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Complete 3-D elasticity solutions in the half-space for constant and linearly varying pressure loads

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ABSTRACT

The 3-D elasticity solutions for the infinite half-space under point loading conditions form the basis of many currently used algorithms for contact stress analysis. Using standard techniques, the point solution, i.e. Green's functions, can be integrated over various contact areas and various loading profiles. For a rectangular contact area, the subsurface solutions for constant and linearly varying pressure loads had been unavailable because of the difficulty in obtaining the closed form integrals. This paper presents the complete solutions to the integrals for constant and linearly varying loads in both the normal and tangential directions.

INTRODUCTION

Love[1],[2] has given the solutions for contact in the elastic half-space for an arbitrary pressure profile in integral form. Johnson[3] has simplified the analysis by applying normal and tangential loads separately. These loading cases can be simplified further and may be written in terms of only three harmonic functions.

DEFINITION OF TERMS

The coordinate system is defined as a right-hand system with the x-axis and y-axis lying on the surface and the z-axis pointing into the body. Greek variables (ξ,η) refer to surface points where loads are applied, and Latin variables (x,y,z) refer to the point of interest. The pressure is assumed applied to a rectangular region on the surface. With no loss of generality, the integration limits are taken as $-a \le \xi \le a$, and $-b \le \eta \le b$. The following terms are defined:

$$\rho = [(\xi - x)^2 + (\eta - y)^2 + z^2] \tag{1}$$

$$\psi = \iint_S \frac{P(\xi,\eta)}{\rho} d\xi d\eta \tag{2}$$

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$$\chi = \iint_S P(\xi,\eta) Log[\rho + z] d\xi d\eta \tag{3}$$

$$\Omega = \iint_S P(\xi,\eta)(zLog[\rho + z] - \rho) d\xi d\eta \tag{4}$$

where $P(\xi,\eta)$ is the pressure profile over the surface. ψ, χ, and Ω are the three harmonic functions which if evaluated in closed form will produce the complete solution to the problem. In the following, it will be assumed that the pressure profile is separable into its orthogonal components and is expressible as a linear combination of polynomials and known coeficients, α. Hence,

$$P(\xi,\eta) = \{P_x(\xi,\eta), P_y(\xi,\eta), P_z(\xi,\eta)\}. \tag{5}$$

$$P_i(\xi,\eta) = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi\eta\ , \text{ for } i = x,y,z \tag{6}$$

DISPLACEMENTS, STRAINS AND STRESSES

Tangential Loading (x-direction)
The displacements for loading tangential to the surface in the x-direction can be derived from the functions described in Equations (1-4) using the analysis of Johnson. For displacements,

$$u_x = \frac{P_x}{4\pi G}\{2\psi + 2\nu\frac{\partial^2\Omega}{\partial x^2} - z\frac{\partial^2\chi}{\partial x^2}\} \tag{7}$$

$$u_y = \frac{P_x}{4\pi G}\{2\nu\frac{\partial^2\Omega}{\partial x\partial y} - z\frac{\partial^2\chi}{\partial x\partial y}\} \tag{8}$$

$$u_z = \frac{P_x}{4\pi G}\{(1-2\nu)\frac{\partial\chi}{\partial x} - z\frac{\partial\psi}{\partial x}\} \tag{9}$$

where G and ν are the shear modulus and poisson's ratio respectively. The strains are obtained using various partial derivatives of displacement, and the stresses are computed to be:

$$\sigma_{xx} = \frac{P_x}{2\pi}\{2(1+\nu)\frac{\partial\psi}{\partial x} + 2\nu\frac{\partial^3\Omega}{\partial x^3} - z\frac{\partial^3\chi}{\partial x^3}\} \tag{10}$$

$$\sigma_{yy} = \frac{P_x}{2\pi}\{2\nu\frac{\partial\psi}{\partial x} + 2\nu\frac{\partial^3\Omega}{\partial x\partial y^2} - z\frac{\partial^3\chi}{\partial x\partial y^2}\} \tag{11}$$

$$\sigma_{zz} = \frac{P_x}{2\pi}\{-z\frac{\partial^2\psi}{\partial x\partial z}\} \quad \tau_{xy} = \frac{P_x}{2\pi}\{\frac{\partial\psi}{\partial y} + 2\nu\frac{\partial^3\Omega}{\partial x^2\partial y} - z\frac{\partial^3\chi}{\partial x^2\partial y}\} \tag{12,13}$$

$$\tau_{xz} = \frac{P_x}{2\pi}\{\frac{\partial\psi}{\partial z} - z\frac{\partial^2\psi}{\partial x^2}\} \qquad \tau_{yz} = \frac{P_x}{2\pi}\{-z\frac{\partial^2\psi}{\partial x\partial y}\} \tag{14,15}$$

Tangential Loading (y-direction)
For loading in the y-direction, the variables x and y are permuted in Equations (7) through (15).

