Tohoku Math. J.
61 (2009), 1–40

COMPLETE CLASSIFICATION OF PARALLEL SURFACES IN 4-DIMENSIONAL LORENTZIAN SPACE FORMS

BANG-YEN CHEN AND JOERI VAN DER VEKEN

(Received November 19, 2007, revised June 13, 2008)

Abstract. In this paper we completely classify parallel non-degenerate surfaces in 4-dimensional Lorentzian space forms. In addition, we also completely classify non-degenerate surfaces with parallel mean curvature vector in 4-dimensional Lorentzian space forms.

1. Introduction. By a Lorentzian manifold we mean a pseudo-Riemannian manifold with index one. A *Lorentzian space form* is by definition a Lorentzian manifold of constant sectional curvature.

Let $\mathbb{E}^n_s$ denote the pseudo-Euclidean n-space with metric tensor given by

$$g_0 = -\sum_{i=1}^{s} dx_i^2 + \sum_{j=s+1}^{n} dx_j^2 \,, \tag{1.1}$$

where $(x_1, \dots, x_n)$ is the rectangular coordinate system of $\mathbb{E}^n_s$. We put

$$\begin{aligned} S^k_s(x_0, c) &= \{\boldsymbol{x} \in \mathbb{E}^{k+1}_s ;\ \langle \boldsymbol{x} - x_0, \boldsymbol{x} - x_0 \rangle = c^{-1} > 0\} \,, \\ H^k_s(x_0, -c) &= \{\boldsymbol{x} \in \mathbb{E}^{k+1}_{s+1} ;\ \langle \boldsymbol{x} - x_0, \boldsymbol{x} - x_0 \rangle = -c^{-1} < 0\} \,, \end{aligned} \tag{1.2}$$

where $\langle\ ,\ \rangle$ is the indefinite inner product on $\mathbb{E}^n_s$. Then $S^k_s(x_0, c)$ and $H^k_s(x_0, -c)$ are complete pseudo-Riemannian manifolds with index s of constant curvature c and $-c$, respectively. We simply denote $S^k_s(x_0, c)$ and $H^k_s(x_0, -c)$ by $S^k_s(c)$ and $H^k_s(-c)$ when x_0 is the origin.

The Lorentzian manifolds $\mathbb{E}^k_1$, $S^k_1(x_0, c)$ and $H^k_1(x_0, -c)$ are complete Lorentzian space forms, which are known as the Minkowski, de Sitter, and anti-de Sitter spaces, respectively.

A vector v is called *space-like* (resp. *time-like*) if $\langle v, v \rangle > 0$ (resp. $\langle v, v \rangle < 0$). A vector v is called *light-like* if it is nonzero and it satisfies $\langle v, v \rangle = 0$. A curve is called a null curve if its tangent vector is light-like at each point.

A submanifold of a pseudo-Riemannian manifold (in particular, in a Riemannian manifold) is called a parallel submanifold if it has parallel second fundamental form. Parallel submanifolds are one of the most fundamental submanifolds. Parallel submanifolds in real (resp. complex) space forms have been classified in [8, 18] (resp. in [14, 15]). Some special classes of parallel submanifolds in Lorentzian space forms have been studied in [1, 9, 10, 12].

2000 *Mathematics Subject Classification.* Primary 53B25; Secondary 53C50, 83C15.

Key words and phrases. Lorentzian space form, parallel second fundamental form, surface with parallel mean curvature vector, parallel surface.

The second named author is a postdoctoral researcher supported by the Research Foundation – Flanders (Belgium) (FWO).

In this article, we completely classify non-degenerate surfaces with parallel second fundamental form in 4-dimensional Lorentzian space forms. In addition, we completely classify non-degenerate surfaces with parallel mean curvature vector in 4-dimensional Lorentzian space forms.

2. Preliminaries.

2.1. Basic notation, formulas and definitions. Let $L_1^4(c)$ denote a Lorentzian space form of constant sectional curvature c. Then the Riemann curvature tensor $\tilde{R}$ of $L_1^4(c)$ is given by

$$\tilde{R}(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\}.$$

Throughout the paper, we *assume that M is a non-degenerate surface in $L_1^4(c)$*, i.e., the induced metric on M is non-degenerate. So, M is either space-like or Lorentzian. We put $\delta = 1$ or $\delta = -1$, according to M being space-like or Lorentzian, respectively.

Denote by ∇ and $\tilde{\nabla}$ the Levi Civita connections on M and $L_1^4(c)$, respectively. Let X and Y denote vector fields tangent to M and let ξ be a normal vector field. Then the formulas of Gauss and Weingarten give a decomposition of the vector fields $\tilde{\nabla}_X Y$ and $\tilde{\nabla}_X \xi$ into a tangent and a normal component (cf. [2, 3, 16]):

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y)\,, \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi\,. \tag{2.2}$$

These formulae define h, A and D, which are called the second fundamental form, the shape operator and the normal connection respectively.

The mean curvature vector is defined by $H = (1/2)\operatorname{trace} h$. For each $\xi \in T_x^\perp M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space $T_x M$ at $x \in M$. The shape operator and the second fundamental form are related by

$$\langle h(X,Y),\xi\rangle = \langle A_\xi X, Y\rangle \tag{2.3}$$

for X, Y tangent to M and ξ normal to M.

The equations of Gauss, Codazzi and Ricci are given respectively by

$$\begin{aligned} \langle R(X,Y)Z,W\rangle = {} & \langle A_{h(Y,Z)}X,W\rangle - \langle A_{h(X,Z)}Y,W\rangle \\ & + c\,(\langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle)\,, \end{aligned} \tag{2.4}$$

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z)\,, \tag{2.5}$$

$$\langle R^D(X,Y)\xi,\eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle \tag{2.6}$$

for X, Y, Z, W tangent to M and ξ, η normal to M, where $\overline{\nabla} h$ is defined by

$$(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)\,. \tag{2.7}$$

The surface M is said to be *totally geodesic* if $h = 0$ holds identically; and *parallel* if we have $\overline{\nabla} h = 0$. The surface is said to have *parallel mean curvature vector* if we have $DH = 0$ identically. It is called *totally umbilical* if its second fundamental form satisfies $h(X,Y) = \langle X,Y\rangle H$.

The *light cone* $\mathcal{LC}^{n-1}(x_0)$ with vertex x_0 in $\mathbb{E}^n_s$ is defined to be

$$\mathcal{LC}^{n-1}(x_0) = \{\boldsymbol{x} \in \mathbb{E}^n_s\, ;\ \langle \boldsymbol{x} - x_0, \boldsymbol{x} - x_0\rangle = 0\}\,. \tag{2.8}$$

We simply denote $\mathcal{LC}^{n-1}(x_0) \subset \mathbb{E}^n_s$ by $\mathcal{LC} \subset \mathbb{E}^n_s$ if x_0 is the origin.

A surface in a pseudo-Riemannian 3-manifold (or in a light cone) is called a *CMC surface* if its mean curvature vector H satisfies $\langle H, H\rangle = constant \neq 0$.

2.2. Moving frames. Let M be a space-like or a Lorentzian surface in a Lorentzian space form $L^4_1(c)$. Put $\delta = 1$ if M is space-like and put $\delta = -1$ if M is Lorentzian. Let $\{e_1, e_2\}$ be a local tangent frame and let $\{e_3, e_4\}$ be a local normal frame, such that

$$\langle e_1, e_1\rangle = \delta\,, \quad \langle e_1, e_2\rangle = 0\,, \quad \langle e_2, e_2\rangle = 1\,, \tag{2.9}$$

$$\langle e_3, e_3\rangle = -\delta\,, \quad \langle e_3, e_4\rangle = 0\,, \quad \langle e_4, e_4\rangle = 1\,. \tag{2.10}$$

We define the one-forms ω_1^2, ω_2^1, ω_3^4 and ω_4^3 by the following equations:

$$\nabla_X e_1 = \omega_1^2(X)e_2\,, \quad \nabla_X e_2 = \omega_2^1(X)e_1\,, \quad D_X e_3 = \omega_3^4(X)e_4\,, \quad D_X e_4 = \omega_4^3(X)e_3\,. \tag{2.11}$$

Then $\omega_2^1 = -\delta\omega_1^2$ and $\omega_4^3 = \delta\omega_3^4$.

If M is space-like, sometimes we use a local normal frame $\{e_3, e_4\}$ on M satisfying

$$\langle e_3, e_3\rangle = \langle e_4, e_4\rangle = 0\,, \quad \langle e_3, e_4\rangle = -1\,. \tag{2.12}$$

We may put

$$D_X e_3 = \theta(X)e_3\,, \quad D_X e_4 = -\theta(X)e_4 \tag{2.13}$$

for some one-form θ.

2.3. Isothermal coordinates and lemmas. Locally, there exists an isothermal coordinate system (u, v) on a space-like (or Lorentzian) surface M so that the metric tensor of M takes the following form:

$$g = E(u, v)(\delta du^2 + dv^2)\,, \tag{2.14}$$

where $\delta = 1$ if M is space-like; and $\delta = -1$ if M is Lorentzian (see [11, page 111] for Lorentzian surfaces).

The following lemmas are obtained in [5].

LEMMA 2.1. *Let M be a non-degenerate surface in a Lorentzian space form $L^4_1(c)$. If M has parallel mean curvature vector, then with respect to the isothermal coordinates (u, v) satisfying* (2.14) *we have*

$$\frac{1}{2}D_{\partial_v}(\delta l - n) = D_{\partial_u}(\delta m)\,, \quad \frac{1}{2}D_{\partial_u}(\delta l - n) = -D_{\partial_v} m\,, \tag{2.15}$$

where $l = h(\partial_u, \partial_u)$, $m = h(\partial_u, \partial_v)$, $n = h(\partial_v, \partial_v)$.

LEMMA 2.2. *Let M be a non-degenerate surface with parallel nonzero mean curvature vector in a Lorentzian space form $L^4_1(c)$. Then we have*:

(1) $R^D = 0$, *i.e.*, *M has flat normal connection*;
(2) $[A_\xi, A_\eta] = 0$ *for $\xi, \eta \in T^\perp_p M$.*

2.4. Marginally trapped surfaces. The concept of *trapped surfaces*, introduced by Penrose in [17] plays a very important role in general relativity. In the theory of cosmic black holes, if there is a massive source inside the surface, then close enough to a massive enough source, the outgoing light rays may also be converging; a *trapped surface*. Everything inside is trapped. Nothing can escape, not even light. It is believed that there is a marginally trapped surface, separating the trapped surfaces from the untrapped ones, where the outgoing light rays are instantaneously parallel. The surface of a *black hole* is located by the marginally trapped surface.

In terms of the mean curvature vector, a codimension-two space-like surface is *future trapped* if its mean curvature vector is time-like and future-pointing at each point (similarly, for passed trapped); and it is *marginally trapped* if the mean curvature vector is light-like at each point on the surface.

3. Space-like surfaces with $DH = 0$. In this section we classify space-like surfaces in $L^4_1(c)$ with $DH = 0$.

THEOREM 3.1. *A space-like surface M with parallel mean curvature vector in the* 4*-dimensional Minkowski space-time $\mathbb{E}^4_1$ is congruent to one of the following twelve types of surfaces*:

(1) *a minimal surface of* $\mathbb{E}^4_1$;

(2) *a CMC surface of the light cone* $\mathcal{LC} \subset \mathbb{E}^4_1$;

(3) *a CMC surface of a Euclidean* 3*-space* $\mathbb{E}^3 \subset \mathbb{E}^4_1$;

(4) *a CMC surface of a* 3*-dimensional Minkowski space-time* $\mathbb{E}^3_1 \subset \mathbb{E}^4_1$;

(5) *a CMC surface of a* 3*-dimensional de Sitter space-time* $S^3_1(c) \subset \mathbb{E}^4_1$;

(6) *a CMC surface of a* 3*-dimensional hyperbolic space* $H^3(-c) \subset \mathbb{E}^4_1$;

(7) *a flat parallel surface given by* $L = a(\cosh u, \sinh u, \cos v, \sin v)$, $a > 0$;

(8) *a flat parallel surface given by*

$$L = \frac{1}{2}((1-b)u^2 + (1+b)v^2, (1-b)u^2 + (1+b)v^2, 2u, 2v), \quad b \in R;$$

(9) *a flat non-parallel surface with constant light-like mean curvature vector*, *which lies in the hyperplane* $\mathcal{H}_0 = \{(t, t, x_3, x_4) \in \mathbb{E}^4_1\}$, *but not in any light cone*;

(10) *a non-parallel flat marginally trapped surface lying in the light cone* $\mathcal{LC}$;

(11) *a non-parallel surface lying in the de Sitter space-time* $S^3_1(c)$ *for some* $c > 0$ *such that the mean curvature vector* H' *of* M *in* $S^3_1(c)$ *satisfies* $\langle H', H' \rangle = -c$;

(12) *a non-parallel surface lying in the hyperbolic space* $H^3(-c)$ *for some* $c > 0$ *such that the mean curvature vector* H' *of* M *in* $H^3(-c)$ *satisfies* $\langle H', H' \rangle = c$.

Surfaces of types (7)–(12) *are marginally trapped in* $\mathbb{E}^4_1$.

PROOF. Let M be a space-like surface in $\mathbb{E}^4_1$. Assume that M has parallel mean curvature vector. Then $\langle H, H \rangle$ is constant. So, one of the following three cases occurs:

(i) $H = 0$,

(ii) H is light-like,

(iii) $\langle H, H\rangle$ is a nonzero constant.

If $H = 0$, we obtain case (1). If H is light-like, then according to Theorem 3.1 of [6], M is one of the surfaces given by cases (7)–(12).

Now, let us assume that $\langle H, H\rangle$ is a nonzero constant. Let (u, v) be isothermal coordinates on M satisfying (2.14). Let us choose e_3, e_4 to be an orthonormal normal frame with $H = be_3$ for a positive number b. Let us put $\varepsilon_3 = \langle e_3, e_3\rangle$, $\varepsilon_4 = \langle e_4, e_4\rangle$. We have $\varepsilon_4 = -\varepsilon_3$. From $DH = 0$, we also have $De_3 = De_4 = 0$. Let us put

$$\alpha = \frac{1}{2}\langle l - n, e_3\rangle\,, \quad \beta = \langle m, e_3\rangle\,, \tag{3.1}$$

where l, m, n are defined as in Lemma 2.1. It follows from Lemma 2.1 and $De_3 = 0$ that α and β satisfy the Cauchy-Riemann condition:

$$\frac{\partial\alpha}{\partial v} = \frac{\partial\beta}{\partial u}\,, \quad \frac{\partial\alpha}{\partial u} = -\frac{\partial\beta}{\partial v}\,. \tag{3.2}$$

Thus, the function $\phi_1 = \alpha + i\beta$ is a holomorphic function in $z = u + iv$.

Similarly, if we put

$$\gamma = \frac{1}{2}\langle l - n, e_4\rangle\,, \quad \delta = \langle m, e_4\rangle\,, \tag{3.3}$$

then $\phi_2 = \gamma + i\delta$ is also holomorphic by the same argument. From the definitions of ϕ_1, ϕ_2, we get

$$\frac{\phi_2}{\phi_1} = \frac{\alpha\gamma + \beta\delta + i(\alpha\delta - \beta\gamma)}{\alpha^2 + \beta^2}\,. \tag{3.4}$$

Let us put

$$h(e_j, e_k) = h^3_{jk}e_3 + h^4_{jk}e_4\,, \tag{3.5}$$

for $j, k = 1, 2$, where $e_1 = \partial_u/\sqrt{E}$, $e_2 = \partial_v/\sqrt{E}$. With respect to e_1, e_2, (2.3) and (3.5) give

$$A_{e_3} = \varepsilon_3 \begin{pmatrix} h^3_{11} & h^3_{12} \\ h^3_{12} & h^3_{22} \end{pmatrix}, \quad A_{e_4} = \varepsilon_4 \begin{pmatrix} h^4_{11} & h^4_{12} \\ h^4_{12} & h^4_{22} \end{pmatrix}. \tag{3.6}$$

Since $H = be_3$, we find $2b = h^3_{11} + h^3_{22}$ and $h^4_{11} + h^4_{22} = 0$.

After applying $[A_{e_3}, A_{e_4}] = 0$ from Lemma 2.2 and using (3.6), we obtain

$$h^4_{12}(h^3_{11} - h^3_{22}) = (h^4_{11} - h^4_{22})h^3_{12}\,. \tag{3.7}$$

It is easy to see that (3.7) is equivalent to $\beta\gamma = \alpha\delta$. Thus, (3.4) implies that the meromorphic function ϕ_2/ϕ_1 is real; hence, it is constant. A straightforward computation yields $\phi_2/\phi_1 = -h^4_{12}/h^3_{12} = \delta/\beta$. Let us put

$$\eta_1 = (\sin r)e_3 - (\cos r)e_4\,, \quad \eta_2 = (\cos r)e_3 - (\sin r)e_4\,, \quad r = \arctan\left(\frac{\phi_2}{\phi_1}\right). \tag{3.8}$$

By applying (3.6)–(3.8), we find

$$A_{\eta_1} = \zeta I\,, \quad \operatorname{trace} A_{\eta_2} = \varepsilon_3 b\cos r\,, \tag{3.9}$$

where $\zeta = \varepsilon_3 b \sin r$ is constant. Since $De_3 = De_4 = 0$ and r, trace A_{e_3} are constant, we have $D\eta_1 = D\eta_2 = 0$.

If $\zeta = 0$, then $\sin r = 0$. In this case, e_4 is a constant unit vector which implies that $\langle L, e_4\rangle$ is constant. Hence, after choosing suitable Minkowskian coordinates we obtain case (3) or case (4) depending on e_4 being time-like or space-like, respectively.

