \mathsf{Z}$ but then $U' \in \mathcal{D}_k$ and $U' + U_1, U' + U \in \mathcal{D}_{k+1}$. This means that $U' \in \boldsymbol{\lambda}^-_{\mathfrak{D}_k}(U_1, U_2, U_3)$ and $U \in \boldsymbol{\lambda}^-_{\mathfrak{D}_k}(U_1, U_2, U')$ which by (2) gives that $U \in \boldsymbol{\Lambda}^-_{\mathfrak{D}_k}(U_1, U_2, U_3)$.

If $n \neq k + 2$, i.e. if our projective space $\mathfrak{D}_H$ is not a plane, then take a point U' not in the plane of Δ (see Fig. 2). There are three lines $U' + U_1, U' + U_2, U' + U \in \mathcal{D}_{k+1}$ as $H + \mathsf{Z}$ is on none of them. We proceed as above for U'.

In case 2.6(iii), as the lines are of size at least 3, take an additional point U' on the side $U_2 + U_3$ (see Fig. 3). As the sole line $U + U_3$ is deleted we have U' as in the previous case.

In case 2.6(iv) no line is deleted so the proof is complete.

Now, consider (ii). Note that $\mathfrak{D}_H$ is a Fano projective plane (lines are of size 3) where either (a): a pencil of lines through $H + \mathsf{Z}$ is deleted, or (b): the point $H + \mathsf{Z}$ and the line W through that point are deleted.

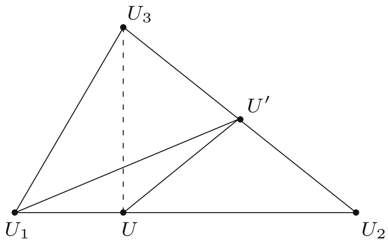

Figure 3. The case where the line $H + \mathsf{Z} = U + U_3$ is deleted

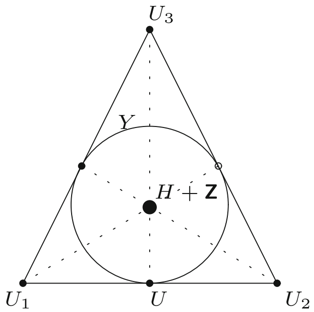

Figure 4. The case where a pencil of lines through $H + \mathsf{Z}$ is deleted

$\supseteq$: In view of 2.8(ii) we need to show that if a point U is on one of the sides Y_i of Δ and on a line Y crossing some other side Y_j of Δ in W, then $U \in \mathrm{S}(H)$. Observe that, U, W together with some vertex, say U_l, form a triangle which shares two sides with Δ. Hence, by 2.2, k-subspaces U, W, U_l cover a $(k-1)$-subspace, actually $U \cap W \cap U_l = H$, which means that $U \in \mathrm{S}(H)$.

$\subseteq$: Let $U \in \mathrm{S}(H)$. We can assume that U is not a vertex of Δ as otherwise we are through.

In case (a) note that the point $H + \mathsf{Z}$ cannot be on any side of Δ, so U must be on some side of Δ. Moreover, U is on a line Y which crosses another side of Δ (see Fig. 4).

In case (b) we have three possibilities (see Fig. 5):

(1) $H + \mathsf{Z}$ is on some side of Δ and W goes through a vertex of Δ,
(2) $H + \mathsf{Z}$ is on some side of Δ but W misses all the vertices of Δ,
(3) $H + \mathsf{Z}$ is on no side of Δ and W goes through a vertex of Δ.

In any of them it is easy to verify that $U \in \mathbf{\Lambda}^{-}_{\mathfrak{D}_k}(U_1, U_2, U_3)$. □

Dually, three lines Y_1, Y_2, Y_3 that are sides of a triangle in $\mathfrak{D}_k$, i.e. such that $\mathbf{\Delta}^{+}_{\mathfrak{D}_k}(Y_1, Y_2, Y_3)$, uniquely span a top $\mathrm{T}(B)$, where $B = Y_1 + Y_2 + Y_3 \in \mathcal{H}_{k+2}$, so we write $\mathrm{T}(Y_1, Y_2, Y_3) := \mathrm{T}(B)$. We have the dual to 2.9 here:

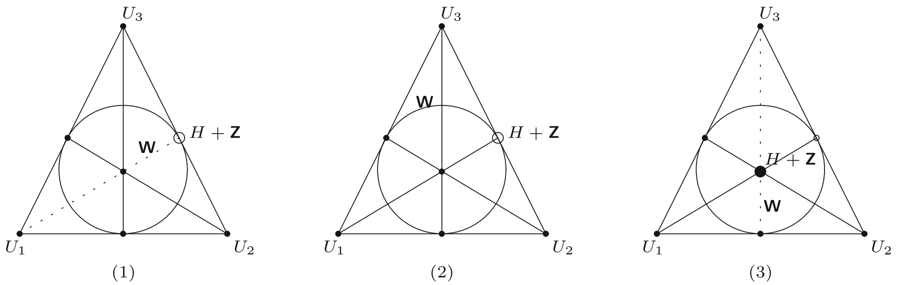

FIGURE 5. The case where the point $H+\mathsf{Z}$ and the line W through that point are deleted

Proposition 2.10. *Let* $\mathbf{\Delta}^+_{\mathfrak{D}_k}(Y_1,Y_2,Y_3)$ *and distinguish a special case where*

(∗) *the ground field is* $\mathbb{Z}_2, k=2,$ *and* $\dim((Y_1+Y_2+Y_3)\cap\mathsf{W})=k+1$.

(i) *If* (∗) *does not hold, then*
$$\mathrm{T}(Y_1,Y_2,Y_3)=\mathbf{\Lambda}^+_{\mathfrak{D}_k}(Y_1,Y_2,Y_3).$$

(ii) *If* (∗) *holds and* U_1,U_2,U_3 *are the vertices of the triangle* Y_1,Y_2,Y_3*, then*
$$\mathrm{T}(Y_1,Y_2,Y_3)=\mathbf{\Lambda}^+_{\mathfrak{D}_k}(Y_1,Y_2,Y_3)\cup\{Y\in\mathcal{D}_{k+1}\colon(\exists W\in\mathcal{D}_{k+1})(\exists U\in\mathcal{D}_k)$$
$$[U\subset Y,W\wedge\big(\vee_{1\le i<j\le 3}(U_i\subset Y\wedge U_j\subset W\wedge U_j\not\subset Y)\big)]\}.$$

Consequently, tops are definable in terms of $\mathfrak{D}_k$.

Note that when $\mathsf{W}=V$ for $H\in\mathcal{H}_{k-1}$ to have $\mathrm{S}(H)\neq\emptyset$ we need $\mathsf{Z}\not\subseteq H$ but this means that $H\in\mathcal{D}_{k-1}$. Dually, for $\mathsf{Z}=\Theta$ and $B\in\mathcal{H}_{k+2}$ we have $\mathrm{T}(B)\neq\emptyset$ iff $B\not\subseteq\mathsf{W}$. Hence, as an immediate consequence of 2.7, 2.9, and 2.10 we get

Proposition 2.11. (i) *If* $\mathsf{W}\neq V$*, then*
$$\big\{\mathrm{S}(H)\colon H\in\mathcal{H}_{k-1}\big\}=\big\{\mathrm{S}(U_1,U_2,U_3)\colon\ \mathbf{\Delta}^-_{\mathfrak{D}_k}(U_1,U_2,U_3)\big\}.\tag{16}$$

(ii) *If* $\mathsf{W}=V$*, then*
$$\big\{\mathrm{S}(U_1,U_2,U_3)\colon\ \mathbf{\Delta}^-_{\mathfrak{D}_k}(U_1,U_2,U_3)\big\}=\big\{\mathrm{S}(H)\colon H\in\mathcal{D}_{k-1}\big\}.\tag{17}$$

So, we can say that in the corresponding cases the classes $\mathcal{H}_{k-1}$ *and* $\mathcal{D}_{k-1}$ *are (up to the map* $H\longmapsto\mathrm{S}(H)$*) definable in* $\mathfrak{D}_k$.

(iii) *If* $\mathsf{Z}\neq\Theta$*, then*
$$\big\{\mathrm{T}(B)\colon B\in\mathcal{H}_{k+2}\big\}=\big\{\mathrm{T}(Y_1,Y_2,Y_3)\colon\ \mathbf{\Delta}^+_{\mathfrak{D}_k}(Y_1,Y_2,Y_3)\big\}.\tag{18}$$

(iv) *If* $\mathsf{Z}=\Theta$*, then*
$$\big\{\mathrm{T}(Y_1,Y_2,Y_3)\colon\ \mathbf{\Delta}^+_{\mathfrak{D}_k}(Y_1,Y_2,Y_3)\big\}=\big\{\mathrm{T}(B)\colon B\in\mathcal{D}_{k+2}\big\}.\tag{19}$$

So, now we can say that in the corresponding cases the classes $\mathcal{H}_{k+2}$ *and* $\mathcal{D}_{k+2}$ *are (up to the map* $B\longmapsto\mathrm{T}(B)$*) definable in* $\mathfrak{D}_k$.