Normal Loading (z-direction)
The complete formulas for displacement are:

$$u_x = \frac{P_z}{4\pi G}\{(2\nu - 1)\frac{\partial\chi}{\partial x} - z\frac{\partial\psi}{\partial x}\} \tag{16}$$

$$u_y = \frac{P_y}{4\pi G}\{(2\nu - 1)\frac{\partial\chi}{\partial y} - z\frac{\partial\psi}{\partial y}\} \tag{17}$$

$$u_z = \frac{P_z}{4\pi G}\{2(1-\nu)\psi - z\frac{\partial\psi}{\partial z}\} \tag{18}$$

The stresses are obtained from the following:

$$\sigma_{xx} = \frac{P_z}{2\pi}\{2\nu\frac{\partial\psi}{\partial z} - z\frac{\partial^2\psi}{\partial x^2} - (1-2\nu)\frac{\partial^2\chi}{\partial x^2} \tag{19}$$

$$\sigma_{yy} = \frac{P_z}{2\pi}\{2\nu\frac{\partial\psi}{\partial z} - z\frac{\partial^2\psi}{\partial y^2} - (1-2\nu)\frac{\partial^2\chi}{\partial y^2} \tag{20}$$

$$\sigma_{zz} = \frac{P_z}{2\pi}\{\frac{\partial\psi}{\partial z} - z\frac{\partial^2\psi}{\partial z^2}\} \tag{21}$$

$$\tau_{xy} = \frac{P_z}{2\pi}\{-(1-2\upsilon)\frac{\partial^2\chi}{\partial x\partial y} - z\frac{\partial^2\psi}{\partial x\partial y}\} \tag{22}$$

$$\tau_{yz} = \frac{P_z}{2\pi}\{-z\frac{\partial^2\psi}{\partial y\partial z}\} \qquad \tau_{xz} = \frac{P_z}{2\pi}\{-z\frac{\partial^2\psi}{\partial x\partial z}\} \tag{23,24}$$

EVALUATION OF INTEGRALS

Constant Load Case
The complete solution is obtainable in closed form, only if the integrals of Equations (2-4) are available explicitly. Once these are obtained, the various derivatives required in Equations (7-24) can also be obtained. Introducing a change of variables $\bar{x} = \xi - x$, and $\bar{y} = \eta - y$, the solutions are given below.

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$$\psi = \{-z\tan^{-1}[\frac{\bar{x}\bar{y}}{z\rho}] + \bar{y}Log[\bar{x}+\rho] + \bar{x}Log[\bar{y}+\rho]\}\Bigg|_{-a}^{+a}\Bigg|_{-b}^{+b} \tag{25}$$

$$\chi = \{\frac{\bar{x}^2}{2}\tan^{-1}[\frac{\bar{y}}{\bar{x}}] + \frac{\bar{y}^2}{2}\tan^{-1}[\frac{\bar{x}}{\bar{y}}] - \frac{z^2}{2}\tan^{-1}[\frac{\bar{x}\bar{y}}{z\rho}] - \frac{\bar{y}^2}{2}\tan^{-1}[\frac{\bar{x}z}{\bar{y}\rho}] \tag{26}$$

$$-\frac{\bar{x}^2}{2}\tan^{-1}[\frac{z\bar{y}}{\bar{x}\rho}] + z\bar{y}Log[\bar{x}+\rho] + z\bar{x}Log[\bar{y}+\rho] + \bar{x}\bar{y}Log[z+\rho] - \frac{3}{2}\bar{x}\bar{y}\}\Bigg|_{-a}^{+a}\Bigg|_{-b}^{+a}$$

$$\Omega = \{-\frac{\bar{x}\bar{y}\rho}{3} - \frac{3\bar{x}\bar{y}z}{2} + \frac{\bar{y}^2 z}{2}\tan^{-1}[\frac{\bar{x}}{\bar{y}}] + \frac{\bar{x}^2 z}{2}\tan^{-1}[\frac{\bar{y}}{\bar{x}}] - \frac{z^3}{6}\tan^{-1}[\frac{\bar{x}\bar{y}}{z\rho}] \tag{27}$$