Now, assume that $\zeta \neq 0$. Let us consider the following map:

$$\psi : M \to \mathbb{E}^4_1 : \; p \mapsto L(p) + \zeta^{-1}\eta_1(p)\,. \tag{3.10}$$

Then, we have $\tilde{\nabla}_X \psi = X - \zeta^{-1} A_{\eta_1} X = 0$ for $X \in TM$. Thus, ψ is constant, say $c_0 \in \mathbb{E}^4_1$. We may assume $c_0 = 0$ by choosing suitable Minkowskian coordinates. Thus, we obtain from (3.8) that

$$\langle L, L\rangle = \frac{\varepsilon_4}{b^2}(\cot^2 r - 1)\,. \tag{3.11}$$

If $\cot^2 r = 1$, then M is a CMC surface of the light cone $\mathcal{LC}$. This gives case (2).

If $c = \varepsilon_4 b^2/(\cot^2 r - 1) > 0$, M is a CMC surface of $S^3_1(c)$, which gives case (5).

Finally, if $c = \varepsilon_4 b^2/(\cot^2 r - 1) < 0$, then M is a CMC surface of $H^3(-c)$, which gives case (6) of the theorem. □

THEOREM 3.2. *A space-like surface with parallel mean curvature vector in the de Sitter space-time $S^4_1(1) \subset \mathbb{E}^5_1$ is congruent to one of the following twelve types of surfaces*:

(1) *a minimal surface of $S^4_1(1)$*;

(2) *a CMC surface in $S^4_1(1) \cap \mathcal{E}$, where $\mathcal{E}$ is a space-like hyperplane in $\mathbb{E}^5_1$*;

(3) *a CMC surface in $S^4_1(1) \cap \mathcal{E}_1$, where $\mathcal{E}_1$ is a Minkowskian hyperplane in $\mathbb{E}^5_1$*;

(4) *a surface M which lies in $S^4_1(1) \cap \mathcal{H}$, where $\mathcal{H}$ is a degenerate hyperplane in $\mathbb{E}^5_1$ such that the normal vector of M in $S^4_1(1) \cap \mathcal{H}$ is light-like*;

(5) *a parallel surface of curvature one given by* $L=(1, \sin u, \cos u \cos v, \cos u \sin v, 1)$ *with* $a, b, c \in R$;

(6) *a flat parallel surface defined by* $L = (1/2)(2u^2 - 1, 2u^2 - 2, 2u, \sin 2v, \cos 2v)$;

(7) *a flat parallel surface defined by*

$$L = \left(\frac{b}{\sqrt{4-b^2}}, \frac{\cos(\sqrt{2-b}u)}{\sqrt{2-b}} \frac{\sin(\sqrt{2-b}u)}{\sqrt{2-b}}, \frac{\cos(\sqrt{2+b}v)}{\sqrt{2+b}}, \frac{\cos(\sqrt{2+b}v)}{\sqrt{2+b}}\right)$$

with $|b| < 2$.

(8) *a flat parallel surface defined by*

$$L = \left(\frac{\cosh(\sqrt{b-2}u)}{\sqrt{b-2}}, \frac{\sinh(\sqrt{b-2}u)}{\sqrt{b-2}}, \frac{\cos(\sqrt{2+b}v)}{\sqrt{2+b}}, \frac{\cos(\sqrt{2+b}v)}{\sqrt{2+b}}, \frac{b}{\sqrt{b^2-4}}\right)$$

with $b > 2$.

(9) *a non-parallel surface of curvature one with constant light-like mean curvature vector, and it lies in $\mathcal{K}_a \cap S^4_1(1)$, with $\mathcal{K}_a = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{E}^5_1; x_5 = x_1 + a\}$ for some $a \in R$, but not in any light cone in $\mathbb{E}^5_1$*;

(10) *a non-parallel marginally trapped surface of curvature one in $S^4_1(1)$ which lies in* $\mathcal{LC}_1 := \{(\boldsymbol{y}, 1) \in \mathbb{E}^5_1; \langle \boldsymbol{y}, \boldsymbol{y}\rangle = 0, \boldsymbol{y} \in \mathbb{E}^4_1\} \subset S^4_1(1)$;

(11) *a non-parallel surface of* $S^4_1(1)$ *which lies in* $S^4_1(1) \cap S^4_1(x_0, c)$ *with* $x_0 \neq 0$ *and* $c > 0$ *such that the mean curvature vector* H' *of* M *in* $S^4_1(1) \cap S^4_1(x_0, c)$ *satisfies* $\langle H', H'\rangle = -c$;

(12) *a non-parallel surface of* $S^4_1(1)$ *which lies in* $S^4_1(1) \cap H^4(x_0, -c)$ *with* $x_0 \neq 0$ *and* $c > 0$ *such that the mean curvature vector* H' *of* M *in* $S^4_1(1) \cap H^4(x_0, -c)$ *satisfies* $\langle H', H'\rangle = c$.

Surfaces of types (5)–(12) *are marginally trapped in* $S^4_1(1)$.

PROOF. Let M be a space-like surface in $S^4_1(1)$ with parallel mean curvature vector. Then $\langle H, H\rangle$ is constant. So, one of the following three cases occurs: (i) $H = 0$, (ii) H is light-like, (iii) $\langle H, H\rangle$ is a nonzero constant.

If $H = 0$, we get case (1) of the theorem. If H is light-like, then [6, Theorem 6.1] implies that M is one of the surfaces given by cases (5)–(12).

Now, assume $\langle H, H\rangle$ is a nonzero constant. Let us put $\varepsilon_3 = \langle e_3, e_3\rangle$, $\varepsilon_4 = \langle e_4, e_4\rangle$. Then we have $\varepsilon_4 = -\varepsilon_3$. From $DH = 0$, we get $De_3 = De_4 = 0$. Let us put

$$\eta_1 = (\sin r)e_3 - (\cos r)e_4\,, \quad \eta_2 = (\cos r)e_3 - (\sin r)e_4\,, \quad r = \arctan\left(\frac{\phi_2}{\phi_1}\right), \tag{3.12}$$

exactly in the same way as in the proof of Theorem 3.1. Then, by applying the same argument as in the proof of Theorem 3.1, we have

$$D\eta_1 = D\eta_2 = 0\,, \quad A_{\eta_1} = \zeta I\,, \quad \operatorname{trace} A_{\eta_2} = \varepsilon_3 b \cos r\,, \tag{3.13}$$

where $\zeta = \varepsilon_3 b \sin r$ is a constant.

If $\zeta = 0$, then $\sin r = 0$. So, e_4 is a constant unit vector which implies that $\langle L, e_4\rangle$ is constant. Hence, after choosing suitable Minkowskian coordinates, we obtain case (2) or case (3) depending on e_4 being time-like or space-like, respectively.

Next, assume that $\zeta \neq 0$. Let us consider the map:

$$\psi : M \to \mathbb{E}^5_1 :\ p \mapsto L(p) + \zeta^{-1}\eta_1(p)\,. \tag{3.14}$$

We have $\tilde\nabla_X \psi = 0$, $X \in TM$. Thus, ψ is constant, say $c_0 \in \mathbb{E}^5_1$. Thus, we obtain

$$\langle L - c_0, L - c_0\rangle = \frac{\varepsilon_4}{b^2}(\cot^2 r - 1)\,. \tag{3.15}$$

Combining this with $\langle L, L\rangle = 1$ gives

$$\langle L, c_0\rangle = \frac{1}{2}\left\{1 + \langle c_0, c_0\rangle - \frac{\varepsilon_4}{b^2}(\cot^2 r - 1)\right\}. \tag{3.16}$$

Thus, we obtain cases (2), (3) or (4) depending on whether c_0 is time-like, space-like or light-like. □

THEOREM 3.3. *A space-like surface with parallel mean curvature vector in the anti de Sitter space-time* $H^4_1(-1) \subset \mathbb{E}^5_2$ *is congruent to one of the following twelve types of surfaces*:

(1) *a minimal surface of* $H^4_1(-1)$;

(2) *a CMC surface in* $H^4_1(-1) \cap \mathcal{E}_1$, *where* $\mathcal{E}_1$ *a Minkowskian hyperplane in* $\mathbb{E}^5_2$;

(3) *a CMC surface in* $H^4_1(-1) \cap \mathcal{E}_2$, *where* $\mathcal{E}_2$ *is a hyperplane with index* 2 *in* $\mathbb{E}^5_2$;

(4) *a surface M which lies in $H_1^4(-1) \cap \mathcal{H}$, where $\mathcal{H}$ is a degenerate hyperplane such that the normal vector of M in $H_1^4(-1) \cap \mathcal{H}$ is light-like*;

(5) *a parallel surface of curvature -1 given by* $L=(1, \cosh u \cosh v, \sinh u, \cosh u \sinh v, 1)$;

(6) *a flat parallel surface defined by* $L = (1/2)(2u^2 + 2, \cosh 2v, 2u, \sinh 2v, 2u^2 + 1)$;

(7) *a flat parallel surface defined by*

$$L=\left(\frac{\cosh(\sqrt{2-b}u)}{\sqrt{2-b}}, \frac{\cosh(\sqrt{2+b}v)}{\sqrt{2+b}}, \frac{\sinh(\sqrt{2-b}u)}{\sqrt{2-b}}, \frac{\sinh(\sqrt{2+b}v)}{\sqrt{2+b}}, \frac{b}{\sqrt{4-b^2}}\right),$$

$|b| < 2$;

(8) *a flat parallel surface defined by*

$$L = \left(\frac{b}{\sqrt{b^2-4}}, \frac{\cosh(\sqrt{b+2}v)}{\sqrt{b+2}}, \frac{\sinh(\sqrt{b+2}v)}{\sqrt{b+2}}, \frac{\cos(\sqrt{b-2}u)}{\sqrt{b-2}}, \frac{\sin(\sqrt{b-2}u)}{\sqrt{b-2}}\right),$$

$b > 2$;

(9) *a non-parallel surface with curvature -1 and constant light-like mean curvature vector, which lies in $\mathcal{G}_b \cap H_1^4(-1)$, with $\mathcal{G}_b = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{E}_2^5;\, x_3 = x_1 + b\}$ for some $b \in \boldsymbol{R}$, but not in any light cone of $\mathbb{E}_2^5$*;

(10) *a non-parallel marginally trapped surface of $H_1^4(-1)$ with curvature -1, and it lies in $\mathcal{LC}_2 := \{(1, \boldsymbol{y}) \in \mathbb{E}_2^5;\, \langle \boldsymbol{y}, \boldsymbol{y}\rangle = 0, \boldsymbol{y} \in \mathbb{E}_1^4\} \subset H_1^4(-1)$*;

(11) *a non-parallel surface lying in $H_1^4(-1) \cap S_2^4(x_0, c)$ with $x_0 \neq 0$ and $c > 0$ such that the mean curvature vector H' in $H_1^4(-1) \cap S_2^4(x_0, c)$ satisfies $\langle H', H'\rangle = -c$*;

(12) *a non-parallel surface lying in $H_1^4(-1) \cap H_1^4(x_0, -c)$ with $x_0 \neq 0$ and $c > 0$ such that mean curvature vector H' in $H_1^4(-1) \cap H_1^4(x_0, -c)$ satisfies $\langle H', H'\rangle = c$.*

Surfaces of types (5)–(12) are marginally trapped in $H_1^4(-1)$.

REMARK 3.1. For the existence and examples of surfaces of types (9)–(12) of Theorem 3.1, Theorem 3.2 and Theorem 3.3, see [6].

4. Lorentzian surfaces with $DH = 0$.

LEMMA 4.1. *Let A be a linear operator on a two-dimensional vector space V, which is symmetric with respect to a Lorentzian inner product $\langle\ ,\ \rangle$ on V. Then there exists a basis $\{e_1, e_2\}$ of V with $\langle e_1, e_1\rangle = -1$, $\langle e_1, e_2\rangle = 0$ and $\langle e_2, e_2\rangle = 1$ such that, with respect to $\{e_1, e_2\}$, A takes one of the following forms*:

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \tag{4.1}$$

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \beta \neq 0, \tag{4.2}$$

$$A = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha+2 \end{pmatrix} \quad or \quad A = \begin{pmatrix} \alpha & 1 \\ -1 & \alpha-2 \end{pmatrix}. \tag{4.3}$$

PROOF. Let $\{u_1, u_2\}$ be a basis of V with $\langle u_1, u_1\rangle = -1$, $\langle u_1, u_2\rangle = 0$ and $\langle u_2, u_2\rangle = 1$. Since A is symmetric, it takes the form $A = \left(\begin{smallmatrix} a & b \\ -b & c \end{smallmatrix}\right)$ with respect to $\{u_1, u_2\}$. Notice that, for a fixed real number t, the basis $\{e_1, e_2\}$ given by

$$e_1 = (\cosh t)u_1 + (\sinh t)u_2\,, \quad e_2 = (\sinh t)u_1 + (\cosh t)u_2 \tag{4.4}$$

also satisfies $\langle e_1, e_1\rangle = -1$, $\langle e_1, e_2\rangle = 0$ and $\langle e_2, e_2\rangle = 1$.

If $b = 0$, we are in case (4.1).

Now, suppose that $b \neq 0$. We distinguish three cases.

If $|(c-a)/2b| > 1$, we can choose the t in (4.4) so that $2b/(c-a) = \tanh(2t)$, and we obtain (4.1).

If $|(c-a)/2b| < 1$, we can choose t in (4.4) so that $(c-a)/2b = \tanh(2t)$, and we obtain (4.2). Notice that we may assume $\beta \neq 0$ for (4.2), since otherwise, it reduces to a special case of (4.1).

Finally, if $c - a = \pm 2b$, we can choose $t = \pm \ln |b|$ in (4.4) and we obtain either

$$A = \begin{pmatrix} \alpha & 1 \\ -1 & \beta \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} \alpha & -1 \\ 1 & \beta \end{pmatrix}. \tag{4.5}$$

By changing e_2 into $-e_2$, we can transform the second matrix of (4.5) into the first matrix of (4.5). If $\beta - \alpha = \pm 2$, we have obtained the form (4.3). If $\beta - \alpha \neq \pm 2$, we can do another transformation to obtain either (4.1) or (4.2). $\square$

REMARK 4.1. With respect to the null-basis $\{v_1 = (e_1+e_2)/\sqrt{2}, v_2 = (e_1-e_2)/\sqrt{2}\}$, the matrices of (4.3) become respectively

$$A = \begin{pmatrix} \alpha+1 & -2 \\ 0 & \alpha+1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \alpha-1 & 0 \\ 2 & \alpha-1 \end{pmatrix}.$$

THEOREM 4.1. *Let M be a Lorentzian surface in the Minkowski space-time $\mathbb{E}^4_1$. Then M has parallel mean curvature vector if and only if, up to suitable choice of Minkowskian coordinates, M is one of the following surfaces*:

(1) *a minimal surface of* $\mathbb{E}^4_1$;

(2) *a CMC surface of a Minkowski space-time* $\mathbb{E}^3_1 \subset \mathbb{E}^4_1$;

(3) *a CMC surface of a de Sitter space-time* $S^3_1(c) \subset \mathbb{E}^4_1$.

PROOF. Let $L : M \to \mathbb{E}^4_1$ be a Lorentzian surface with $DH = 0$ in $\mathbb{E}^4_1$. If $H = 0$, we obtain case (1). Next, assume $H \neq 0$ and choose an orthonormal normal frame $\{e_3, e_4\}$ with $H = \kappa e_3$. Then κ is a nonzero constant and $De_3 = De_4 = 0$.

Since A_{e_4} is a symmetric operator with trace zero, it follows from Lemma 4.1 that we can choose $\{e_1, e_2\}$ which satisfies (2.9) and

$$A_{e_4} = \begin{pmatrix} -\gamma & 0 \\ 0 & \gamma \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix} \quad \text{or} \quad A_{e_4} = \begin{pmatrix} \pm 1 & 1 \\ -1 & \mp 1 \end{pmatrix}. \tag{4.6}$$

Case (a): *A_{e_4} takes the first form of* (4.6). From Lemma 2.2 we get $[A_{e_3}, A_{e_4}] = 0$ and hence we have

$$A_{e_3} = \begin{pmatrix} -\alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} -\gamma & 0 \\ 0 & \gamma \end{pmatrix} \tag{4.7}$$

for some functions α, β, γ. Since $H = \kappa e_3 \neq 0$, we get that

$$\beta - \alpha = 2\kappa \tag{4.8}$$

is a nonzero constant. From (2.3), (2.9) and (4.7), we obtain

$$h(e_1, e_1) = \alpha e_3 + \gamma e_4\,, \quad h(e_1, e_2) = 0\,, \quad h(e_2, e_2) = \beta e_3 + \gamma e_4\,. \tag{4.9}$$

By applying (2.11), (4.9) and the equation of Codazzi, we find

$$\begin{aligned} e_1\gamma &= -2\gamma\omega_1^2(e_2)\,, \quad e_1\beta = -(\alpha+\beta)\omega_1^2(e_2)\,, \\ e_2\gamma &= -2\gamma\omega_1^2(e_1)\,, \quad e_2\alpha = -(\alpha+\beta)\omega_1^2(e_1)\,, \end{aligned} \tag{4.10}$$

which imply $e_j(\ln\gamma) = e_j(\ln(\alpha+\beta))$ for $j = 1, 2$ and thus $\gamma/(\alpha+\beta)$ is constant.