3. Reconstructions and automorphisms

Lemma 3.1. *If $\mathsf{W} \neq V$, then the underlying projective space $\mathfrak{P}$ can be recovered from the Grassmannian $\mathbf{G}_k(\mathcal{D})$.*

Proof. Assume that $\mathsf{W} \neq V$. The class $\mathcal{H}_{k-1}$ is definable in $\mathfrak{D}_k$ by 2.11(i). This means that we get the structure $\mathfrak{M} = \langle \mathcal{H}_{k-1}, \mathcal{D}_k, \subset \rangle$ defined in $\mathfrak{D}_k$.

In case $k > 2$, applying the operation $\mathbf{\Lambda}^-$ in $\mathfrak{M}$ we find that

$$\{\mathbf{\Lambda}^-_{\mathfrak{M}}(U_1, U_2, U_3)\colon\ \mathbf{\Delta}^-_{\mathfrak{M}}(U_1, U_2, U_3)\} = \{\mathrm{S}(H)\colon H \in \mathcal{H}_{k-2}\}$$

as the same operation $\mathbf{\Lambda}^-$ applied in a substructure $\mathfrak{D}_{k-1}$ of $\mathfrak{M}$ would give the class $\mathcal{H}_{k-2}$ according to 2.11(i). Hence the class $\mathcal{H}_{k-2}$ is defined in $\mathfrak{D}_k$. So, the projective Grassmannian $\mathbf{G}_{k-2}(\mathcal{H}) = \langle \mathcal{H}_{k-2}, \mathcal{H}_{k-1}, \subset \rangle$ can be recovered in $\mathfrak{D}_k$. In view of 2.1 we are done.

If $k = 2$, then since $\mathfrak{D}_k \cong \mathfrak{D}_{n-k}$ and by (11) we have $n \geq 5$, so we can apply $\mathbf{\Lambda}^+$ in $\mathfrak{D}_2$. By 2.11(iii),(iv) we get the class $\mathcal{H}_4$ or $\mathcal{D}_4$ defined. Then, applying $\mathbf{\Lambda}^-$ on $\langle \mathcal{D}_3, \mathcal{H}_4 \rangle$ or $\langle \mathcal{D}_3, \mathcal{D}_4 \rangle$, respectively, in view of 2.11(i) the class $\mathcal{H}_2$ is defined. So, we have just recovered the projective space $\langle \mathcal{H}_1, \mathcal{H}_2, \subset \rangle \cong \mathfrak{P}$. □

Theorem 3.2. *The underlying projective space $\mathfrak{P}$ and the interval $[\mathsf{Z}_{\max}, \mathsf{W}_{\min}]$ can be recovered from the Grassmannian $\mathbf{G}_k(\mathcal{D})$.*

Proof. By 3.1 only the case where $\mathsf{W} = V$ needs to be solved to have $\mathfrak{P}$ recovered. In that case, however, note that $\mathsf{Z} \neq \Theta$ and we can use the reasoning dual to that of 3.1. Now, we actually have the underlying space with an additional structure, i.e. $(\mathfrak{P}, \mathcal{D}_k, \mathcal{D}_{k+1})$ the projective space $\mathfrak{P}$ with two families $\mathcal{D}_k$ and $\mathcal{D}_{k+1}$ of its subspaces distinguished. For $U \in \mathcal{H}_i$ it is trivially seen that $U \in [\mathsf{Z}_{\max}, \mathsf{W}_{\min}]$ iff $U \notin \mathcal{D}_i$ for $i = k, k+1$. Consequently, both $\mathsf{Z}_{\max}$ and $\mathsf{W}_{\min}$ are definable. □

The above statement can be rephrased in the language of automorphisms. Recall that an automorphism of a Grassmannian $\mathbf{G}_k(\mathcal{P}) = \langle \mathcal{P}_k, \mathcal{P}_{k+1}, \subset \rangle$ is a pair (f, g) where $f\colon \mathcal{P}_k \longrightarrow \mathcal{P}_k$, $g\colon \mathcal{P}_{k+1} \longrightarrow \mathcal{P}_{k+1}$, and $f(U) \subset g(Y)$ iff $U \subset Y$.