$$-\frac{\bar{y}^3}{6}Log[\bar{x}+\rho] - \frac{\bar{x}^3}{6}Log[\bar{y}+\rho] - \frac{\bar{y}^2 z}{2}\tan^{-1}[\frac{\bar{x}z}{\bar{y}\rho}] - \frac{\bar{x}^2 z}{2}\tan^{-1}[\frac{\bar{y}z}{\bar{x}\rho}]$$

$$+\frac{\bar{y}z^2}{2}Log[\bar{x}+\rho] + \frac{\bar{x}z^2}{2}Log[\bar{y}+\rho] + \bar{x}\bar{y}zLog[z+\rho]\}\Bigg|_{-a}^{+a}\Bigg|_{-b}^{+b}$$

<u>Required Derivatives for Constant Load Case</u>

The derivatives required by Equations (7-24), when the loading is constant over a rectangular area are given below. For brevity, the limits of integration have not been show. The definite integral is obtained by substituting the four limits of integration. In general, derivatives w.r.t. y are not shown, since these can be obtained by permutation.

$$\frac{\partial\psi}{\partial x} = -Log[\bar{y}+\rho], \quad \frac{\partial\psi}{\partial z} = -\tan^{-1}[\frac{\bar{x}\bar{y}}{z\rho}] \quad \frac{\partial^2\psi}{\partial x^2} = \frac{\bar{x}}{(\bar{y}+\rho)\rho} \tag{28,29,30}$$

$$\frac{\partial^2\psi}{\partial z^2} = -\frac{\bar{x}}{(\bar{y}+\rho)\rho} - \frac{\bar{y}}{(\bar{x}+\rho)\rho} \quad \frac{\partial^2\psi}{\partial x\partial y} = \frac{1}{\rho} \quad \frac{\partial^2\psi}{\partial x\partial z} = \frac{\bar{y}z}{(\rho^2-\bar{y}^2)\rho} \tag{31,32,33}$$

$$\frac{\partial\chi}{\partial x} = -\bar{x}\tan^{-1}[\frac{\bar{y}}{\bar{x}}] + \bar{x}\tan^{-1}[\frac{\bar{y}z}{\bar{x}\rho}] - zLog[\rho+\bar{y}] - \bar{y}Log[\rho+z] \tag{34}$$

$$\frac{\partial^2 \chi}{\partial x^2} = \tan^{-1}[\frac{\bar{y}}{\bar{x}}] - \tan^{-1}[\frac{\bar{y}z}{\bar{x}\rho}] \qquad \frac{\partial^3 \chi}{\partial x^3} = \frac{\bar{y}}{\rho(\rho+z)} - \frac{\bar{y}z}{\rho(\rho^2-\bar{y}^2)} \qquad (35,36)$$

$$\frac{\partial^3 \chi}{\partial x^2 \partial y} = \frac{-\bar{x}}{\rho(\rho+z)} \qquad \frac{\partial^3 \Omega}{\partial x^3} = -\frac{\bar{y}}{(\rho+z)} + Log[\rho+\bar{y}] \qquad (37,38)$$

$$\frac{\partial^2 \Omega}{\partial x^2} = z\tan^{-1}[\frac{\bar{y}}{\bar{x}}] - z\tan^{-1}[\frac{\bar{y}z}{\bar{x}\rho}] - \bar{x}Log[\bar{y}+\rho] \qquad \frac{\partial^3 \Omega}{\partial x^2 \partial y} = \frac{\bar{x}}{(\rho+z)} \qquad (39,40)$$

<u>Linear Ramp Load Case</u>

For ramp load distributions, $\xi \cdot P_i = [(\xi - x) - x] \cdot P_i$, with $x \cdot P_i$ obtained by multiplying x by the constant load solutions. The harmonic functions for $(\xi - x) \cdot P_i$ are given below. By permutation, $\eta \cdot P_i$ loading may be obtained.