Let us put

$$\hat{e}_3 = (\cos t)e_3 + (\sin t)e_4\,, \quad \hat{e}_4 = (\sin t)e_3 - (\cos t)e_4\,, \quad t = \arctan\left(\frac{2\gamma}{\alpha+\beta}\right). \tag{4.11}$$

Then $\hat{e}_3, \hat{e}_4$ are orthonormal parallel normal vector fields such that

$$\operatorname{trace} A_{\hat{e}_3} = 2\kappa\cos t\,, \quad A_{\hat{e}_4} = (\kappa\sin t)I\,. \tag{4.12}$$

Hence, trace $A_{\hat{e}_3}$ and $\kappa\sin t$ are constant.

If $\gamma = 0$, we get $\hat{e}_4 = e_4$ and $A_{e_4} = 0$. Combining this with $De_4 = 0$ shows that e_4 is a constant vector. Thus, $\langle L, e_4\rangle$ is constant, where L is the immersion of M in $\mathbb{E}^4_1$. So, after choosing a suitable Minkowskian coordinate system we get case (2).

Next, let us assume that $\gamma \neq 0$. It follows from (4.12), the constancy of $\kappa\sin t$ and $D\hat{e}_4 = 0$ that $\tilde{\nabla}_X\hat{e}_4 = -(\kappa\sin t)X$ for any $X \in TM$. Hence

$$\psi : M \to \mathbb{E}^4_1 : \; p \mapsto L(p) + \frac{\hat{e}_4(p)}{\kappa\sin t} \tag{4.13}$$

is a constant map. So, after choosing ψ to be the origin we obtain

$$\langle L, L\rangle = \kappa^{-2}\csc^2 t = \text{constant}. \tag{4.14}$$

Thus, M is a CMC surface of a de Sitter space-time. So, we obtain case (3).

Case (b): *A_{e_4} takes the second form of* (4.6). Since $[A_{e_3}, A_{e_4}] = 0$ from Lemma 2.2, we have

$$A_{e_3} = \begin{pmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix} \tag{4.15}$$

for some functions α, β, γ. Hence

$$h(e_1, e_1) = \alpha e_3\,, \quad h(e_1, e_2) = -\beta e_3 - \gamma e_4\,, \quad h(e_2, e_2) = -\alpha e_3\,. \tag{4.16}$$

This implies that $2H = -2\alpha e_3$ and hence $\alpha = -\kappa$ is constant. The equation of Codazzi gives

$$\begin{aligned} e_1\beta &= -2\beta\omega_1^2(e_2)\,, \quad e_1\gamma = -2\gamma\omega_1^2(e_2)\,, \\ e_2\beta &= -2\beta\omega_1^2(e_1)\,, \quad e_2\gamma = -2\gamma\omega_1^2(e_1)\,. \end{aligned} \tag{4.17}$$

These equations imply that β/γ is constant. Remark that we may assume $\gamma \neq 0$, because the case $\gamma = 0$ was already solved above. Let us put

$$\hat{e}_3 = (\cos t)e_3 + (\sin t)e_4\,, \quad \hat{e}_4 = (\sin t)e_3 - (\cos t)e_4\,, \quad t = \arctan\left(\frac{\beta}{\gamma}\right). \tag{4.18}$$

Then $\hat{e}_3, \hat{e}_4$ are orthonormal normal vector fields, which are parallel in the normal bundle, such that $A_{\hat{e}_3}$ and $A_{\hat{e}_4}$ satisfy (4.12). We can now proceed as in case (a).

Case (c): *A_{e_4} takes the third form of* (4.6). Since $[A_{e_3}, A_{e_4}] = 0$ from Lemma 2.2, we have

$$A_{e_3} = \begin{pmatrix} \alpha & \pm\dfrac{\alpha-\beta}{2} \\ \mp\dfrac{\alpha-\beta}{2} & \beta \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} \pm 1 & 1 \\ -1 & \mp 1 \end{pmatrix} \tag{4.19}$$

for some functions α, β. Hence

$$h(e_1, e_1) = -\alpha e_3 \mp e_4\,, \quad h(e_1, e_2) = \frac{\alpha-\beta}{2}e_3 - e_4\,, \quad h(e_2, e_2) = \beta e_3 \mp e_4\,. \tag{4.20}$$

Remark that $2H = (\alpha+\beta)e_3 = 2\kappa e_3$ and hence $(\alpha+\beta)/2 = \kappa$ is a nonzero constant.

From the equation of Codazzi, we obtain that $\omega_1^2 = 0$ and that α and β are constant. Now put

$$\hat{e}_3 = (\cos t)e_3 + (\sin t)e_4\,, \quad \hat{e}_4 = (\sin t)e_3 - (\cos t)e_4\,, \quad t = \arctan\left(\frac{\beta}{\gamma}\right). \tag{4.21}$$

Then $\hat{e}_3, \hat{e}_4$ are orthonormal parallel normal vector fields such that $A_{\hat{e}_3}$ and $A_{\hat{e}_4}$ satisfy (4.12). We can now proceed as in the previous cases.

The converse is easy to verify. □

THEOREM 4.2. *Let M be a Lorentzian surface in $S_1^4(1) \subset \mathbb{E}_1^5$. Then M has parallel mean curvature vector if and only if M is one of the following surfaces*:

(1) *a minimal Lorentzian surface in $S_1^4(1)$*;

(2) *a CMC surface in $S_1^4(1) \cap \mathcal{E}_1$, where $\mathcal{E}_1$ is a Lorentzian hyperplane in $\mathbb{E}_1^5$.*

PROOF. Under the hypothesis, if $H = 0$, we get case (1). So, we assume $H \neq 0$ and choose an orthonormal normal frame $\{e_3, e_4\}$ with $H = \kappa e_3$. Then, κ is a nonzero constant, $De_3 = De_4 = 0$ and $[A_{e_3}, A_{e_4}] = 0$.

Since trace $A_{e_4} = 0$, it follows from Lemma 4.1 that there exist e_1 and e_2 satisfying (2.9) and that A_{e_4} is given by one the three forms in (4.6).

Case (a): *A_{e_4} takes the first form of* (4.6). From $[A_{e_3}, A_{e_4}] = 0$ we have

$$A_{e_3} = \begin{pmatrix} -\alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} -\gamma & 0 \\ 0 & \gamma \end{pmatrix} \tag{4.22}$$

for some functions α, β, γ. Since $H = \kappa e_3 \neq 0$, we get $\beta - \alpha = 2\kappa$ is a nonzero constant. From (2.3), (2.9) and (4.22), we obtain

$$h(e_1, e_1) = \alpha e_3 + \gamma e_4\,, \quad h(e_1, e_2) = 0\,, \quad h(e_2, e_2) = \beta e_3 + \gamma e_4\,. \tag{4.23}$$

By applying (2.11), (4.23) and the equation of Codazzi, we may prove that $\gamma/(\alpha+\beta)$ is constant.

If $\gamma = 0$, we get $A_{e_4} = 0$. Combining this with $De_4 = 0$ shows that e_4 is a constant space-like vector. So, $\langle L, e_4\rangle$ is constant, say b. Hence, M lies the Lorentzian hyperplane $\mathcal{E}_1$ given by $\langle L, e_4\rangle = b$. This gives case (2).

If $\gamma \neq 0$, let us put $\hat{e}_4 = (\sin t)e_3 - (\cos t)e_4$ with $t = \arctan(2\gamma/(\alpha+\beta))$. Then $\hat{e}_4$ is a space-like parallel unit normal vector fields with $A_{\hat{e}_4} = (\kappa \sin t)I$. Hence, it follows from (4.12), the constancy of $\kappa \sin t$ and $D\hat{e}_4 = 0$ that $L + \kappa^{-1}(\csc t)\hat{e}_4$ is a constant vector, say c_0. So, we have

$$\langle L - c_0, L - c_0\rangle = \kappa^{-2}\csc^2 t = \text{constant}\,. \tag{4.24}$$

If $c_0 = 0$, then we have $L = -\kappa^{-1}(\csc t)\hat{e}_4$, which is impossible since $\hat{e}_4$ is tangent to $S^4_1(1)$. Thus, we must have $c_0 \neq 0$. So, it follows from $\langle L, L\rangle = 1$ and (4.24) that M lies in the hyperplane $\mathcal{E}$ given by $2\langle L, c_0\rangle = 1 + \langle c_0, c_0\rangle - \kappa^2 \csc^2 t$. Since M is Lorentzian, the hyperplane $\mathcal{E}$ must be Lorentzian. Thus, we obtain case (2) again.

Case (b): *A_{e_4} takes the second form of* (4.6). This can be reduced to case (a) just like case (b) in the proof of Theorem 4.1.

Case (c): *A_{e_4} takes the third form of* (4.6). Similarly, one can proceed as in the previous cases.

The converse is easy to verify. □

THEOREM 4.3. *Let M be a Lorentzian surface in $H^4_1(-1) \subset \mathbb{E}^5_2$. Then M has parallel mean curvature vector if and only if M is one of the following surfaces*:

(1) *a minimal Lorentzian surface in $H^4_1(-1)$;*
(2) *a CMC surface in $H^4_1(-1) \cap \mathcal{E}_1$, where $\mathcal{E}_1$ is a Lorentzian hyperplane in $\mathbb{E}^5_2$;*
(3) *a CMC surface in $H^4_1(-1) \cap \mathcal{E}_2$, where $\mathcal{E}_2$ is a hyperplane of index 2 in $\mathbb{E}^5_2$;*
(4) *a CMC surface in $H^4_1(-1) \cap \mathcal{H}$, where $\mathcal{H}$ is a degenerate hyperplane in $\mathbb{E}^5_2$.*

5. Parallel surfaces in 3-dimensional Lorentzian space forms. Now, we classify parallel surfaces in 3-dimensional Lorentzian space forms. These classifications serve as auxiliary results for the classification in 4-dimensional Lorentzian space forms.

THEOREM 5.1. *A non-degenerate parallel surface in $\mathbb{E}^3_1$ is congruent to an open part of one of the following eight types of surfaces*:

(1) *a Euclidean plane $\mathbb{E}^2$ in $\mathbb{E}^3_1$ given by $L = (0, u, v)$;*
(2) *a totally umbilical hyperbolic plane H^2 in $\mathbb{E}^3_1$ given by*

$$L = b(\cosh u \cosh v, \cosh u \sinh v, \sinh u)\,, \quad b > 0;$$

(3) *a flat cylinder $H^1 \times \mathbb{E}^1$ in $\mathbb{E}^3_1$ defined by $L = (a\cosh u, a \sinh u, v)$ with $a > 0$;*

(4) *a Lorentzian plane* $\mathbb{E}^2_1$ *in* $\mathbb{E}^3_1$ *given by* $L = (u, v, 0)$;

(5) *a totally umbilical de Sitter space* S^2_1 *in* $\mathbb{E}^3_1$ *given by*

$$L = b(\sinh u, \cosh u \cos v, \cosh u \sin v)\,, \quad b > 0;$$

(6) *a flat cylinder* $\mathbb{E}^1_1 \times S^1$ *in* $\mathbb{E}^3_1$ *given by* $L = (u, a\cos v, a\sin v)$ *with* $a > 0$;

(7) *a flat cylinder* $S^1_1 \times \mathbb{E}^1$ *given by* $L = (a\sinh u, a\cosh u, v)$ *with* $a > 0$;

(8) *a flat minimal Lorentzian surface in* $\mathbb{E}^3_1$ *given by*

$$L = \left(\frac{1}{6}(u-v)^3 + u, \frac{1}{6}(u-v)^3 + v, \frac{1}{2}(u-v)^2\right).$$

PROOF. We distinguish the cases that M is space-like and that M is Lorentzian.

Case (a): *M is a space-like parallel surface in* $\mathbb{E}^3_1$. Let e_3 be a unit time-like normal vector field and $\{e_1, e_2\}$ an orthonormal frame field on M which diagonalizes the shape operator associated to e_3, say $Ae_1 = \alpha e_1$ and $Ae_2 = \beta e_2$. A direct computation shows that the surface is parallel if and only if α and β are constant and $(\alpha - \beta)\omega_1^2 = 0$.

If $\alpha = \beta = 0$, then M is totally geodesic which gives case (1) of the theorem.

If $\alpha = \beta = a \neq 0$, then the second fundamental form of M in $\mathbb{E}^3_1$ satisfies

$$h(e_1, e_1) = -ae_3\,, \quad h(e_1, e_2) = 0\,, \quad h(e_2, e_2) = -ae_3\,. \tag{5.1}$$

Thus, if follows from the equation of Gauss that $K = -a^2$. If we choose coordinates (x, y) with $g = dx^2 + \cosh^2(ax)dy^2$. Then, we have

$$L_{xx} = -ae_3\,, \quad L_{xy} = a\tanh(ax)L_y\,, \quad L_{yy} = -a\cosh^2(ax)e_3 - \frac{a}{2}\sinh(2ax)L_x\,,$$
$$\tilde{\nabla}_{\partial_x} e_3 = -aL_x\,, \quad \tilde{\nabla}_{\partial_y} e_3 = -aL_y\,.$$

The solution of this system is

$$L(u, v) = c_1\cosh(ax)\cosh(ay) + c_2\cosh(ax)\sinh(ay) + c_3\sinh(ax) + c_4\,,$$

with $c_1, c_2, c_3, c_4 \in \mathbb{E}^3_1$. After choosing suitable Minkowskian coordinates and making a suitable reparametrization, we obtain case (2) of the theorem.

If $\alpha \neq \beta$, then $\omega_1^2 = 0$. So, M is flat and it follows from the equation of Gauss that $\alpha\beta = 0$. Without loss of generality, we may assume $\beta = 0$. Now, choose coordinates (u, v) on M with $\partial_u = e_1$ and $\partial_v = e_2$. Then, the immersion L satisfies

$$L_{uu} = -\alpha e_3\,, \quad L_{uv} = 0\,, \quad L_{vv} = 0\,, \quad \tilde{\nabla}_{\partial_u} e_3 = -\alpha L_u\,, \quad \tilde{\nabla}_{\partial_v} e_3 = 0\,.$$

The solution of this system of equations is, up to a translation, given by

$$L(u, v) = c_1\cosh(\alpha u) + c_2\sinh(\alpha u) + c_3 v\,,$$

with $c_1, c_2, c_3 \in \mathbb{E}^3_1$. After reparametrizing and choosing suitable Minkowskian coordinates, we obtain case (3) of the theorem.

Case (b): *M is a Lorentzian parallel surface in* $\mathbb{E}^3_1$. Let e_3 be a unit space-like normal vector field. In general, the shape operator A_{e_3} cannot be diagonalized, but it follows from Lemma 4.1 that, after a suitable choice of a frame $\{e_1, e_2\}$ with $\langle e_1, e_1\rangle = -1$, $\langle e_1, e_2\rangle = 0$ and $\langle e_2, e_2\rangle = 1$, there are three cases to consider.

Case (b.1): *A takes the form* (4.1). In this case, we have $K = \alpha\beta$ and

$$h(e_1, e_1) = -\alpha e_3\,, \quad h(e_1, e_2) = 0\,, \quad h(e_2, e_2) = \beta e_3\,. \tag{5.2}$$

From (5.2) and (2.7) we know that $\overline{\nabla} h = 0$ if and only if

$$d\alpha = d\beta = (\alpha - \beta)\omega_1^2 = 0\,. \tag{5.3}$$

In particular, α, β are constant if $\overline{\nabla} h = 0$ holds.

Case (b.1.i): $\alpha = \beta = 0$. In this case, we obtain case (4).

Case (b.1.ii): $\alpha = \beta = a \neq 0$. In this case, M is a totally umbilical surface with $K = a^2$. By choosing coordinates (x, y) with $g = -dx^2 + \cosh^2(ax)dy^2$, we obtain

$$L_{xx} = -ae_3\,, \quad L_{xy} = a\tanh(ax)L_y\,, \quad L_{yy} = a\cosh^2(ax)e_3 + \frac{a}{2}\sinh(2ax)L_x\,,$$
$$\tilde{\nabla}_{\partial_x} e_3 = -aL_x\,, \quad \tilde{\nabla}_{\partial_y} e_3 = -aL_y\,.$$

Solving this system leads, after a suitable reparametrization, to case (5).

Case (b.1.iii): $\alpha \neq \beta$. In this case, we have $\omega_1^2 = 0$ and the surface is flat; so $\alpha\beta = 0$. A similar calculation as in the space-like case yields case (6) of the theorem if $\alpha = 0$ and case (7) if $\beta = 0$.

Case (b.2): *A takes the form* (4.2). The second fundamental form satisfies

$$h(e_1, e_1) = -\alpha e_3\,, \quad h(e_1, e_2) = -\beta e_3\,, \quad h(e_2, e_2) = \alpha e_3 \tag{5.4}$$

for some functions α, β with $\beta \neq 0$. Hence, by applying the assumption $\overline{\nabla} h = 0$, (2.7) and (2.11), we have

$$d\alpha = d\beta = \omega_1^2 = 0\,. \tag{5.5}$$

Thus, α, β are constant and the surface is flat.

On the other hand, equation (2.4) of Gauss together with (5.4) gives $K = \alpha^2 + \beta^2$, which is a contradiction since $\beta \neq 0$.

Case (b.3): *A takes one of the forms* (4.3). Since the surface is parallel, we obtain that α is constant and $\omega_1^2 = 0$. Hence the surface is flat. But from the equation of Gauss, we obtain that the Gaussian curvature is given by $K = (\alpha \pm 1)^2$.

Case (b.3.i): *A takes the first form of* (4.3) *with* $\alpha = -1$. If we take coordinates (u, v) with $\partial_u = e_1$ and $\partial_v = e_2$, $g = -du^2 + dv^2$, then the formulas of Gauss and Weingarten yield the following system of equations:

$$L_{uu} = e_3\,, \quad L_{uv} = -e_3\,, \quad L_{vv} = e_3\,, \quad \tilde{\nabla}_{\partial_u} e_3 = L_u + L_v\,, \quad \tilde{\nabla}_{\partial_v} e_3 = -L_u - L_v\,.$$

Solving this system leads, after a reparametrization, to case (8) of the theorem.