Theorem 3.3. *Each automorphism $F = (f, g)$ of the Grassmannian $\mathbf{G}_k(\mathcal{D})$ can be extended to an automorphism $F' = (f', g')$ of the projective Grassmannian $\mathbf{G}_k(\mathcal{H})$ such that f' preserves $[\mathsf{Z}, \mathsf{W}]_k$ and g' preserves $[\mathsf{Z}, \mathsf{W}]_{k+1}$. Hence f and g are both induced by a semilinear map on V that preserves $[\mathsf{Z}_{\max}, \mathsf{W}_{\min}]$.*

4. A special case of ideals (and dually of filters)

One specific case, or two if we also think of the dual one, is worth taking a closer look at. If $\mathsf{Z} = \Theta$, then our interval $[\mathsf{Z}, \mathsf{W}]$ becomes the principal ideal

$(\mathsf{W}]$ which can be basically identified with W. The dual case is when we take $\mathsf{W} = V$, then we get the principal filter $[\mathsf{Z})$.

Besides the class $\mathcal{D} = \{U \in \mathcal{H} \colon U \not\subset \mathsf{W}\}$ of outer subspaces, in this specific case here, the class of *skew* subspaces:

$$\mathcal{G} := \{U \in \mathcal{H} \colon U \cap \mathsf{W} = \Theta\}, \qquad \mathcal{G}_k := \mathcal{G} \cap \mathcal{H}_k,$$

seems to be interesting. Note that $\mathcal{D}_1 = \mathcal{G}_1$ and an element of $\mathcal{D}$ is contained in an element of $\mathcal{G}$ or it contains an element of $\mathcal{G}$. Moreover

$$\mathcal{G}_k \neq \emptyset \quad \text{iff} \quad 1 \le k \le \operatorname{codim}(\mathsf{W}), \qquad \mathcal{D}_k \neq \emptyset \quad \text{iff} \quad 1 \le k \le n.$$

The integer

$$\operatorname{ind} := \operatorname{ind}(\mathfrak{P}, \mathsf{W}) := \max\{k \colon \mathcal{G}_k \neq \emptyset\} \tag{20}$$

is well defined, and $\operatorname{ind} = \operatorname{codim}(\mathsf{W})$.

Recall that the class $\mathcal{G}_k$ consists of the strong k-subspaces of the *projective slit space*

$$\mathfrak{G}_1 := \mathbf{G}_1(\mathcal{G}) = \langle \mathcal{G}_1, \mathcal{G}_2, \subset \rangle.$$

If W is a secundum in $\mathfrak{P}$, then $\mathfrak{P} \setminus \mathsf{W}$ is a partial geometry (or its generalization to an arbitrary ground field) in the language of [11].

Evidently, the structure

$$\mathfrak{D}_1 := \mathbf{G}_1(\mathcal{D}) = \langle \mathcal{D}_1, \mathcal{D}_2, \subset \rangle$$

called a *slit space* (cf. [5,6]), is a restriction of $\mathfrak{P}$ to the set $\mathcal{D}_1$. If $\operatorname{ind} = 1$, then it is an affine space. For a line Y in $\mathfrak{D}_1$ we always have $\dim(Y \cap \mathsf{W}) \le 1$, so there is a natural parallelism, namely if Y_1, Y_2 are lines in $\mathfrak{D}_1$, then

$$Y_1 \parallel Y_2 \quad \text{iff} \quad Y_1 \cap \mathsf{W} = Y_2 \cap \mathsf{W}.$$

This is an equivalence relation but it does not need to satisfy Euclid's postulate in full. If $\mathfrak{D}_1$ is not an affine space, then there is a point U and a line Y in $\mathfrak{D}_1$ such that no line through U is parallel to Y. In the class of *affine* lines $\mathcal{A}_1 = \{Y \in \mathcal{D}_2 \colon Y \parallel Y\} = \{Y \in \mathcal{D}_2 \colon \dim(Y \cap \mathsf{W}) = 1\}$, however, the parallelism satisfies Euclid's postulate.