$$\psi = \{\frac{\bar{y}\rho}{2} + \frac{(\rho^2-\bar{y}^2)}{2}Log[\rho+\bar{y}]\}\Big|_{-a}^{+a}\Big|_{-b}^{+b} \qquad (41)$$

$$\chi = \{\frac{-7\bar{x}^2\bar{y}}{12} + \frac{\bar{y}z\rho}{3} - \frac{\bar{x}^3}{3}\tan^{-1}[\frac{\bar{x}}{\bar{y}}] - \frac{\bar{x}^3}{3}\tan^{-1}[\frac{\bar{y}z}{\bar{x}\rho}] + \frac{\bar{x}^2 z}{2}Log[\rho+\bar{y}] \qquad (42)$$

$$+\frac{\bar{x}^2\bar{y}}{2}Log[\rho+z] + \frac{z^3}{12}Log[(\rho^2-\bar{x}^2)(\rho+\bar{y})^2] + \frac{\bar{y}^3}{12}Log[(\rho^2-\bar{x}^2)(\rho+z)^2]\}\Big|_{-a}^{+a}\Big|_{-b}^{+b}$$

$$\Omega = \{\frac{-7\bar{x}^2\bar{y}z}{12} + \frac{\bar{y}z^2\rho}{3} - \frac{\bar{x}^3 z}{3}\tan^{-1}[\frac{\bar{x}}{\bar{y}}] - \frac{\bar{x}^3 z}{3}\tan^{-1}[\frac{\bar{y}z}{\bar{x}\rho}] + \frac{\bar{x}^2 z^2}{2}Log[\rho+\bar{y}] \qquad (43)$$

$$+\frac{\bar{x}^2\bar{y}z}{2}Log[\rho+z] + \frac{z^4}{12}Log[(\rho^2-\bar{x}^2)(\rho+\bar{y})^2] + \frac{\bar{y}^3 z}{12}Log[(\rho^2-\bar{x}^2)(\rho+z)^2]$$

$$-\frac{\rho\bar{y}(5\rho^2-3\bar{y}^2)}{24} - \frac{(\rho^2-\bar{y}^2)^2}{8}Log[\rho+\bar{y}]\}\Big|_{-a}^{+a}\Big|_{-b}^{+b}$$

<u>Required Derivatives for Linear Ramp Load Case</u>

$$\frac{\partial\psi}{\partial x} = -\bar{x}Log[\rho+\bar{y}], \quad \frac{\partial\psi}{\partial y} = -\rho, \quad \frac{\partial\psi}{\partial z} = zLog[\rho+\bar{y}] \qquad (44,45,46)$$

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$$\frac{\partial^2\psi}{\partial x^2}=\frac{\bar{x}^2}{\rho(\rho+\bar{y})}+Log[\rho+\bar{y}],\quad \frac{\partial^2\psi}{\partial z^2}=\frac{z^2}{\rho(\rho+\bar{y})}+Log[\rho+\bar{y}] \tag{47,48}$$

$$\frac{\partial^2\psi}{\partial x\partial y}=\frac{\bar{x}}{\rho},\quad \frac{\partial^2\psi}{\partial y^2}=\frac{\bar{y}}{\rho},\quad \frac{\partial^2\psi}{\partial y\partial z}=-\frac{z}{\rho},\quad \frac{\partial^2\psi}{\partial x\partial z}=-\frac{\bar{x}z}{\rho(\rho+\bar{y})} \tag{49,50,51,52}$$

$$\frac{\partial\chi}{\partial x}=\bar{x}\bar{y}+\bar{x}^2\tan^{-1}[\frac{\bar{x}}{y}]+\bar{x}^2\tan^{-1}[\frac{\bar{y}z}{x\rho}]-\bar{x}zLog[\rho+\bar{y}]-\bar{x}\bar{y}Log[\rho+z] \tag{53}$$

$$\frac{\partial\chi}{\partial y}=-\frac{z\rho}{2}-\frac{\bar{x}^2}{2}Log[\rho+z]-\frac{\bar{y}^2}{4}Log[(\rho^2-\bar{x}^2)(\rho+z)^2] \tag{54}$$

$$\frac{\partial^2\chi}{\partial x\partial y}=\bar{x}Log[\rho+z],\quad \frac{\partial^2\chi}{\partial y^2}=\frac{y}{2}Log[(\rho^2-\bar{x}^2)(\rho+z)^2] \tag{55}$$

$$\frac{\partial^2\chi}{\partial x^2}=-2\bar{x}\tan^{-1}[\frac{\bar{x}}{y}]+zLog[\rho+\bar{y}]+\bar{y}Log[\rho+z]-2\bar{x}\tan^{-1}[\frac{\bar{y}z}{x\rho}] \tag{56}$$

$$\frac{\partial^3\chi}{\partial x^3}=-\frac{\bar{x}\bar{y}z}{\rho(\rho^2-\bar{y}^2)}+\frac{\bar{x}\bar{y}}{\rho(\rho+z)}+2\tan^{-1}[\frac{\bar{x}}{y}]+2\tan^{-1}[\frac{\bar{y}z}{x\rho}] \tag{57}$$