Case (b.3.ii): *A takes the second form of* (4.3) *with* $\alpha = 1$. Proceeding in the same way as in the previous case, we obtain a surface congruent to case (8). □

REMARK 5.1. The flat minimal Lorentzian surface given in case (8) of Theorem 5.1 is a B-scroll in the sense of [10] since it has degenerate relative nullity (see [10, page 391]; see also [13]).

THEOREM 5.2. *A non-degenerate parallel surface in* $S^3_1(c) \subset \mathbb{E}^4_1$, $c > 0$, *is congruent to an open part of one of the following six types of surfaces*:

(1) *a totally umbilical sphere* S^2 *in* $S^3_1(c)$ *locally given by*

$$L = (a, b \sin u, b \cos u \cos v, b \cos u \sin v)\,, \quad b^2 - a^2 = c^{-1}\,;$$

(2) *a totally umbilical Euclidean plane* $\mathbb{E}^2$ *in* $S^3_1(c)$ *given by*

$$L = \frac{1}{\sqrt{c}}\left(u^2 + v^2 - \frac{3}{4}, u^2 + v^2 - \frac{5}{4}, u, v\right);$$

(3) *a totally umbilical hyperbolic plane* H^2 *in* $S^3_1(c)$ *given by*

$$L = (a \cosh u \cosh v, a \cosh u \sinh v, a \sinh u, b)\,, \quad b^2 - a^2 = c^{-1}\,;$$

(4) *a flat surface* $H^1 \times S^1$ *in* $S^3_1(c)$ *defined by*

$$L = (a \cosh u, a \sinh u, b \cos v, b \sin v)\,, \quad b^2 - a^2 = c^{-1}\,;$$

(5) *a totally umbilical de Sitter space* S^2_1 *in* $S^3_1(c)$ *given by*

$$L = (a \sinh u, a \cosh u \cos v, a \cosh u \sin v, b)\,, \quad a^2 + b^2 = c^{-1}\,;$$

(6) *a flat surface* $S^1_1 \times S^1$ *in* $S^3_1(c)$ *given by*

$$L = (a \sinh u, a \cosh u, b \cos v, b \sin v)\,, \quad a^2 + b^2 = c^{-1}\,.$$

PROOF. First, we classify non-degenerate parallel surfaces in $S^3_1(1)$ and then we apply the dilation $L \mapsto L/\sqrt{c}$ on $\mathbb{E}^4_1$ to obtain the desired results.

Case (a): *M is a space-like parallel surface in $S^3_1(1)$.* Let e_3 be a normal vector field of M in $S^3_1(1)$ with $\langle e_3, e_3 \rangle = -1$ and let $\{e_1, e_2\}$ be an orthonormal frame field on the surface which diagonalizes the shape operator A associated to e_3 so that $Ae_1 = \alpha e_1$ and $Ae_2 = \beta e_2$. A straightforward computation shows that the surface is parallel if and only if $d\alpha = d\beta = (\alpha - \beta)\omega^2_1 = 0$.

Case (a.1). If $\alpha = \beta = 0$, the surface is a totally geodesic unit 2-sphere. Hence, we get case (1) of the theorem with $a = 0$.

Case (a.2). If $\alpha = \beta = a \neq 0$, the surface is totally umbilical. So, the second fundamental form of M in $S^3_1(1)$ satisfies

$$h(e_1, e_1) = -ae_3\,, \quad h(e_1, e_2) = 0\,, \quad h(e_2, e_2) = -ae_3\,. \tag{5.6}$$

From (5.6) and the equation of Gauss, we obtain $K = 1 - a^2$.

If $a^2 = 1$, then M is flat. In this case, we may choose coordinates (u, v) with $g = du^2 + dv^2$. Then, we obtain from (5.6) that

$$L_{uu} = -e_3 - L\,, \quad L_{uv} = 0\,, \quad L_{vv} = -e_3 - L\,, \quad \tilde{\nabla}_{\partial_u} e_3 = -L_u\,, \quad \tilde{\nabla}_{\partial_v} e_3 = -L_v\,.$$

After solving this system, choosing suitable Minkowskian coordinates and making a suitable reparametrization, we obtain case (2).

If $a^2 < 1$, then M is of positive curvature $1-a^2$. In this case, we may choose coordinates (x, y) with $g = dx^2 + \cos^2(\sqrt{1-a^2}x)dy^2$. Then, we have

$$L_{xx} = -ae_3 - L\,, \quad L_{xy} = -\sqrt{1-a^2}\tan(\sqrt{1-a^2}x)L_y\,,$$
$$L_{yy} = -\cos^2(\sqrt{1-a^2}x)(ae_3+L) + \frac{\sqrt{1-a^2}}{2}\sin(2\sqrt{1-a^2}x)L_x\,,$$
$$\tilde{\nabla}_{\partial_x}e_3 = -aL_x\,, \quad \tilde{\nabla}_{\partial_y}e_3 = -aL_y\,.$$

Solving this system leads, after a reparametrization, to case (1) with $a \neq 0$.

Similarly, for $a^2 > 1$, we may obtain case (3) of the theorem.

Case (a.3). If the surface is not totally umbilical, we have $\omega_1^2 = 0$. So, M is flat and we find $\beta = \alpha^{-1}$ from the equation of Gauss. Without loss of generality, we may assume $\alpha > 0$. By taking coordinates (x, y) with $\partial_x = e_1$, $\partial_y = e_2$, we have

$$\begin{gathered} L_{xx} = -(\alpha e_3 + L)\,, \quad L_{xy} = 0\,, \quad L_{yy} = -(\alpha^{-1}e_3 + L)\,, \\ \tilde{\nabla}_{\partial_x}e_3 = -\alpha L_x\,, \quad \tilde{\nabla}_{\partial_y}e_3 = -\alpha^{-1}L_y\,. \end{gathered} \tag{5.7}$$

If $\varphi = \alpha^2 - 1 > 0$, then solving (5.7) gives

$$L = c_1\cosh(\sqrt{\varphi}x) + c_2\sinh(\sqrt{\varphi}x) + c_3\cos(\sqrt{\varphi}y/\alpha) + c_4\sin(\sqrt{\varphi}y/\alpha)\,,$$

with $c_1, c_2, c_3, c_4 \in \mathbb{E}^4_1$. After choosing suitable Minkowskian coordinates and making a suitable reparametrization, we obtain case (4).

If $\varphi = \alpha^2 - 1 < 0$, with a similar approach, we also obtain case (4).

Case (b): *M is a Lorentzian parallel surface in $S^3_1(1)$.* Let e_1, e_2 be as in Lemma 4.1, and e_3 be a unit normal vector field in $S^3_1(1)$ with associated shape operator A. Then it follows from Lemma 4.1 that there are three cases to consider.

Case (b.1): *A takes the form* (4.1). In this case, we have $K = 1 + \alpha\beta$ and

$$h(e_1, e_1) = -\alpha e_3\,, \quad h(e_1, e_2) = 0\,, \quad h(e_2, e_2) = \beta e_3\,. \tag{5.8}$$

From (5.8) and (2.7) we know that $\overline{\nabla}h = 0$ if and only if $d\alpha = d\beta = (\alpha - \beta)\omega_1^2 = 0$. In particular, α, β are constant if $\overline{\nabla}h = 0$ holds.

Case (b.1.i): $\alpha = \beta = a$. The surface is totally umbilical with $K = 1+a^2$. By choosing coordinates (x, y) with $g = -dx^2 + \cosh^2(\sqrt{1+a^2}x)dy^2$, we obtain

$$L_{xx} = -ae_3 + L\,, \quad L_{xy} = \sqrt{1+a^2}\tanh(\sqrt{1+a^2}x)L_y\,,$$
$$L_{yy} = \cosh^2(\sqrt{1+a^2}x)(ae_3 - L) + \frac{\sqrt{1+a^2}}{2}\sinh(2\sqrt{1+a^2}x)L_x\,,$$
$$\tilde{\nabla}_{\partial_x}e_3 = -aL_x\,,\ \tilde{\nabla}_{\partial_y}e_3 = -aL_y\,.$$

Solving this system leads, after a reparametrization, to case (5) of the theorem.

Case (b.1.ii): $\alpha \neq \beta$. In this case, we have $\omega_1^2 = 0$. Thus, M is flat. So, we get $\beta = -\alpha^{-1}$. If we choose coordinates (x, y) with $\partial_x = e_1$, $\partial_y = e_2$, we have

$$L_{xx} = -\alpha e_3 + L\,,\ \ L_{xy} = 0\,,\ \ L_{yy} = -\alpha^{-1}e_3 - L\,,\ \ \tilde{\nabla}_{\partial_x}e_3 = -\alpha L_x\,,\ \ \tilde{\nabla}_{\partial_y}e_3 = \alpha^{-1}L_y\,.$$

Solving this system leads, after a reparametrization, to case (6).

Case (b.2): *A takes the form* (4.2). In this case, we also have (5.4) and (5.5). Thus, the surface is flat. On the other hand, equation (2.4) of Gauss together with (5.2) gives $K = 1 + \alpha^2 + \beta^2$, which contradicts $K = 0$.

Case (b.3): *A takes one of the forms* (4.3). If the surface is not totally umbilical, we obtain a contradiction similarly as in case (b.2). □

REMARK 5.2. It was proved in [7] that cases (2) and (4) of Theorem 5.2 are the only isometric immersions of $\mathbb{E}^2$ into S^3_1 and case (6) is the only isometric immersion of $\mathbb{E}^2_1$ into S^3_1.

THEOREM 5.3. *A non-degenerate parallel surface in $H^3_1(-c) \subset \mathbb{E}^4_2$, $c > 0$, is congruent to an open part of one of the following ten types of surfaces*:

(1) *a totally umbilical hyperbolic plane H^2 in $H^3_1(-c)$ given by*

$$L = (b, a\cosh u \cosh v, a\cosh u \sinh v, a\sinh u)\,, \quad a^2 + b^2 = c^{-1}\,;$$

(2) *a surface $H^1 \times H^1$ in $H^3_1(-c)$ given by*

$$L = (a\cosh u, b\cosh v, a\sinh u, b\sinh v)\,, \quad a^2 + b^2 = c^{-1}\,;$$

(3) *a totally umbilical de Sitter space S^2_1 in $H^3_1(-c)$ given by*

$$L = (b, a\sinh u, a\cosh u \sin v, a\cosh u \cos v)\,, \quad b^2 - a^2 = c^{-1}\,;$$

(4) *a totally umbilical Lorentzian plane $\mathbb{E}^2_1$ in $H^3_1(-c)$ given by*

$$L = \frac{1}{\sqrt{c}}\left(u^2 - v^2 - \frac{5}{4}, u, v, u^2 - v^2 - \frac{3}{4}\right);$$

(5) *a totally umbilical anti-de Sitter space H^2_1 in $H^3_1(-c)$ given by*

$$L = (a\sin u, a\cos u \cosh v, a\cos u \sinh v, b)\,, \quad a^2 - b^2 = c^{-1}\,;$$

(6) *a surface $S^1_1 \times H^1$ in $H^3_1(-c)$ given by*

$$L = (a\sinh u, b\cosh v, a\cosh u, b\sinh v)\,, \quad b^2 - a^2 = c^{-1}\,;$$

(7) *a flat surface $H^1_1 \times S^1$ in $H^3_1(-c)$ defined by*

$$L = (a\cos u, a\sin u, b\cos v, b\sin v)\,, \quad a^2 - b^2 = c^{-1}\,;$$

(8) *a Lorentzian plane $\mathbb{E}^2_1$ immersed in $H^3_1(-c)$ by*

$$L = \frac{1}{\sqrt{c}}(\cos u \cosh v - \tan k \sin u \sinh v, \sec k \sin u \cosh v,$$
$$\cos u \sinh v - \tan k \sin u \cosh v, \sec k \sin u \sinh v)\,, \quad k \in R\,;$$

(9) *a Lorentzian plane $\mathbb{E}^2_1$ immersed in $H^3_1(-c)$ by*

$$L = \frac{1}{\sqrt{c}}\left(\cos v - \frac{u-v}{2}\sin v, \sin v + \frac{u-v}{2}\cos v, \frac{u-v}{2}\sin v, \frac{u-v}{2}\cos v\right);$$

(10) *a Lorentzian plane* $\mathbb{E}^2_1$ *immersed in* $H^3_1(-c)$ *by*

$$L = \frac{1}{\sqrt{c}}\left(\cosh v - \frac{u+v}{2}\sinh v, \frac{u+v}{2}\cosh v, \sinh v - \frac{u+v}{2}\cosh v, \frac{u+v}{2}\sinh v\right).$$

PROOF. Just like the proof of Theorem 5.2, we may first classify non-degenerate parallel surfaces in $H^3_1(-1)$ and then apply the dilation $L \mapsto L/\sqrt{c}$ on $\mathbb{E}^4_2$ to obtain the desired results.

Case (a): *M is a space-like parallel surface in* $H^3_1(-1)$. Let e_3 be a time-like unit normal vector field in $S^3_1(1)$ with associated shape operator A and let $\{e_1, e_2\}$ be an orthonormal tangent frame diagonalizing A, say $Ae_1 = \alpha e_1$ and $Ae_2 = \beta e_2$. In this case, we have $K = -\alpha\beta - 1$ and the second fundamental form satisfies

$$h(e_1, e_1) = -\alpha e_3\,, \quad h(e_1, e_2) = 0\,, \quad h(e_2, e_2) = -\beta e_3\,. \tag{5.9}$$

From (5.9) and (2.7) we know that $\overline{\nabla} h = 0$ if and only if $d\alpha = d\beta = (\alpha - \beta)\omega_1^2 = 0$. In particular, α, β are constant if $\overline{\nabla} h = 0$ holds.

Case (a.1): $\alpha = \beta = a$. In this case, we get $K = -(1 + a^2)$. By choosing coordinates (x, y) with $g = dx^2 + \cosh^2(\sqrt{1+a^2}x)dy^2$, we obtain

$$L_{xx} = L - ae_3\,, \quad L_{xy} = \sqrt{1+a^2}\tanh(\sqrt{1+a^2}x)L_y\,,$$
$$L_{yy} = \cosh^2(\sqrt{1+a^2}x)(L - ae_3) - \frac{\sqrt{1+a^2}}{2}\sinh(2\sqrt{1+a^2}x)L_x\,,$$
$$\tilde{\nabla}_{\partial_x} e_3 = -aL_x\,, \quad \tilde{\nabla}_{\partial_y} e_3 = -aL_y\,.$$

After solving this system, choosing suitable Minkowskian coordinates, and making a suitable reparametrization, we obtain case (1) of the theorem.

Case (a.2): $\alpha \neq \beta$. We have $\omega_1^2 = 0$. So, M is flat and we find $\beta = -\alpha^{-1}$. By applying a similar method as before, we obtain case (2).

Case (b): *M is Lorentzian*. We take a unit normal e_3 on M tangent to $H^3_1(-1)$ and we consider the three possibilities for the shape operator A associated to e_3.

Case (b.1): *A takes the form* (4.1). In this case, we have $K = \alpha\beta - 1$ and the second fundamental form satisfies

$$h(e_1, e_1) = -\alpha e_3\,, \quad h(e_1, e_2) = 0\,, \quad h(e_2, e_2) = \beta e_3\,. \tag{5.10}$$

From (5.10) and (2.7) we know that $\overline{\nabla} h = 0$ if and only if $d\alpha = d\beta = (\alpha - \beta)\omega_1^2 = 0$.

Case (b.1.i): $\alpha = \beta = a$. In this case, M is a totally umbilical surface with $K = a^2 - 1$. So, we divide this into 3 cases.

Case (b.1.i.α): $a^2 > 1$. In this case, if we choose coordinates (x, y) such that $g = -dx^2 + \cosh^2(\sqrt{a^2-1}x)dy^2$, then we obtain

$$L_{xx} = -ae_3 - L\,, \quad L_{xy} = \sqrt{a^2-1}\tanh(\sqrt{a^2-1}x)L_y\,,$$
$$L_{yy} = \cosh^2(\sqrt{a^2-1}x)(ae_3+L) + \frac{\sqrt{a^2-1}}{2}\sinh(2\sqrt{a^2-1}x)L_x\,,$$
$$\tilde{\nabla}_{\partial_x}e_3 = -aL_x\,, \quad \tilde{\nabla}_{\partial_y}e_3 = -aL_y\,.$$

Solving this system leads, after a reparametrization, to case (3).

Case (b.1.i.β): $a^2 = 1$. Without loss of generality, we may assume $a = 1$. If we choose coordinates (x, y) such that $g = -dx^2 + dy^2$, then we obtain

$$L_{xx} = -e_3 - L\,, \quad L_{xy} = 0\,, \quad L_{yy} = e_3 + L\,, \quad \tilde{\nabla}_{\partial_x}e_3 = -L_x\,, \quad \tilde{\nabla}_{\partial_y}e_3 = -L_y\,.$$

After solving this system and choosing suitable initial conditions, we get case (4).

Case (b.1.i.γ): $a^2 < 1$. With an analogous computation as (b.1.i.α), we obtain case (5) of the theorem.

Case (b.1.ii): $\alpha \neq \beta$. In this case, we have $\omega_1^2 = 0$. Thus, M is flat. So, we get $\beta = -\alpha^{-1}$. With an analogous computation as above, we obtain cases (6) and (7).

Case (b.2): *A takes the form* (4.2). In this case, we have (5.4) and (5.5). Thus, the surface is flat and α, β are constant with $\beta \neq 0$.