Clearly, 3.2 remains true in these more specific settings but we can do a bit more here for a new k-th Grassmannian $\mathfrak{G}_k := \mathbf{G}_k(\mathcal{G})$ over skew subspaces in $\mathfrak{P}$ with distinguished W as well as for our k-th Grassmannian $\mathfrak{D}_k = \mathbf{G}_k(\mathcal{D})$ over outer subspaces. Let us begin with a routine observation for $k = 1$.

Proposition 4.1. *Assume that* $2 \le \operatorname{ind}$*, i.e.* $\mathcal{G}_2, \mathcal{D}_2 \neq \emptyset$ *which means that* $\mathfrak{G}_1$ *and* $\mathfrak{D}_1$ *make sense. Then the three structures:* $(\mathfrak{P}, \mathsf{W}), \mathbf{G}_1(\mathcal{G})$*, and* $\mathbf{G}_1(\mathcal{D})$ *are mutually definable.*

When $1 \le k \le \operatorname{ind} - 1, n - 2$ the skew Grassmannian $\mathfrak{G}_k$ is a non-trivial partial linear space. It can be easily checked that $\mathfrak{G}_k$ is not a gamma space (cf. [2]) in general but it satisfies *none–one–all_except_one–or–all* of the axioms.

Stars and tops are key tools when it comes to Grassmannians and we have seen that already, but in this case their internal definitions in $\mathfrak{D}_k$ become a lot more simple: for $U_1, U_2, U_3 \in \mathcal{D}_k$ with $\mathbf{\Delta}^-_{\mathfrak{D}_k}(U_1, U_2, U_3)$

$$\mathrm{S}(U_1, U_2, U_3) = \big\{U \in \mathcal{D}_k\colon\ \mathbf{\Delta}^-_{\mathfrak{D}_k}(U_1, U_2, U) \vee \mathbf{\Delta}^-_{\mathfrak{D}_k}(U_2, U_3, U) \vee \mathbf{\Delta}^-_{\mathfrak{D}_k}(U_3, U_1, U)\big\} \tag{21}$$

and for $Y_1, Y_2, Y_3 \in \mathcal{D}_{k+1}$ with $\mathbf{\Delta}^+_{\mathfrak{D}_k}(Y_1, Y_2, Y_3)$ we have

$$\mathrm{T}(Y_1, Y_2, Y_3) = \big\{Y \in \mathcal{D}_{k+1}\colon\ \mathbf{\Delta}^+_{\mathfrak{D}_k}(Y_1, Y_2, Y) \vee \mathbf{\Delta}^+_{\mathfrak{D}_k}(Y_2, Y_3, Y) \vee \mathbf{\Delta}^+_{\mathfrak{D}_k}(Y_3, Y_1, Y)\big\}. \tag{22}$$

Recall that by 3.2 we can recover $\mathfrak{P}$ and $[\mathsf{Z}, \mathsf{W}]$ from our Grassmannian $\mathfrak{D}_k$. So, when $k \le \dim(\mathsf{W})$, i.e. when both $\mathbf{G}_k(\mathcal{D})$ and $\mathbf{G}_k(\mathcal{G})$ are not $\mathbf{G}_k(\mathcal{H})$, we immediately get the following theorem. We will give, however, a direct proof without the assumption on the size of lines in $\mathfrak{P}$.

Theorem 4.2. *If* $2 \le k \le n-2, \mathrm{ind}-1$*, then the Grassmannian* $\mathbf{G}_k(\mathcal{G})$ *can be recovered from the Grassmannian* $\mathbf{G}_k(\mathcal{D})$*.*

Proof. The reasoning runs through a sequence of simple steps.

(i) Let $Y_1, Y_2 \in \mathcal{D}_{k+1}$ be lines in $\mathfrak{D}_k$. We write

$$Y_1 \parallel Y_2 \iff (\exists \mathcal{T} \in \mathcal{T}(\mathfrak{D}_k))[Y_1, Y_2 \in \mathcal{T}] \wedge \neg(\exists U \in \mathcal{D}_k)[U \subset Y_1, Y_2], \tag{23}$$

where $\mathcal{T}(\mathfrak{D}_k)$ stands for the class of tops over $\mathfrak{D}_k$. Assume that $Y_1 \parallel Y_2$. Then $Y_1, Y_2 \subset B$ for some $B \in \mathcal{D}_{k+2}$ and $U := Y_1 \cap Y_2 \in \mathcal{H}_k \setminus \mathcal{D}_k$. Note that $U \notin \mathcal{D}_k$ means that $U \subset \mathsf{W}$ and hence $U = Y_1 \cap \mathsf{W} = Y_2 \cap \mathsf{W}$. So, $\dim(Y \cap \mathsf{W}) = k$ iff $Y \parallel Y'$ for some Y'.