$$\frac{\partial^3\chi}{\partial x^2\partial y}=-\frac{\bar{x}^2}{\rho(\rho+z)}-Log(\rho+z),\quad \frac{\partial^3\chi}{\partial x\partial y^2}=-\frac{\bar{x}\bar{y}}{\rho(\rho+z)} \tag{58,59}$$

$$\frac{\partial^3\chi}{\partial y^3}=-\frac{\bar{y}^2}{\rho(\rho+z)}-\frac{1}{2}Log[(\rho^2-\bar{x}^2)(\rho+z)^2] \tag{60}$$

$$\frac{\partial^2\Omega}{\partial x^2}=-\frac{\bar{y}\rho}{2}-2\bar{x}z\tan^{-1}[\frac{\bar{x}}{y}]-2\bar{x}z\tan[\frac{\bar{y}z}{x\rho}]+\frac{(z^2-3\bar{x}^2)}{2}Log[\rho+\bar{y}]+\bar{y}zLog[\rho+z] \tag{61}$$

$$\frac{\partial^2\Omega}{\partial x\partial y}=\bar{x}zLog[\rho+z]-\bar{x}\rho,\quad \frac{\partial^3\Omega}{\partial x\partial y^2}=+\frac{\bar{x}\bar{y}}{(\rho+z)} \tag{62,63}$$

$$\frac{\partial^2\Omega}{\partial y^2}=\frac{\bar{y}z}{2}Log[(\rho^2-\bar{x}^2)(\rho+z)^2]-\bar{y}\rho\quad \frac{\partial^3\Omega}{\partial x^2\partial y}=+\rho+\frac{\bar{x}^2}{\rho+z}-zLog[\rho+z] \tag{64,65}$$

$$\frac{\partial^3\Omega}{\partial x^3}=-\frac{\bar{x}\bar{y}}{(\rho+z)}+2z\tan^{-1}[\frac{\bar{x}}{y}]+2z\tan^{-1}[\frac{\bar{y}z}{x\rho}]+3\bar{x}Log[\rho+\bar{y}] \tag{66}$$

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$$\frac{\partial^3\Omega}{\partial y^3} = +\rho + \frac{\bar{y}^2}{\rho+z} - \frac{z}{2}Log[(\rho^2 - \bar{x}^2)(\rho+z)^2] \tag{67}$$

Bilinear Load Case

For the bilinear load case, $\xi\cdot\eta\cdot P_i = [(\xi-x)(\eta-y)+\eta x+\xi y-xy]P_i$. The following harmonic functions for $(\xi-x)(\eta-y)\cdot P_i$ are obtained:

$$\psi = \frac{\rho^3}{3}\Bigg|_{-a}^{+a}\Bigg|_{-b}^{+b} \tag{68}$$

$$\chi = \{-\frac{3}{16}\bar{x}^2\bar{y}^2 + \frac{z\rho}{24}(5\rho^2 - 3z^2) + \frac{\bar{x}^2\bar{y}^2}{4}Log[\rho+z] \tag{69}$$

$$+\frac{\bar{x}^4}{16}Log[(\rho^2-\bar{y}^2)(\rho+z)^2] + \frac{\bar{y}^2}{16}Log[(\rho^2-\bar{x}^2)(\rho+z)^2]\}\Bigg|_{-a}^{+a}\Bigg|_{-b}^{+b}$$

$$\Omega = \{-\frac{3}{16}\bar{x}^2\bar{y}^2 z + \frac{z^2\rho}{24}(5\rho^2-3z^2) + \frac{\bar{x}^2\bar{y}^2 z}{4}Log[\rho+z] \tag{70}$$

$$+\frac{\bar{x}^4 z}{16}Log[(\rho^2-\bar{y}^2)(\rho+z)^2] + \frac{\bar{y}^4 z}{16}Log[(\rho^2-\bar{x}^2)(\rho+z)^2] - \frac{\rho^5}{15}\}\Bigg|_{-a}^{+a}\Bigg|_{-b}^{+b}$$