On the other hand, equation (2.4) of Gauss and (5.4) give $K = -\alpha^2 - \beta^2 - 1$. Thus, we have $\alpha^2 + \beta^2 = 1$. Thus, we may put $\alpha = \sin k$, $\beta = \cos k$ for some $k \in \boldsymbol{R}$.

Since $\omega_1^2 = 0$, there exist coordinates (x, y) with $\partial_x = e_1$, $\partial_y = e_2$. So, the metric tensor is $g = -dx^2 + dy^2$. Thus, we obtain

$$L_{xx} = -(\sin k)e_3 - L\,, \quad L_{xy} = -(\cos k)e_3\,, \quad L_{yy} = (\sin k)e_3 + L\,,$$
$$\tilde{\nabla}_{e_1}e_3 = -\sin k e_1 + \cos k e_2\,, \quad \tilde{\nabla}_{e_2}e_3 = -\cos k e_1 - \sin k e_2\,, \quad \cos k \neq 0\,.$$

Solving this system leads to case (8) of the theorem after the reparametrization

$$u = \sqrt{\cos k}\left(x\cos\frac{k}{2} - y\sin\frac{k}{2}\right)\,, \quad v = \sqrt{\cos k}\left(y\cos\frac{k}{2} + x\sin\frac{k}{2}\right)\,.$$

Case (b.3): *A takes one of the forms* (4.3).

Case (b.3.i): *A takes the first form of* (4.3). Then the second fundamental form of M in $H_1^3(-1)$ satisfies

$$h(e_1, e_1) = -\alpha e_3\,, \quad h(e_1, e_2) = -e_3\,, \quad h(e_2, e_2) = (\alpha+2)e_3\,. \tag{5.11}$$

Hence, by applying $\overline{\nabla}h = 0$, (2.7), (2.11) and (5.11), we have $d\alpha = \omega_1^2 = 0$. Thus, $K = 0$ and we can choose coordinates (x, y) with $\partial_x = e_1$, $\partial_y = e_2$. So, the metric tensor is $g = -dx^2 + dy^2$.

On the other hand, from the equation of Gauss and (5.11) we find $K = \alpha(\alpha+2)$. Hence, we have either $\alpha = 0$ or $\alpha = -2$.

Case (b.3.i.α): $\alpha = 0$. The immersion $L : M \to H^3(-1) \subset \mathbb{E}^4_2$ satisfies

$$L_{xx} = -L\,, \quad L_{xy} = -e_3\,, \quad L_{yy} = 2e_3 + L\,, \quad \tilde{\nabla}_{e_1} e_3 = e_2\,, \quad \tilde{\nabla}_{e_2} e_3 = -e_1 - 2e_2\,.$$

Solving this system yields case (9) of the theorem.

Case (b.3.i.β): $\alpha = -2$. In this case, $L : M \to H^3(-1) \subset \mathbb{E}^4_2$ satisfies

$$L_{xx} = 2e_3 - L\,, \quad L_{xy} = -e_3\,, \quad L_{yy} = L\,, \quad \tilde{\nabla}_{e_1} e_3 = 2e_1 + e_2\,, \quad \tilde{\nabla}_{e_2} e_3 = -e_1\,.$$

Solving this system gives case (10) of the theorem.

Case (b.3.ii): *A takes the second form of* (4.3). Then we have

$$h(e_1, e_1) = -\alpha e_3\,, \quad h(e_1, e_2) = -e_3\,, \quad h(e_2, e_2) = (\alpha - 2)e_3\,.$$

As before, the equations of and Gauss and Codazzi, and $\overline{\nabla} h = 0$ yield $\omega_1^2 = 0$ and $\alpha(\alpha - 2) = 0$. Thus, after choosing (x, y) as before, we have one of the following:

$$\begin{cases} L_{xx} = -L\,, \quad L_{xy} = -e_3\,, \quad L_{yy} = -2e_3 + L\,, \\ \tilde{\nabla}_{\partial_x} e_3 = e_2\,, \quad \tilde{\nabla}_{\partial_y} e_3 = -e_1 + 2e_2\,; \end{cases}$$

$$\begin{cases} L_{xx} = -2L_x - L\,, \quad L_{xy} = -e_3\,, \quad L_{yy} = L\,, \\ \tilde{\nabla}_{\partial_x} e_3 = -2e_1 + e_2\,, \quad \tilde{\nabla}_{\partial_y} e_3 = -e_1\,. \end{cases}$$

After solving these systems, choosing suitable Minkowskian coordinates and replacing y and x by $-y$ and $-x$, respectively, we obtain cases (9) and (10) as well. □

REMARK 5.3. It was proved in [7, page 93] that if $\phi : \mathbb{E}^2_1 \to H^3_1(-1)$ is an isometric immersion with parallel shape operator, then ϕ is the B-scroll of a complete curve of torsion 1 and constant curvature (see [7] for details). Consequently, for $c = 1$ case (4) and cases (6)–(10) of Theorem 5.3 are such B-scrolls.

6. Parallel surfaces in the light cone.

THEOREM 6.1. *Let M be a non-degenerate parallel surface of $\mathbb{E}^4_1$. If M lies in the light cone $\mathcal{LC} = \{\boldsymbol{x} \in \mathbb{E}^4_1;\ \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0\}$, then M is space-like and the immersion is congruent to an open part of one of the following four types of surfaces*:

(1) *a totally umbilical surface of positive curvature defined by*

$$L = a(1, \cos u \cos v, \cos u \sin v, \sin u)\,, \quad a > 0\,;$$

(2) *a totally umbilical surface of negative curvature defined by*

$$L = a(\cosh u \cosh v, \cosh u \sinh v, \sinh u, 1)\,, \quad a > 0\,;$$

(3) *a flat totally umbilical surface defined by*

$$L = \left(u^2 + v^2 + \frac{1}{4}, u^2 + v^2 - \frac{1}{4}, u, v\right);$$

(4) *a flat surface defined by* $L = a(\cosh u, \sinh u, \cos v, \sin v)$, $a > 0$.

PROOF. Let $L : M \to \mathbb{E}^4_1$ be a non-degenerate parallel surface. If M is immersed in $\mathcal{LC}$, we can regard M as a parallel surface of $H^4_1(-1)$ via the following inclusion:

$$\iota : \mathcal{LC} \to H^4_1(-1) \subset \mathbb{E}^5_2 \;:\; x \mapsto (1, x)\,. \tag{6.1}$$

Since M admits a light-like normal vector in $H^4_1(-1)$, each normal space is a Lorentzian plane. Thus, M is a space-like surface.

Let $\hat{L} = \iota \circ L : M \to \mathcal{LC} \to H^4_1(-1) \subset \mathbb{E}^5_2$ be the composition of L and ι. Put

$$e_3 = \hat{L} - (1, 0, 0, 0, 0)\,. \tag{6.2}$$

Then e_3 is a light-like normal vector of M in $H^4_1(-1)$. Let e_4 be another light-like normal vector of M in $H^4_1(-1)$ such that $\langle e_3, e_4\rangle = -1$.

It follows from (2.13) and (6.2) that $A_{e_3} = -I$ and $De_3 = De_4 = 0$. Thus, we may choose an orthonormal frame $\{e_1, e_2\}$ such that

$$A_{e_3} = -I\,, \quad A_{e_4} = \begin{pmatrix} \gamma & 0 \\ 0 & \varepsilon \end{pmatrix}. \tag{6.3}$$

Then the second fundamental form of M in $H^4_1(-1)$ satisfies

$$h(e_1, e_1) = -\gamma e_3 + e_4\,, \quad h(e_1, e_2) = 0\,, \quad h(e_2, e_2) = -\varepsilon e_3 + e_4\,. \tag{6.4}$$

Since M is a parallel surface in $H^4_1(-1)$, it follows from (6.4) that

$$d\gamma = d\varepsilon = (\gamma - \varepsilon)\omega_1^2 = 0\,.$$

From the equation of Gauss and (6.4) we find $K = \gamma + \varepsilon - 1$.

Case (a): $\gamma = \varepsilon$. We have $A_{e_3} = -I$, $A_{e_4} = \gamma I$ and $K = 2\gamma - 1$.

Case (a.1): $2\gamma > 1$. In this case, we shall choose coordinates (u, v) on M with $g = du^2 + \cos^2(\sqrt{2\gamma - 1}u)dv^2$, then the immersion $\hat{L}$ of M in $\mathbb{E}^5_2$ satisfies

$$\hat{L}_{uu} = -\gamma e_3 + e_4 + \hat{L}\,, \quad \hat{L}_{uv} = -\sqrt{2\gamma - 1}\tan(\sqrt{2\gamma - 1}u)\hat{L}_v\,,$$
$$\hat{L}_{vv} = \frac{1}{2}\sqrt{2\gamma - 1}\sin(2\sqrt{2\gamma - 1}u)\hat{L}_u + \cos^2(\sqrt{2\gamma - 1}u)(-\gamma e_3 + e_4 + \hat{L})\,,$$
$$\tilde{\nabla}_{\partial_u} e_3 = \hat{L}_u\,, \quad \tilde{\nabla}_{\partial_v} e_3 = \hat{L}_v\,, \quad \tilde{\nabla}_{\partial_u} e_4 = -\gamma \hat{L}_u\,, \quad \tilde{\nabla}_{\partial_v} e_4 = -\gamma \hat{L}_v\,.$$

After solving this system and choosing suitable Minkowskian coordinates, we obtain

$$\hat{L} = \frac{1}{\sqrt{2\gamma - 1}}(\sqrt{2\gamma - 1}, 1, \cos u \cos v, \cos u \sin v, \sin u)\,,$$

which gives rise to case (1) of the theorem for L.

Case (a.2): $2\gamma < 1$. Similarly, we obtain case (2) for L.

Case (a.3): $2\gamma = 1$. We choose coordinates (u, v) on M with $g = du^2 + dv^2$, then the immersion $\hat{L}$ of M in $\mathbb{E}^5_2$ satisfies

$$\hat{L}_{uu} = -\frac{e_3}{2} + e_4 + \hat{L}\,, \quad \hat{L}_{uv} = 0\,, \quad \hat{L}_{vv} = -\frac{e_3}{2} + e_4 + \hat{L}\,,$$
$$\tilde{\nabla}_{\partial_u} e_3 = \hat{L}_u\,, \quad \tilde{\nabla}_{\partial_v} e_3 = \hat{L}_v\,, \quad \tilde{\nabla}_{\partial_u} e_4 = -\frac{1}{2}\hat{L}_u\,, \quad \tilde{\nabla}_{\partial_v} e_4 = -\frac{1}{2}\hat{L}_v\,.$$

Solving this system leads, after a reparametrization, to case (3) of the theorem.

Case (b): $\gamma \neq \varepsilon$. We get $\omega_1^2 = 0$. Thus, M is flat and $\varepsilon = 1 - \gamma$. Since $\gamma \neq \varepsilon$, we have $\gamma \neq 1/2$. If we choose coordinates (u, v) with $\partial_u = e_1$, $\partial_v = e_2$, we obtain

$$\hat{L}_{uu} = -\gamma e_3 + e_4 + \hat{L}\,, \quad \hat{L}_{uv} = 0\,, \quad \hat{L}_{vv} = (\gamma - 1)e_3 + e_4 + \hat{L}\,,$$
$$\tilde{\nabla}_{\partial_u} e_3 = \hat{L}_u\,, \quad \tilde{\nabla}_{\partial_v} e_3 = \hat{L}_v\,, \quad \tilde{\nabla}_{\partial_u} e_4 = -\gamma \hat{L}_u\,, \quad \tilde{\nabla}_{\partial_v} e_4 = (\gamma - 1)\hat{L}_v\,.$$

Solving this system leads, after a reparametrization, to case (4) of the theorem. □

THEOREM 6.2. *Let M be a non-degenerate parallel surface of $\mathbb{E}^4_2$. If M lies in the light cone $\mathcal{LC} = \{x \in \mathbb{E}^4_2;\ \langle x, x\rangle = 0\}$, then M is Lorentzian and the immersion is congruent to an open part of one of the following eight types of surfaces*:

(1) *a totally umbilical surface of positive curvature defined by*

$$L = a(\sinh u, 1, \cosh u \cos v, \cosh u \sin v)\,, \quad a > 0\,;$$

(2) *a totally umbilical surface of negative curvature defined by*

$$L = a(\sin u, \cos u \cosh v, 1, \cos u \sinh v)\,, \quad a > 0\,;$$

(3) *the flat totally umbilical surface defined by*

$$L = \left(u, u^2 - v^2 - \frac{1}{4}, u^2 - v^2 + \frac{1}{4}, v\right);$$

(4) *a flat surface defined by* $L = a(\sinh u, \cosh v, \cosh u, \sinh v)$, $a > 0$;

(5) *a flat surface defined by* $L = a(\sin u, \cos u, \cos v, \sin v)$, $a > 0$;

(6) *a flat surface defined by*

$$L = a(\sinh u \cos v + \sinh u \sin v, \cosh u \sin v - \sinh u \cos v\,,$$
$$\cosh u \cos v - \sinh u \sin v, \cosh u \sin v + \sinh u \cos v)\,, \quad a > 0;$$

(7) *a flat surface defined by*

$$L = a(\cos v - u \sin v, \sin v + u \cos v, \cos v + u \sin v, \sin v - u \cos v)\,, \quad a > 0;$$

(8) *a flat surface defined by*

$$L = a(\cosh u - v \sinh u, \sinh u + v \cosh u, \cosh u + v \sinh u, \sinh u - v \cosh u)$$

with $a > 0$.

PROOF. Let $L : M \to \mathbb{E}^4_2$ be a non-degenerate parallel surface. Assume that M is immersed in $\mathcal{LC}$. Then M is immersed in $S^4_2(1) \subset \mathbb{E}^5_2$ via the following inclusion:

$$\iota : \mathcal{LC} \to S^4_2(1) \subset \mathbb{E}^5_2 \ :\ \boldsymbol{x} \mapsto (\boldsymbol{x}, 1)\,. \tag{6.5}$$

Since M admits a light-like normal vector in S^4_2, each normal space is a Lorentzian plane. Hence, the surface M is Lorentzian.

Let $\hat{L} = \iota \circ L : M \to \mathcal{LC} \to S^4_2(1) \subset \mathbb{E}^5_2$ be the composition of L and ι. Put

$$e_3 = \hat{L} - (0, 0, 0, 0, 1)\,. \tag{6.6}$$

Then e_3 is a light-like normal vector in $S^4_2(1)$. Let e_4 be another light-like normal vector of M in $S^4_2(1)$ with $\langle e_3, e_4\rangle = -1$. It follows from (2.13) and (6.6) that

$$A_{e_3} = -I\,, \quad De_3 = De_4 = 0 \tag{6.7}$$

Let e_1, e_2 be a frame on M with $\langle e_1, e_1\rangle = -1$, $\langle e_2, e_2\rangle = 1$ and $\langle e_1, e_2\rangle = 0$ as in Lemma 4.1. Then it follows from Lemma 4.1 that there are three cases to consider.

Case (a): A_{e_4} *takes the form* (4.1). In this case, we obtain

$$h(e_1, e_1) = \alpha e_3 - e_4\,, \quad h(e_1, e_2) = 0\,, \quad h(e_2, e_2) = -\beta e_3 + e_4\,, \tag{6.8}$$

which implies $K = 1 + \alpha + \beta$. Since M is a parallel surface in $S^4_2(1)$, (6.8) implies that $d\alpha = d\beta = (\alpha - \beta)\omega_1^2 = 0$.

Case (a.i): $\alpha = \beta$. We have $A_{e_3} = -I$, $A_{e_4} = \alpha I$ and $K = 1 + 2\alpha$.

Case (a.i.1): $2\alpha > -1$. In this case, we shall choose coordinates (u, v) on M with $g = -du^2 + \cosh^2(\sqrt{1+2\alpha}u)dv^2$. Then the immersion $\hat{L}$ of M in $\mathbb{E}^5_2$ satisfies

$$\begin{gathered}
\hat{L}_{uu} = \alpha e_3 - e_4 + \hat{L}\,, \quad \hat{L}_{uv} = \sqrt{1+2\alpha}\tanh(\sqrt{1+2\alpha}u)\hat{L}_v\,,\\
\hat{L}_{vv} = \frac{1}{2}\sqrt{1+2\alpha}\sinh(2\sqrt{1+2\alpha}u)\hat{L}_u + \cosh^2(\sqrt{1+2\alpha}u)(e_4 - \alpha e_3 - \hat{L})\,,\\
\tilde{\nabla}_{\partial_u} e_3 = \hat{L}_u\,, \quad \tilde{\nabla}_{\partial_v} e_3 = \hat{L}_v\,, \quad \tilde{\nabla}_{\partial_u} e_4 = -\alpha\hat{L}_u\,, \quad \tilde{\nabla}_{\partial_v} e_4 = -\alpha\hat{L}_v\,.
\end{gathered}$$

After solving this system, choosing suitable Minkowskian coordinates and making a suitable reparametrization, we obtain case (1) of the theorem for L.

Case (a.i.2): $2\alpha < -1$. Similarly, we obtain case (2).

Case (a.i.3): $2\alpha = -1$. We choose coordinates (u, v) with $g = du^2 + dv^2$. Then the immersion $\hat{L}$ of M in $\mathbb{E}^5_2$ satisfies

$$\begin{gathered}
\hat{L}_{uu} = -\frac{e_3}{2} - e_4 + \hat{L}\,, \quad \hat{L}_{uv} = 0\,, \quad \hat{L}_{vv} = \frac{e_3}{2} + e_4 - \hat{L}\,,\\
\tilde{\nabla}_{\partial_u} e_3 = \hat{L}_u\,, \quad \tilde{\nabla}_{\partial_v} e_3 = \hat{L}_v\,, \quad \tilde{\nabla}_{\partial_u} e_4 = \frac{1}{2}\hat{L}_u\,, \quad \tilde{\nabla}_{\partial_v} e_4 = \frac{1}{2}\hat{L}_v\,.
\end{gathered}$$

Solving this system leads, after a reparametrization, to case (3) of the theorem.