(ii) Let $U_1, U_2 \in \mathcal{D}_k$ be points of $\mathfrak{D}_k$. We write

$$U_1 \parallel U_2 \iff (\exists Y_1, Y_2, Y_0 \in \mathcal{D}_{k+1})[Y_1 \parallel Y_2 \wedge U_1 \subset Y_0, Y_1 \wedge U_2 \subset Y_0, Y_2]. \tag{24}$$

Observe that if $U_1, U_2 \in \mathcal{D}_k, Y_1, Y_2 \in \mathcal{D}_{k+1}, Y_1 \parallel Y_2$, and there is $Y_0 \in \mathcal{D}_{k+1}$ such that $U_i \subset Y_0, Y_i$, then $\dim(U_1 \cap U_2) = k-1$, as U_1, U_2 are collinear in $\mathfrak{D}_k$, and $U_1 \cap U_2 \subset Y_1 \cap Y_2 \subset \mathsf{W}$ which, according to (23), formally means that $U_1 \parallel U_2$ for U_1, U_2 are treated as lines of $\mathfrak{D}_{k-1}$.

(iii) Let us write

$$\mathcal{F}_{l,m} := \big\{X \in \mathcal{D}_l\colon\ \dim(X \cap \mathsf{W}) = m\big\} \quad \text{for} \quad 0 \le m \le \min\{l, \dim(\mathsf{W})\}.$$

In view of (i), $\mathcal{F}_{k+1,k} = \big\{Y \in \mathcal{D}_{k+1}\colon (\exists Y')[Y \parallel Y']\big\}$ is definable in $\mathfrak{D}_k$. Also $\mathcal{F}_{k,k-1} = \big\{U \in \mathcal{D}_k\colon (\exists Y \in \mathcal{F}_{k+1,k})[U \subset Y]\big\}$ is definable in $\mathfrak{D}_k$.

(iv) Let $Y \in \mathcal{F}_{k+1,m}$ and $\mathcal{D}_k \ni U \subset Y$. Set $Y_0 = Y \cap \mathsf{W}$; by assumption, $\dim(Y_0) = m$. Clearly, $U \cap \mathsf{W} = U \cap Y_0$. Then U is a hyperplane in Y and thus $\dim(U \cap Y_0) \in \{m, m-1\}$. Finally,

$$\mathcal{F}_{k,m-1} = \big\{U \in \mathcal{D}_k\colon U \notin \mathcal{F}_{k,m} \wedge (\exists Y \in \mathcal{F}_{k+1,m})\big[U \subset Y\big]\big\}. \tag{25}$$

(v) Let $U \in \mathcal{F}_{k,m}$ and $U \subset Y \in \mathcal{D}_{k+1}$. Set $U_0 = U \cap \mathsf{W}$; from the assumption, $\dim(U_0) = m$. Then, $U_0 \subset Y \cap \mathsf{W}$ and thus $\dim(Y \cap \mathsf{W}) \geq m$. This yields

$$\mathcal{F}_{k+1,m} = \left\{ Y \in \mathcal{D}_{k+1} : Y \notin \left(\bigcup_{i=m+1}^{k} \mathcal{F}_{k+1,i} \right) \wedge (\exists U \in \mathcal{F}_{k,m}) \big[U \subset Y \big] \right\}. \tag{26}$$

(vi) Strictly speaking, in (25) and (26) only inclusions $\supset$ were justified in (iv) and (v), respectively. The converse inclusions are evident, though. So, we have got the following inductive definability schemata:

– $\mathcal{F}_{k,m-1}$ is definable, when $\mathcal{F}_{k+1,m}$ and $\mathcal{F}_{k,m}$ are given,

and

– $\mathcal{F}_{k+1,m-1}$ is definable, when $\mathcal{F}_{k,m-1}$ and $\mathcal{F}_{k+1,m}, \ldots \mathcal{F}_{k+1,k}$ are given.

Taking (iii) into account we conclude with the following: $\mathcal{F}_{k,m}$ and $\mathcal{F}_{k+1,m}$ are definable in $\mathfrak{D}_k$ for each admissible m, in particular, for $m = 0$, when $\mathcal{G}_k \neq \emptyset$, which completes the proof. □

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