Required Derivatives for Bilinear Load Case

Once again, derivatives w.r.t. y are not shown and may be obtained by permutation of x and y. The required derivatives work out to be:

$$\frac{\partial\psi}{\partial x} = -\bar{x}\rho,\quad \frac{\partial\psi}{\partial z} = z\rho,\quad \frac{\partial^2\psi}{\partial x\partial y} = \frac{\bar{x}\bar{y}}{\rho},\quad \frac{\partial^2\psi}{\partial x^2} = \rho + \frac{\bar{x}^2}{\rho} \tag{71,72,73,74}$$

$$\frac{\partial^2\psi}{\partial x\partial z} = -\frac{\bar{x}z}{\rho},\quad \frac{\partial^2\psi}{\partial z^2} = \rho + \frac{z^2}{\rho},\quad \frac{\partial^2\chi}{\partial x\partial y} = \bar{x}\bar{y}Log[\rho+z] \tag{75,76,77}$$

$$\frac{\partial\chi}{\partial x} = \frac{\bar{x}\bar{y}^2}{4} - \frac{\bar{x}z\rho}{2} - \frac{\bar{x}\bar{y}^2}{2}Log[\rho+z] - \frac{\bar{x}^3}{4}Log[(\rho^2-\bar{y}^2)(\rho+z)^2] \tag{78}$$

$$\frac{\partial^2\chi}{\partial x^2} = \frac{\rho z}{2} + \frac{\bar{y}^2}{2}Log[\rho+z] + \frac{3\bar{x}^2}{4}Log[(\rho^2-\bar{y}^2)(\rho+z)^2] \tag{79}$$

$$\frac{\partial^3\chi}{\partial x^3} = -\frac{\bar{x}^3}{\rho(\rho+z)} - \frac{3\bar{x}}{2} Log[(\rho^2 - y^2)(\rho+z)^2] \tag{80}$$

$$\frac{\partial^3\chi}{\partial x^2\partial y} = -\frac{\bar{x}^2\bar{y}}{\rho(\rho+z)} - \bar{y}Log[\rho+z] \qquad \frac{\partial^2\Omega}{\partial x\partial y} = \bar{x}\bar{y}zLog[\rho+z] - \bar{x}\bar{y}\rho \tag{81,82}$$

$$\frac{\partial^2\Omega}{\partial x^2} = \rho(-\frac{4}{3}\bar{x}^2 - \frac{\bar{y}^2}{3} + \frac{z^2}{6}) + \frac{\bar{y}^2 z}{2} Log[\rho+z] + \frac{3\bar{x}^2 z}{4} Log[(\rho^2 - \bar{y}^2)(\rho+z)^2] \tag{83}$$

$$\frac{\partial^3\Omega}{\partial x^3} = \rho\bar{x} + \frac{\bar{x}^3}{\rho+z} - \frac{3\bar{x}z}{2} Log[(\rho^2 - y^2)(\rho+z)^2] \qquad \frac{\partial^3\Omega}{\partial x^2\partial y} = \rho\bar{y} + \frac{\bar{x}^2\bar{y}}{\rho+z} - \bar{y}zLog[\rho+z] \tag{84,85}$$

Example Calculation

The complete solution to any constant or linearly varying ramp load can be obtained by substituting the calculated values for Equations (28-85) into Equation (7-24)as needed. For example, suppose it is desired to evaluate $\partial\psi / \partial x$ for a linear loading function. Then,

$$P(\xi,\eta) = \alpha_2\xi \tag{86}$$

$$\psi(x,\bar{x}) = \iint \frac{\xi}{\rho(\bar{x})} d\xi\, d\eta = \iint \frac{\bar{x}}{\rho(\bar{x})} d\xi\, d\eta + x\iint \frac{1}{\rho(\bar{x})} d\xi\, d\eta \tag{87}$$

with x factoring out of the last integral because integration is with respect to the surface variables. ψ is written as a function of both x and $\bar{x}$ and the chain rule is invoked. Hence,

$$\frac{\partial\psi}{\partial x}(x,\bar{x}) = \frac{\partial\psi}{\partial\bar{x}}\cdot\frac{\partial\bar{x}}{\partial x} + \frac{\partial\psi}{\partial x}\cdot\frac{\partial x}{\partial x} \quad \text{where } \frac{\partial\bar{x}}{\partial x} = -1 \text{ and } \frac{\partial x}{\partial x} = 1 \tag{88}$$

The final result is:

$$\frac{\partial\psi}{\partial x} = \{-xLog[\bar{y}+\rho] - z\tan^{-1}[\frac{\bar{x}\bar{y}}{z\rho}] + \bar{y}Log[\bar{x}+\rho]\}\Bigg|_{-a}^{+a}\Bigg|_{-b}^{+b} \tag{89}$$

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