Case (a.ii): $\alpha \neq \beta$. We get $\omega_1^2 = 0$. So, M is flat and $\beta = -1 - \alpha$. Since $\alpha \neq \beta$, we find $\alpha \neq -1/2$. If we choose coordinates (u, v) with $\partial_u = e_1$, $\partial_v = e_2$, we get

$$\begin{gathered}
\hat{L}_{uu} = \alpha e_3 - e_4 + \hat{L}\,, \quad \hat{L}_{uv} = 0\,, \quad \hat{L}_{vv} = (1+\alpha)e_3 + e_4 - \hat{L}\,,\\
\tilde{\nabla}_{\partial_u} e_3 = \hat{L}_u\,, \quad \tilde{\nabla}_{\partial_v} e_3 = \hat{L}_v\,, \quad \tilde{\nabla}_{\partial_u} e_4 = -\alpha\hat{L}_u\,, \quad \tilde{\nabla}_{\partial_v} e_4 = (1+\alpha)\hat{L}_v\,.
\end{gathered}$$

Solving this system leads, after a reparametrization, to case (4) or case (5) depending on whether $1 + 2\alpha > 0$ or $1 + 2\alpha < 0$.

Case (b): A_{e_4} *takes the form* (4.2). In this case, we have

$$h(e_1, e_1) = \alpha e_3 - e_4\,, \quad h(e_1, e_2) = \beta e_3\,, \quad h(e_2, e_2) = -\alpha e_3 + e_4\,. \tag{6.9}$$

Since M is a parallel surface, (6.9) yields $d\alpha = d\beta = \omega_1^2 = 0$. Thus, we have $0 = K = 1+2\alpha$ and hence $\alpha = -1/2$.

If we choose coordinates (x, y) with $\partial_x = e_1$, $\partial_y = e_2$, we obtain

$$\hat{L}_{xx} = \frac{1}{2}(\hat{L} + c_0) - e_4\,, \quad \hat{L}_{xy} = \beta(\hat{L} - c_0)\,, \quad \hat{L}_{yy} = e_4 - \frac{1}{2}(\hat{L} + c_0)\,,$$
$$\tilde{\nabla}_{\partial_x} e_4 = \frac{1}{2}\hat{L}_x + \beta \hat{L}_y\,, \quad \tilde{\nabla}_{\partial_y} e_4 = -\beta\hat{L}_x + \frac{1}{2}\hat{L}_y\,, \quad c_0 = (0, 0, 0, 0, 1)\,.$$

If $\beta = k^2 > 0$, then after solving this system, we obtain

$$\hat{L} = c_0 + \cosh u(c_1 \cos v + c_2 \sin v) + \sinh u(c_3 \cos v + c_4 \sin v)$$

with $c_1, c_2, c_3, c_4 \in \mathbb{E}^5_2$ and $u = k(x + y)/\sqrt{2}$, $v = k(x - y)/\sqrt{2}$. After choosing suitable Minkowskian coordinates we obtain case (6) of the theorem.

Similarly, if $\beta = -k^2 < 0$, we also obtain case (6) of the theorem.

Case (c): *A_{e_4} takes one of the forms* (4.3).

Case (c.1): *A_{e_4} takes the first form of* (4.3). In this case, we have

$$h(e_1, e_1) = \alpha e_3 - e_4\,, \quad h(e_1, e_2) = e_3\,, \quad h(e_2, e_2) = -(\alpha + 2)e_3 + e_4\,. \tag{6.10}$$

Because M is a parallel surface, (6.10) yields $d\alpha = \omega_1^2 = 0$. Thus, it follows from (6.10) that $0 = K = 3 + 2\alpha$. Thus, $\alpha = -3/2$.

If we choose coordinates (x, y) with $\partial_x = e_1$, $\partial_y = e_2$, we obtain

$$\hat{L}_{xx} = \frac{3}{2}c_0 - e_4 - \frac{1}{2}\hat{L}\,, \quad \hat{L}_{xy} = \hat{L} - c_0\,, \quad \hat{L}_{yy} = \frac{1}{2}c_0 + e_4 - \frac{3}{2}\hat{L}\,,$$
$$\tilde{\nabla}_{\partial_x} e_4 = \frac{3}{2}\hat{L}_x + \hat{L}_y\,, \quad \tilde{\nabla}_{\partial_y} e_4 = -\hat{L}_x + \frac{1}{2}\hat{L}_y\,,$$

with $c_0 = (0, 0, 0, 0, 1)$. Solving this system gives

$$\hat{L} = c_0 + (c_1 + c_2(x + y))\cos(x - y) + (c_3 + c_4(x + y))\sin(x - y)\,,$$

with $c_1, c_2, c_3, c_4 \in \mathbb{E}^5_2$. After choosing suitable Minkowskian coordinates and a reparametrization, we get case (7).

Case (c.2): *A_{e_4} takes the second form of* (4.3). In this case, we have

$$h(e_1, e_1) = \alpha e_3 - e_4\,, \quad h(e_1, e_2) = e_3\,, \quad h(e_2, e_2) = (2 - \alpha)e_3 + e_4\,. \tag{6.11}$$

Because M is a parallel surface, (6.11) yields $d\alpha = \omega_1^2 = 0$. Thus, it follows from (6.11) that $0 = K = 2\alpha - 1$. Thus, $\alpha = 1/2$.

If we choose coordinates (x, y) with $\partial_x = e_1$, $\partial_y = e_2$, we obtain

$$\hat{L}_{xx} = \frac{3}{2}\hat{L} - \frac{1}{2}c_0 - e_4\,, \quad \hat{L}_{xy} = \hat{L} - c_0\,, \quad \hat{L}_{yy} = \frac{1}{2}\hat{L} - \frac{3}{2}c_0 + e_4\,,$$
$$\tilde{\nabla}_{\partial_x} e_4 = -\frac{1}{2}\hat{L}_x + \hat{L}_y\,, \quad \tilde{\nabla}_{\partial_y} e_4 = -\hat{L}_x + \frac{3}{2}\hat{L}_y\,, \quad c_0 = (0, 0, 0, 0, 1)\,.$$

Solving this system leads, after a reparametrization, to case (8). □

7. Classification of space-like parallel surfaces in $L^4_1(c)$. Now, we classify space-like parallel surfaces in 4-dimensional Lorentzian space forms.

LEMMA 7.1. *Suppose that M is a space-like surface in a Lorentzian space form $L_1^4(c)$. Let $\{e_3, e_4\}$ be a normal frame satisfying $\langle e_3, e_3\rangle = -1$, $\langle e_3, e_4\rangle = 0$, $\langle e_4, e_4\rangle = 1$ and let $\{e_1, e_2\}$ be an orthonormal tangent frame with respect to which the shape operators are given by*

$$A_{e_3} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} \gamma & \delta \\ \delta & \varepsilon \end{pmatrix}.$$

Define $\omega_1^2 = -\omega_2^1$ and $\omega_3^4 = \omega_4^3$ as in (2.11). *We have*

(1) *M is parallel if and only if the following equations hold*:

$$d\alpha = \gamma\omega_3^4, \quad d\beta = \varepsilon\omega_3^4, \quad d\gamma = \alpha\omega_3^4 + 2\delta\omega_1^2,$$

$$d\delta = (\varepsilon - \gamma)\omega_1^2, \quad d\varepsilon = \beta\omega_3^4 - 2\delta\omega_1^2, \quad \delta\omega_3^4 = (\alpha - \beta)\omega_1^2.$$

(2) *The Gaussian curvature of M is given by $K = c - \alpha\beta + \gamma\varepsilon - \delta^2$.*

(3) *The normal curvature of the immersion is given by*

$$K^\perp = \langle R^D(e_1, e_2)e_3, e_4\rangle = \delta(\beta - \alpha).$$

PROOF. The first statement follows by direct computation of the vectors $(\overline{\nabla}_X h)(e_1, e_1)$, $(\overline{\nabla}_X h)(e_1, e_2)$ and $(\overline{\nabla}_X h)(e_2, e_2)$ for an arbitrary vector X. The second and the third statement follow from the equations of Gauss and Ricci, respectively. □

Now, we classify parallel space-like surfaces in $\mathbb{E}_1^4$.

THEOREM 7.1. *If M is a space-like parallel surface in $\mathbb{E}_1^4$, then M is congruent to an open part of one of the following nine types of surfaces*:

(1) *the totally geodesic plane $\mathbb{E}^2$ given by $L = (0, u, v, 0)$;*

(2) *a totally umbilical sphere S^2 given locally by*

$$L = a(0, \cos u \cos v, \cos u \sin v, \sin u), \quad a > 0;$$

(3) *a flat cylinder $\mathbb{E}^1 \times S^1$ given by*

$$L = (0, u, a\cos v, a\sin v), \quad a > 0;$$

(4) *a flat cylinder $H^1 \times \mathbb{E}^1$ given by*

$$L(u, v) = (a\cosh u, a\sinh u, v, 0), a > 0;$$

(5) *a flat surface $H^1 \times S^1$ given by*

$$L = (a\cosh u, a\sinh u, b\cos v, b\sin v), \quad a, b > 0;$$

(6) *a totally umbilical hyperbolic plane H^2 given by*

$$L = a(\cosh u \cosh v, \cosh u \sinh v, \sinh u, 0), \quad a > 0;$$

(7) *the minimal flat surface given by*

$$L = (u^2 - v^2, u^2 - v^2, u, v);$$

(8) *the flat totally umbilical surface defined by*

$$L = \left(u^2 + v^2 + \frac{1}{4}, u^2 + v^2 - \frac{1}{4}, u, v\right);$$

(9) *the flat surface given by*

$$L = \frac{1}{2}((1-b)u^2 + (1+b)v^2, (1-b)u^2 + (1+b)v^2, 2u, 2v), \quad b \in R.$$

Conversely, each surface of the nine types given above is space-like and parallel.

PROOF. Since M is a parallel surface, we have $DH = 0$. Hence, Theorem 3.1 implies that M is a marginally trapped surface or one of the following non-marginally trapped surfaces:

(a) a parallel minimal surface;
(b) a parallel surface lying in the light cone $\mathcal{LC} \subset \mathbb{E}^4_1$;
(c) a parallel surface in a 3-dimensional Euclidean space $\mathbb{E}^3 \subset \mathbb{E}^4_1$;
(d) a parallel surface in a 3-dimensional Minkowski space-time $\mathbb{E}^3_1 \subset \mathbb{E}^4_1$;
(e) a parallel surface in a 3-dimensional de Sitter space-time in $\mathbb{E}^4_1$;
(f) a parallel surface in a 3-dimensional hyperbolic space in $\mathbb{E}^4_1$.

If M is marginally trapped, then it is congruent to case (5) with $a = b$ or to case (9) according to Theorem 3.1.

Now, assume that M is non-marginally trapped.

Case (a): *M is minimal and parallel.* Let $\{e_3, e_4\}$ be a normal frame such that $\langle e_3, e_3\rangle = -1$, $\langle e_3, e_4\rangle = 0$ and $\langle e_4, e_4\rangle = 1$. Choose an orthonormal tangent frame $\{e_1, e_2\}$ such that A_{e_3} is diagonal. Hence, with respect to the basis $\{e_1, e_2\}$, we have

$$A_{e_3} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \quad A_{e_4} = \begin{pmatrix} \gamma & \delta \\ \delta & -\gamma \end{pmatrix} \tag{7.1}$$

for some functions α, γ and δ. Thus, we obtain

$$h(e_1, e_1) = -h(e_2, e_2) = -\alpha e_3 + \gamma e_4, \quad h(e_1, e_2) = \delta e_4. \tag{7.2}$$

From Lemma 7.1 we know that $\overline{\nabla} h = 0$ if and only if the following equalities hold:

$$d\alpha = \gamma\omega_3^4, \quad d\gamma = \alpha\omega_3^4 + 2\delta\omega_1^2, \quad d\delta = -2\gamma\omega_1^2, \quad \delta\omega_3^4 = 2\alpha\omega_1^2. \tag{7.3}$$

Case (a.1): $\alpha = 0$. We have $A_{e_3} = 0$ and $\beta\omega_3^4 = \gamma\omega_3^4 = 0$.

If $\beta = \gamma = 0$, then M is totally geodesic. So, it is congruent to case (1).

If β, γ are not both zero, then we have $\omega_3^4 = 0$. So, the equation of Weingarten implies that e_3 is parallel in $\mathbb{E}^4_1$ and hence, if $L : M \to \mathbb{E}^4_1$ denotes the immersion, then $\langle L, e_3\rangle$ is constant. Thus, M lies in a Euclidean 3-space $\mathbb{E}^3 \subset \mathbb{E}^4_1$. Since M is parallel and minimal in $\mathbb{E}^4_1$, M must be parallel and minimal in $\mathbb{E}^3$. Hence, M is totally geodesic in $\mathbb{E}^3$, which gives case (1).

Case (a.2): $\alpha \neq 0$. In this case, (7.3) can be written as

$$d\alpha = \gamma\omega_3^4, \quad d\gamma = \frac{\alpha^2 + \delta^2}{\alpha}\omega_3^4, \quad d\delta = -\frac{\gamma\delta}{\alpha}\omega_3^4, \quad \omega_1^2 = \frac{\delta}{2\alpha}\omega_3^4. \tag{7.4}$$

Differentiating both sides of the last equation in (7.4) and using the others give

$$\delta d\omega_3^4 = 2\alpha d\omega_1^2\,, \tag{7.5}$$

which implies $\delta K^{\perp} = -2\alpha K$. Thus, from Lemma 7.1 we find

$$\alpha^2 - \gamma^2 - 2\delta^2 = 0\,. \tag{7.6}$$

Combining the equations in (7.4) yields that

$$\alpha\delta = C_1\,, \quad \alpha^2 - \gamma^2 - \delta^2 = C_2 \tag{7.7}$$

are both constant. From (7.6) and (7.7) we obtain $\delta^2 = C_2$, which is constant.

Case (a.2.i): $\delta = 0$. From (7.6) we get $\alpha^2 = \gamma^2$. We may assume $\alpha = \gamma \neq 0$ by changing the orientation of e_4 if necessary. From (7.4) we have $\omega_1^2 = 0$. Thus, there exist local coordinates (u, v) on M with $\partial_u = e_1, \partial_v = e_2$. Then

$$\begin{gathered}
L_{uu} = \alpha(e_4 - e_3)\,, \quad L_{uv} = 0\,, \quad L_{vv} = 2\alpha(e_3 - e_4)\,, \\
\tilde{\nabla}_{\partial_u} e_3 = -\alpha L_u + \frac{\alpha_u}{\alpha} e_4\,, \quad \tilde{\nabla}_{\partial_v} e_3 = \alpha L_v + \frac{\alpha_v}{\alpha} e_4\,, \\
\tilde{\nabla}_{\partial_u} e_4 = -\alpha L_u + \frac{\alpha_u}{\alpha} e_3\,, \quad \tilde{\nabla}_{\partial_v} e_4 = \alpha L_v + \frac{\alpha_v}{\alpha} e_3\,.
\end{gathered}$$

After solving this system, choosing suitable Minkowskian coordinates and making a suitable reparametrization, we obtain case (7), which is a flat and minimal surface, that is not totally geodesic.

Case (a.2.ii): $\delta \neq 0$. Since δ is constant, (7.7) implies that α and γ are constant. Using (7.4)–(7.7), we get $\omega_1^2 = 0$. So, the surface is flat and thus Lemma 7.1 yields $\alpha^2 - \gamma^2 - \delta^2 = 0$. Combining this with (7.6) gives $\delta = 0$, which is a contradiction.

Case (b): *M is a parallel surface in $\mathcal{LC}$.* In this case, Theorem 6.1 implies that M is congruent to cases (2), (5), (6) or (8).

Case (c): *M is a non-minimal parallel surface in $\mathbb{E}^3$.* In this case, M is an open portion of a round sphere or a circular cylinder. So, M is congruent to (2) or (3).

Case (d): *M is a non-minimal parallel surface in $\mathbb{E}^3_1$.* From Theorem 5.1 we know that it is congruent to either case (4) or case (6).

Case (e): *M is a non-minimal parallel surface in a de Sitter space.* By Theorem 5.2, the surface is congruent to cases (2), (5), (6) or (8) .

Case (f): *M is a parallel surface in a hyperbolic space.* We may prove again that either M is totally umbilical in $H^3(-c)$ or M is flat, but not totally umbilical in $H^3(-c)$. In the first case, it is congruent to (2), (6) or (8). In the second case, we proceed as in $S^3_1(c)$ to obtain that M is congruent to (5).

The converse can be easily verified. □

THEOREM 7.2. *If M is a space-like parallel surface in $S^4_1(1) \subset \mathbb{E}^5_1$, then M is congruent to an open part of one of the following ten types of surfaces*:

(1) *a totally umbilical sphere S^2 given locally by*

$$L = (c, b\cos u\cos v, b\cos u\sin v, b\sin u, a)\,, \quad a^2 + b^2 - c^2 = 1\,;$$

(2) *a totally umbilical hyperbolic plane H^2 given by*

$$L = (a\cosh u\cosh v, a\cosh u\sinh v, a\sinh u, b, c)\,, \quad b^2+c^2-a^2=1\,;$$

(3) *a torus $S^1\times S^1$ given by*

$$L = (a, b\cos u, b\sin u, c\cos v, c\sin v)\,, \quad b^2+c^2-a^2=1\,;$$

(4) *a flat surface $H^1\times S^1$ given by*

$$L = (b\cosh u, b\sinh u, c\cos v, c\sin v, a)\,, \quad a^2+c^2-b^2=1\,;$$

(5) *a flat totally umbilical surface defined by*

$$L = \left(u^2+v^2+a^2+\frac{1}{4}, u^2+v^2+a^2-\frac{1}{4}, u, v, \sqrt{1+a^2}\right);$$

(6) *a flat surface defined by*

$$L = \left(v^2-\frac{3}{4}+a^2, a\cos u, a\sin u, v, v^2-\frac{5}{4}+a^2\right), \quad a>0\,;$$

(7) *a flat surface defined by*

$$L = \frac{1}{\sqrt{1+a^2}}\left(u^2+v^2-\frac{3}{4}, u^2+v^2-\frac{5}{4}, u, v, a\right), \quad a\in\boldsymbol{R}\,;$$

(8) *the flat marginally trapped surface defined by*

$$L = \frac{1}{2}(2u^2-1, 2u^2-2, 2u, \sin v, \cos v)\,;$$

(9) *a flat marginally trapped surface defined by*

$$L = \left(\frac{b}{\sqrt{4-b^2}}, \frac{\cos u}{\sqrt{2-b}}, \frac{\sin u}{\sqrt{2-b}}, \frac{\cos v}{\sqrt{2+b}}, \frac{\sin v}{\sqrt{2+b}}\right), \quad |b|<2\,;$$

(10) *a flat marginally trapped surface defined by*

$$L = \left(\frac{\cosh u}{\sqrt{b-2}}, \frac{\sinh u}{\sqrt{b-2}}, \frac{\cos v}{\sqrt{2+b}}, \frac{\sin v}{\sqrt{2+b}}, \frac{b}{\sqrt{b^2-4}}\right), \quad b>2\,.$$

Conversely, each surface of the ten types given above is space-like and parallel.

PROOF. Let M be a parallel space-like surface in $S^4_1(1)\subset\mathbb{E}^5_1$. Then $DH=0$ and so, Theorem 3.2 implies that M is either a marginally trapped surface given by cases (8)–(10) or (7) with $a=1$, or M is non-marginally trapped given by one of the following:

(a) M is a parallel minimal surface of $S^4_1(1)$;

(b) M is a parallel surface in $S^4_1(1)\cap\mathcal{E}$, where $\mathcal{E}$ is a space-like hyperplane;

(c) M is a parallel surface in $S^4_1(1)\cap\mathcal{E}_1$, where $\mathcal{E}_1$ is a Lorentzian hyperplane;

(d) M is a parallel surface in $S^4_1(1)\cap\mathcal{H}$, where $\mathcal{H}$ is a degenerate hyperplane.

Now, assume that M is non-marginally trapped.

Case (a): *M is a minimal parallel surface of $S^4_1(1)$.* As in the proof of Theorem 7.1, we choose $\{e_3, e_4\}$ as in Lemma 7.1 and let $\{e_1, e_2\}$ be an orthonormal tangent frame such that (7.1) and (7.2) hold. Thus, from Lemma 7.1, we have (7.3).

Case (a.1): $\alpha = 0$. We choose $\{e_1, e_2\}$ so that $\delta = 0$. Thus, (7.1) and (7.3) give

$$A_{e_3} = 0\,, \quad A_{e_4} = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}, \quad d\gamma = 0\,, \quad \gamma\omega_1^2 = 0\,. \tag{7.8}$$

If $\gamma = 0$, the surface is totally geodesic, which gives a special case of (1).

If $\gamma \neq 0$, then (7.8) gives $\omega_1^2 = 0$. Hence, M is flat. Lemma 7.1 then yields that $K = 1 - \gamma^2 = 0$. After replacing e_4 by $-e_4$ if necessary, we have $\gamma = 1$. We can choose Euclidean coordinates (u, v) with $e_1 = \partial_u$ and $e_2 = \partial_v$. Then we have

$$L_{uu} = e_4 - L\,, \quad L_{uv} = 0\,, \quad L_{vv} = -e_4 - L\,, \quad \tilde{\nabla}_{\partial_u} e_4 = -L_u\,, \quad \tilde{\nabla}_{\partial_v} e_4 = -L_v\,.$$

After solving this system, choosing suitable Minkowskian coordinates, and making a suitable reparametrization, we obtain a special case of case (3).

Case (a.2): $\alpha \neq 0$. From (7.3), we have (7.4), which implies that

$$\alpha\delta = C_1\,, \quad \alpha^2 - \gamma^2 - \delta^2 = C_2 \tag{7.9}$$

are both constant. Applying the de Rham operator to both sides of the last equation of (7.4) gives $-2\alpha K = \delta K^{\perp}$. Thus, from Lemma 7.1, we obtain

$$1 + \alpha^2 - \gamma^2 - 2\delta^2 = 0\,. \tag{7.10}$$

Combining (7.9) and (7.10) shows that δ is constant. A direct computation using (7.4), (7.10) and the expression for K in Lemma 7.1 yields $\omega_1^2 = \delta = 0$. So, (7.10) gives $\gamma^2 = 1 + \alpha^2$. After choosing coordinates (u, v) as in case (A.1), we obtain

$$\begin{gathered} L_{uu} = -\alpha e_3 + \gamma e_4 - L\,, \quad L_{uv} = 0\,, \quad L_{vv} = \alpha e_3 - \gamma e_4 - L\,, \\ \tilde{\nabla}_{\partial_u} e_3 = -\alpha L_u + \frac{\alpha_u}{\gamma} e_4\,, \quad \tilde{\nabla}_{\partial_v} e_3 = \alpha L_v + \frac{\alpha_v}{\gamma} e_4\,, \\ \tilde{\nabla}_{\partial_u} e_4 = -\gamma L_u + \frac{\alpha_u}{\gamma} e_3\,, \quad \tilde{\nabla}_{\partial_v} e_4 = \gamma L_v + \frac{\alpha_v}{\gamma} e_3\,. \end{gathered}$$

After solving this system, we obtain the same solution as in case (a.1). Thus, we obtain a special case of (3) again.

Now, let us assume that M is non-minimal in $S_1^4(1)$.

Case (b): *M is a parallel surface in $S_1^4(1) \cap \mathcal{E}$, where $\mathcal{E}$ is a space-like hyperplane.* We may assume that $\mathcal{E}$ is defined by $x_1 = a$. Then $S_1^4(1) \cap \mathcal{E} = S^3(c)$ with $c = 1/(1 + a^2)$. Since M is a parallel surface of $S_1^4(1)$, M is also a parallel surface in $S^3(c) = \{(a, x_2, x_3, x_4, x_5) \in \mathbb{E}_1^5;\, x_2^2 + \cdots + x_5^2 = 1 + a^2\}$. Thus, M is either a totally umbilical surface or a torus in $S^3(c)$. In the first case, we obtain case (1). In the second case, we obtain case (3).

Case (c): *M is a parallel surface in $S_1^4(1) \cap \mathcal{E}_1$, where $\mathcal{E}_1$ is a Lorentzian hyperplane.* We may assume that $\mathcal{E}_1$ is defined by $x_5 = a$ with $a \geq 0$. There are three possibilities for $S_1^4(1) \cap \mathcal{E}_1$; namely,

(1) if $a \in [0, 1)$, then $S_1^4(1) \cap \mathcal{E}_1 = S_1^3(c)$, $c = 1/(1 - a^2)$;
(2) if $a > 1$, then $S_1^4(1) \cap \mathcal{E}_1 = H^3(-c)$, $c = 1/(1 - a^2)$, and
(3) if $a = 1$, then $S_1^4(1) \cap \mathcal{E}_1$ is the light cone in $\mathcal{E}_1 = \mathbb{E}_1^4$.

Case (c.1): $a \in [0, 1)$. In this case, M is a parallel surface in $S^3_1(c)$. Thus, Theorem 5.2 implies that M is congruent to (7) or special cases of (1), (2) and (4).

Case (c.2): $a > 1$. In this case, M is either totally umbilical in $H^3(-c)$ or flat. In the first case, we obtain special cases of (1), (2) and (5). The second case gives a special case of (4).

Case (c.3): $a = 1$. In this case, $S^4_1(1) \cap \mathcal{E}_1$ is the light cone $\mathcal{LC}$ in $\mathcal{E}_1 = \mathbb{E}^4_1$. Hence, Theorem 6.1 implies that M is congruent to special cases of (1), (2), (4) or (5).

Case (d): *M is a parallel surface in $S^4_1(1) \cap \mathcal{H}$, where $\mathcal{H}$ is a degenerate hyperplane.* We may assume that $\mathcal{H}$ is the hyperplane $\mathcal{K}_a = \{(x_1, \ldots, x_5) \in \mathbb{E}^5_1;\, x_5 = x_1 + a\}$.

Case (d.1): $a = 0$. Since M lies in $\mathcal{K}_0$, we have $x_5 = x_1$. Then $e_3 = (1, 0, 0, 0, 1)$ is a light-like normal vector of M in $S^4_1(1)$. Let e_4 be a light-like normal vector field with $\langle e_3, e_4\rangle = -1$. Then there is an orthonormal frame field $\{e_1, e_2\}$ such that

$$A_{e_3} = 0\,, \quad A_{e_4} = \begin{pmatrix} \gamma & 0 \\ 0 & \varepsilon \end{pmatrix}, \tag{7.11}$$
$$h(e_1, e_1) = -\gamma e_3\,, \quad h(e_1, e_2) = 0\,, \quad h(e_2, e_2) = -\varepsilon e_3\,.$$

On the other hand, it follows from (7.11) and the equation of Gauss that $K = 1$. Thus, $\omega^2_1 \neq 0$. Now (7.11) and $\overline{\nabla} h = 0$ imply that $\gamma = \varepsilon$ is a constant, say r. Thus, if we choose coordinates (u, v) with $g = du^2 + \cos^2 u dv^2$, then we obtain

$$L_{uu} = -re_3 - L\,, \;\; L_{uv} = -\tan u L_v\,, \;\; L_{vv} = \frac{L_u}{2}\sin 2u - \cos^2 u(re_3 + L)\,, \;\; \tilde{\nabla} e_3 = 0\,.$$

Solving this system leads, after a reparametrization, to special case of case (1).

Case (d.2): $a \neq 0$. Put $e_3 = L - a^{-1}(1, 0, 0, 0, 1)$. Then we have $\langle e_3, L\rangle = 0$ and $\langle e_3, e_3\rangle = -1$. Thus, e_3 is a time-like unit normal vector field of M in $S^4_1(1)$ with $De_3 = 0$. Let e_4 be a unit space-like normal vector field in $S^4_1(1)$ with $\langle e_3, e_4\rangle = 0$. With respect to a suitable orthonormal tangent frame $\{e_1, e_2\}$, we have

$$A_{e_3} = -I\,, \quad A_{e_4} = \begin{pmatrix} \gamma & 0 \\ 0 & \varepsilon \end{pmatrix}.$$

Since $De_3 = 0$, $\overline{\nabla} h = 0$ implies that $d\gamma = d\varepsilon = (\varepsilon - \gamma)\omega^2_1 = 0$.

Case (d.2.i): $\varepsilon = \gamma = 0$. In this case, M is totally umbilical with $K = 0$. Thus, if we choose coordinates (u, v) with $g = du^2 + dv^2$, then we obtain

$$L_{uu} = e_3 - L\,, \quad L_{uv} = 0\,, \quad L_{vv} = e_3 - L\,, \quad \tilde{\nabla}_{\partial_u} e_3 = L_u\,, \quad \tilde{\nabla}_{\partial_v} e_3 = L_v\,.$$

Solving this system gives case (7) with $a = 1$.

Case (d.2.ii): $\varepsilon = \gamma = r \neq 0$. In this case, M is totally umbilical with $K = \gamma^2$. So, if we choose coordinates (x, y) with $g = dx^2 + \cos^2(rx)dy^2$, then we obtain

$$L_{xx} = e_3 + re_4 - L\,, \quad L_{xy} = -r\tan(rx)L_y\,,$$
$$L_{yy} = r\sin(rx)\cos(rx)L_x + \cos^2(rx)(e_3 + re_4 - L)\,,$$
$$\tilde{\nabla}_{\partial_x} e_3 = L_x\,, \quad \tilde{\nabla}_{\partial_y} e_3 = L_y\,, \quad \tilde{\nabla}_{\partial_x} e_4 = -rL_x\,, \quad \tilde{\nabla}_{\partial_y} e_4 = -rL_y\,.$$

After solving this system, choosing suitable Minkowskian coordinates and making a suitable reparametrization, we obtain a special case of (1).

Case (d.2.iii): $\varepsilon \neq \gamma$. In this case, we have $\omega_1^2 = 0$. Thus, M is flat and we get $K = \gamma\varepsilon = 0$. Without loss of generality, we may assume $\varepsilon = 0$, $\gamma \neq 0$. So, after choosing Euclidean coordinates (u, v) with $e_1 = \partial_u$, $e_2 = \partial_v$, we obtain

$$L_{uu} = e_3 + \gamma e_4 - L\,, \quad L_{uv} = 0\,, \quad L_{vv} = e_3 - L\,,$$
$$\tilde{\nabla}_{\partial_u} e_3 = L_u\,, \quad \tilde{\nabla}_{\partial_v} e_3 = L_v\,, \quad \tilde{\nabla}_{\partial_u} e_4 = -\gamma L_u\,, \quad \tilde{\nabla}_{\partial_v} e_4 = 0\,.$$

Solving this system leads, after a reparametrization, to case (6) of the theorem.

The converse can be verified by a straightforward computation. □

The following result completely classifies space-like parallel surface in $H_1^4(-1)$.

THEOREM 7.3. *If M is a space-like parallel surface in $H_1^4(-1) \subset \mathbb{E}_2^5$, then M is congruent to an open part of one of the following ten types of surfaces*:

(1) *a totally umbilical sphere S^2 given locally by*

$$L = (a, c, b\sin u, b\cos u\cos v, b\cos u\sin v)\,, \qquad a^2 - b^2 + c^2 = 1\,;$$

(2) *a totally umbilical hyperbolic plane H^2 given locally by*

$$L = (a, b\cosh u\cosh v, b\cosh u\sinh v, b\sinh u, c)\,, \qquad a^2 + b^2 - c^2 = 1\,;$$

(3) *a flat surface $H^1 \times S^1$ given by*

$$L = (a, b\cosh u, b\sinh u, c\cos v, c\sin v)\,, \qquad a^2 + b^2 - c^2 = 1\,;$$

(4) *a flat surface $H^1 \times H^1$ given by*

$$L = (b\cosh u, c\cosh v, b\sinh u, c\sinh v, a)\,, \qquad b^2 + c^2 - a^2 = 1\,;$$

(5) *a flat totally umbilical surface defined by*

$$L = \left(\sqrt{1-a^2}, u^2 + v^2 + a^2 + \frac{1}{4}, u^2 + v^2 + a^2 - \frac{1}{4}, u, v\right);$$

(6) *a flat surface defined by*

$$L = \left(a, b\left(u^2 + v^2 - \frac{3}{4}\right), b\left(u^2 + v^2 - \frac{5}{4}\right), bu, bv\right), \qquad a^2 = 1 + b^2 > 1\,;$$

(7) *a flat surface defined by*

$$L = \left(v^2 + \frac{5}{4} - a^2, a\cosh u, a\sinh u, v, v^2 + \frac{3}{4} - a^2\right), \qquad a \neq 0\,;$$

(8) *the flat marginally trapped surface defined by*

$$L = \left(u^2 + 1, \frac{1}{2}\cosh v, u, \frac{1}{2}\sinh v, u^2 + \frac{1}{2}\right);$$

(9) *a flat marginally trapped surface defined by*

$$L = \left(\frac{\cosh u}{\sqrt{2-b}}, \frac{\cosh v}{\sqrt{2+b}}, \frac{\sinh u}{\sqrt{2-b}}, \frac{\sinh v}{\sqrt{2+b}}, \frac{b}{\sqrt{4-b^2}}\right), \qquad |b| < 2\,;$$

(10) *a flat marginally trapped surface defined by*

$$L = \left(\frac{b}{\sqrt{b^2-4}}, \frac{\cosh v}{\sqrt{b+2}}, \frac{\sinh v}{\sqrt{b+2}}, \frac{\cos u}{\sqrt{b-2}}, \frac{\sin u}{\sqrt{b-2}} \right), \quad b > 2\,.$$

Conversely, each surface of the ten types given above is space-like and parallel.

PROOF. Let M be a parallel space-like surface in $H^4_1(-1)$. Then $DH = 0$ and according to Theorem 3.3, M is either a marginally trapped surface given by cases (8)–(10) or a special case of (2), or M is non-marginally trapped and one of the following:

(a) M is a parallel minimal surface of $H^4_1(-1)$;
(b) M is a parallel surface in $H^4_1(-1) \cap \mathcal{E}_1$, where $\mathcal{E}_1$ is a Lorentzian hyperplane;
(c) M is a parallel surface in $H^4_1(-1) \cap \mathcal{E}_2$, where $\mathcal{E}_2$ is an index 2 hyperplane;
(d) M is a parallel surface in $H^4_1(-1) \cap \mathcal{H}$, where $\mathcal{H}$ is a degenerate hyperplane.

Now, assume that M is non-marginally trapped.

Case (a): *M is a minimal parallel surface of $H^4_1(-1)$.* As in the proof of Theorem 7.1, we may choose $\{e_3, e_4\}$ as in Lemma 7.1 and let $\{e_1, e_2\}$ be an orthonormal tangent frame such that (7.1) and (7.2) hold. Thus, from Lemma 7.1, we have (7.3).

Case (a.1): $\alpha = 0$. Choose $\{e_1, e_2\}$ such that $\delta = 0$. Then (7.1) and (7.3) give

$$A_{e_3} = 0\,, \quad A_{e_4} = \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix}, \quad d\gamma = 0\,, \quad \gamma\omega_1^2 = 0\,. \tag{7.12}$$

If $\gamma = 0$, the surface is totally geodesic, which gives case (2).

If $\gamma \neq 0$, then (7.13) gives $\omega_1^2 = 0$. Hence the surface is flat. Lemma 7.1 then yields that $K = -1 - \gamma^2 = 0$, which is impossible.

Case (a.2): $\alpha \neq 0$. By similar arguments as in case (A.2) of the proof of Theorem 7.2, we obtain $\omega_1^2 = \delta = 0$ and $\gamma^2 = \alpha^2 - 1$. After choosing Euclidean coordinates (u, v) with $e_1 = \partial_u$, $e_2 = \partial_v$, we have

$$\begin{gathered} L_{uu} = -\alpha e_3 + \gamma e_4 + L\,, \quad L_{uv} = 0\,, \quad L_{vv} = \alpha e_3 - \gamma e_4 + L\,, \\ \tilde{\nabla}_{\partial_u} e_3 = -\alpha L_u + \frac{\alpha_u}{\gamma} e_4\,, \quad \tilde{\nabla}_{\partial_v} e_3 = \alpha L_v + \frac{\alpha_v}{\gamma} e_4\,, \\ \tilde{\nabla}_{\partial_u} e_4 = -\gamma L_u + \frac{\alpha_u}{\gamma} e_3\,, \quad \tilde{\nabla}_{\partial_v} e_4 = \gamma L_v + \frac{\alpha_v}{\gamma} e_3\,. \end{gathered}$$

Solving this system yields case (4) of the theorem with $a = 0$ and $b = c = 1/\sqrt{2}$.

Case (b): *M lies in $H^4_1(-1) \cap \mathcal{E}_1$, $\mathcal{E}_1$ is a Lorentzian hyperplane.* We may assume that $\mathcal{E}_1$ is defined by $x_1 = a$ with $a \geq 0$. Then, $H^4_1(-1) \cap \mathcal{E}_1$ is given by

$$\{(a, x_2, \ldots, x_5) \in \mathbb{E}^5_2\,;\; -x_2^2 + x_3^2 + x_4^2 + x_5^2 = a^2 - 1\}\,.$$

Case (b.1): $a^2 > 1$. In this case, $H^4_1(-1) \cap \mathcal{E}_1 = S^3_1(c)$ with $c = 1/(a^2 - 1)$. Thus, M is a parallel surface of $S^3_1(c)$. Hence, by applying Theorem 5.2, we obtain special cases of (1), (2), (3) or (6).

Case (b.2): $a^2 < 1$. In this case, $H^4_1(-1) \cap \mathcal{E}_1 = H^3(-c)$ with $c = 1/(1 - a^2)$. So, M is a parallel surface of a hyperbolic 3-space $H^3(-c)$. Hence, M is either totally umbilical in

$H^3(-c)$ or flat. In the first case, we obtain special cases of (1), (2) and (5). The second case gives a special case of (3).

Case (b.3): $a = 1$. In this case, $H_1^4(-1) \cap \mathcal{E}_1 = \mathcal{LC} \subset \mathbb{E}_1^4 \,(= \mathcal{E}_1)$. Thus, M is a parallel surface lying in the light cone $\mathcal{LC} \subset \mathbb{E}_1^4$. Hence, by Theorem 6.1, we obtain special cases of (1), (2), (3) and (5).

Case (c): *M is a parallel surface in $H_1^4(-1) \cap \mathcal{E}_2$, where $\mathcal{E}_2$ is a hyperplane of index* 2. We may assume that $\mathcal{E}_2$ is given by $x_5 = a$, $a \geq 0$. Thus, $H_1^4(-1) \cap \mathcal{E}_2 = H_1^3(-c) \subset \mathbb{E}_1^4$ with $c = 1/(1+a^2)$ defined by $-x_1 - x_2^2 + x_3^2 + x_4^2 = -(1+a^2) < 0$. Hence, we may apply Theorem 5.3 to obtain special cases of (2) and (4).

Case (d): *M is a parallel surface in $H_1^4(-1) \cap \mathcal{H}$, $\mathcal{H}$ a degenerate hyperplane.* We may assume that $\mathcal{H}$ is the hyperplane $\mathcal{G}_a = \{(x_1, \ldots, x_5) \in \mathbb{E}_2^5;\, x_5 = x_1 + a\}$.

Case (d.1): $a = 0$. Since M lies in $\mathcal{G}_0$, we have $x_5 = x_1$. Put $e_3 = (1,0,0,0,1)$ and let e_4 be another light-like normal vector field of M in $H_1^4(-1)$ with $\langle e_3, e_4\rangle = -1$, $\langle e_4, e_4\rangle = 0$. With respect to a suitable orthonormal frame field $\{e_1, e_2\}$, we have

$$\begin{aligned} &A_{e_3} = 0\,, \quad A_{e_4} = \begin{pmatrix} \gamma & 0 \\ 0 & \varepsilon \end{pmatrix}, \\ &h(e_1,e_1) = -\gamma e_3\,, \quad h(e_1,e_2) = 0\,, \quad h(e_2,e_2) = -\varepsilon e_3\,. \end{aligned} \tag{7.13}$$

From the equation of Gauss and (7.13), we get $K = -1$. Thus, $\omega_1^2 \neq 0$. Since $De_3 = 0$, it follows from $\overline{\nabla} h = 0$ that $\gamma = \varepsilon$ which is constant, say r.

If we choose coordinates (u, v) with $g = du^2 + \cosh^2 u\, dv^2$, then we obtain

$$L_{uu} = L - re_3\,, \quad L_{uv} = \tanh u L_v\,, \quad L_{vv} = \cosh^2 u(L - re_3) - \frac{L_x}{2}\sinh 2u\,, \quad \tilde{\nabla} e_3 = 0\,.$$

Solving this system leads, after a reparametrization, to a special case of case (2).

Case (d.2): $a \neq 0$. Put $e_3 = L + a^{-1}(1,0,0,0,1)$. Then, by applying $\langle L, L\rangle = -1$, we find $\langle e_3, L\rangle = 0$ and $\langle e_3, e_3\rangle = 1$. Thus, e_3 is a space-like unit normal vector field of M in $H_1^4(-1)$ with $De_3 = 0$. Let e_4 be a unit time-like normal vector field in $H_1^4(-1)$ with $\langle e_3, e_4\rangle = 0$. With respect to a suitable orthonormal tangent frame $\{e_1, e_2\}$, we have $A_{e_3} = -I$ and $A_{e_4} = \left(\begin{smallmatrix} \gamma & 0 \\ 0 & \varepsilon \end{smallmatrix}\right)$. Hence, we obtain $K = -\gamma\varepsilon$ and

$$h(e_1,e_1) = -e_3 - \gamma e_4\,, \quad h(e_1,e_2) = 0\,, \quad h(e_2,e_2) = -e_3 - \varepsilon e_4\,. \tag{7.14}$$

From $De_3 = De_4 = 0$, (7.14) and $\overline{\nabla} h = 0$, we find $(\varepsilon - \gamma)\omega_1^2 = d\gamma = d\varepsilon = 0$.

Case (d.2.i): $\varepsilon = \gamma = 0$. In this case, M is totally umbilical with $K = 0$. Thus, if we choose coordinates (u, v) with $g = du^2 + dv^2$, then we obtain

$$L_{uu} = -e_3 + L\,, \quad L_{uv} = 0\,, \quad L_{vv} = -e_3 + L\,, \quad \tilde{\nabla}_{\partial_u} e_3 = L_u\,, \quad \tilde{\nabla}_{\partial_v} e_3 = L_v\,.$$

Solving this system leads to a special case of case (5) .

Case (d.2.ii). If $\varepsilon = \gamma = r \neq 0$, then M is totally umbilical with $K = -\gamma^2$. So, if we choose coordinates (x, y) with $g = dx^2 + \cosh^2(\gamma x)dy^2$, then we obtain

$$L_{xx} = -e_3 - re_4 + L\,, \quad L_{xy} = r\tanh(rx)L_y\,,$$
$$L_{yy} = \cosh^2(rx)(L - e_3 - re_4) - \sinh(rx)\cosh(rx)L_x\,,$$
$$\tilde{\nabla}_{\partial_x} e_3 = L_x\,, \quad \tilde{\nabla}_{\partial_y} e_3 = rL_y\,, \quad \tilde{\nabla}_{\partial_x} e_4 = -rL_x\,, \quad \tilde{\nabla}_{\partial_y} e_4 = -rL_y\,.$$

Solving this system leads, after a reparametrization, to a special case of case (2).

Case (d.2.iii): If $\varepsilon \neq \gamma$. In this case, we have $\omega_1^2 = 0$. Thus, M is flat and the equation of Gauss yields $K = -\gamma\varepsilon = 0$. We may assume $\varepsilon = 0$, $\gamma \neq 0$. So, after choosing Euclidean coordinates (u, v) with $e_1 = \partial_u$, $e_2 = \partial_v$, we obtain

$$L_{uu} = -e_3 - \gamma e_4 + L\,, \quad L_{uv} = 0\,, \quad L_{vv} = -e_3 + L\,,$$
$$\tilde{\nabla}_{\partial_u} e_3 = L_u\,, \quad \tilde{\nabla}_{\partial_v} e_3 = L_v\,, \quad \tilde{\nabla}_{\partial_u} e_4 = -\gamma L_u\,, \quad \tilde{\nabla}_{\partial_v} e_4 = 0\,.$$

Solving this system leads, after a reparametrization, to case (7) of the theorem

The converse can be verified by a straightforward computation. □

REMARK 7.1. Cases (8), (9) and (10) of Theorem 7.2 are special cases of Cases (6), (3) and (4), respectively.

REMARK 7.2. Cases (8), (9) and (10) of Theorem 7.3 are special cases of Cases (6), (4) and (3), respectively.

8. Classification of Lorentzian parallel surfaces in $L_1^4(c)$. In this section, we classify Lorentzian parallel surfaces in Lorentzian space forms.

LEMMA 8.1. *Let M be a Lorentzian minimal parallel surface in a Lorentzian space form $L_1^4(c)$. Then M is a Lorentzian minimal parallel surface of a totally geodesic Lorentzian $L_1^3(c) \subset L_1^4(c)$.*

PROOF. Assume that M is a Lorentzian minimal parallel surface in a Lorentzian space form $L_1^4(c)$. Choose an orthonormal normal frame $\{e_3, e_4\}$ such that

$$A_{e_3} = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix}, \qquad A_{e_4} = \begin{pmatrix} \beta & -\gamma \\ \gamma & -\beta \end{pmatrix},$$

with respect to a pseudo-orthonormal tangent frame $\{e_1, e_2\}$, satisfying $\langle e_1, e_1\rangle = -1$, $\langle e_1, e_2\rangle = 0$ and $\langle e_2, e_2\rangle = 1$. We have

$$h(e_1, e_1) = -\alpha e_3\,, \quad h(e_1, e_2) = \gamma e_4\,, \quad h(e_2, e_2) = -\alpha e_3 - \beta e_4\,.$$

A straightforward computation shows that the surface is parallel if and only if

$$d\alpha = \beta\omega_3^4\,, \quad d\beta = -2\gamma\omega_1^2 - \alpha\omega_3^4\,, \quad d\gamma = -2\beta\omega_1^2\,, \quad 2\alpha\omega_1^2 = \gamma\omega_3^4\,. \tag{8.1}$$

Differentiating both sides of the last equality of (8.1) gives $2\alpha d\omega_1^2 = \gamma d\omega_3^4$, and combining this with the equations of Gauss and Ricci yields

$$\alpha(\alpha^2 + \beta^2 - 2\gamma^2 - c) = 0\,.$$

Case (a): $\alpha = 0$. From (8.1), we have $\beta\omega_3^4 = \gamma\omega_3^4 = 0$.

Case (a.1): $\omega_3^4 = 0$. In this case, e_3 is parallel in $L_1^4(c)$, and hence we obtain from the reduction theorem that the surface lies in a totally geodesic subspace $L_1^3(c)$.

Case (a.2): $\omega_3^4 \neq 0$. In this case, we have $\beta = \gamma = 0$ and hence the surface is totally geodesic, which implies the result.

Case (b): $\alpha^2 + \beta^2 - 2\gamma^2 = c$. It follows from (8.1) that $\alpha^2 + \beta^2 - \gamma^2$ is constant and hence γ is constant. From the third equation of (8.1), we obtain $\beta\omega_1^2 = 0$.

Case (b.1): $\beta = 0$. From (8.1), we obtain $2\gamma\omega_1^2 + \alpha\omega_3^4 = 0$ and $2\alpha\omega_1^2 - \gamma\omega_3^4 = 0$. Combining these equalities with $\alpha^2 - 2\gamma^2 = c$ yields $(c - 3\alpha^2)\omega_1^2 = 0$. For $\omega_1^2 = 0$, we refer to case (B.2). If $c = 3\alpha^2$, then $c \geq 0$ and it follows from $\alpha^2 - 2\gamma^2 = c$ that $c = -3\gamma^2$. Hence, $c = \alpha = \gamma = 0$ and the surface is totally geodesic.

Case (b.2): $\omega_1^2 = 0$. Since the surface is flat, we have $K = 0$, or, equivalently, $\alpha^2 + \beta^2 - \gamma^2 - c = 0$. Together with $\alpha^2 + \beta^2 - 2\gamma^2 - c = 0$, we obtain $\gamma = 0$ and hence both A_{e_3} and A_{e_4} are diagonal. After a suitable rotation of $\{e_3, e_4\}$ in the normal plane, we may assume that $\alpha = 0$. It follows from the first equality of (8.1) that $\beta\omega_3^4 = 0$. If $\beta = 0$, the surface is totally geodesic. If $\omega_3^4 = 0$, then e_3 is parallel in $L_1^4(c)$ and the result follows form the reduction theorem. $\square$

Making use of Lemma 4.1 and similar techniques as in the previous section, we can prove the following theorems.

THEOREM 8.1. *If M is a Lorentzian parallel surface in $\mathbb{E}_1^4$, then M is congruent to an open part of one of the following six types of surfaces*:

(1) *the totally geodesic plane $\mathbb{E}_1^2$ given by $L = (u, v, 0, 0)$*;

(2) *a totally umbilical de Sitter space S_1^2 given by*

$$L = b(\sinh u, \cosh u \cos v, \cosh u \sin v, 0)\,, \quad b > 0\,;$$

(3) *a flat cylinder $\mathbb{E}_1^1 \times S^1$ given by*

$$L = (u, a\cos v, a\sin v, 0)\,, \quad a > 0\,;$$

(4) *a flat cylinder $S_1^1 \times \mathbb{E}^1$ given by*

$$L = (a\sinh u, a\cosh u, v, 0)\,, \quad a > 0\,;$$

(5) *a flat surface $S_1^1 \times S^1$ defined by*

$$L = (a\sinh u, a\cosh u, b\cos v, b\sin v)\,, \quad a, b > 0\,;$$

(6) *the flat minimal surface in $\mathbb{E}_1^3 \subset \mathbb{E}_1^4$ given by*

$$L = \left(\frac{1}{6}(u-v)^3 + u, \frac{1}{6}(u-v)^3 + v, \frac{1}{2}(u-v)^2, 0\right)\,.$$

Conversely, each surface defined above is a Lorentzian parallel surface in $\mathbb{E}_1^4$.

PROOF. Since M is a parallel surface, we have $DH = 0$. Hence, Theorem 4.1 implies that M is one of the following:

(a) a parallel surface of a Minkowski space-time $\mathbb{E}^3_1 \subset \mathbb{E}^4_1$;

(b) a parallel surface of a de Sitter space-time $S^3_1(c) \subset \mathbb{E}^4_1$ for some $c > 0$;

(c) a minimal parallel surface of $\mathbb{E}^4_1$.

Case (a): *M is parallel surface of* $\mathbb{E}^3_1 \subset \mathbb{E}^4_1$. According to Theorem 5.1, M is congruent to cases (1), (2), (3), (4) or (6).

Case (b): *M is parallel surface* $S^3_1(c) \subset \mathbb{E}^4_1$. Theorem 5.2 implies that M is congruent to case (2) or (5).

Case (c): *M is minimal and parallel.* This reduces to case (a) by Lemma 8.1 □

THEOREM 8.2. *If M is a Lorentzian parallel surface in $S^4_1(1) \subset \mathbb{E}^5_1$, then M is congruent to an open part of one of the following two types of surfaces*:

(1) *a totally umbilical de Sitter space S^2_1 in $S^4_1(1)$ given by*

$$L = (a \sinh u, a \cosh u \cos v, a \cosh u \sin v, b, 0)\,, \quad a^2 + b^2 = 1\,;$$

(2) *a flat surface $S^1 \times S^1$ given by*

$$L = (a \sinh u, a \cosh u, b \cos v, b \sin v, 0)\,, \quad a^2 + b^2 = 1\,.$$

Conversely, each surface defined above is a Lorentzian parallel surface in $S^4_1(1)$.

PROOF. Since M is a parallel surface, we have $DH = 0$. Hence, Theorem 4.2 implies that M is one of the following:

(a) a parallel surface of $S^4_1(1) \cap \mathcal{E}_1$, where $\mathcal{E}_1$ is a Lorentzian hyperplane in $\mathbb{E}^5_1$;

(b) a minimal parallel surface of $S^4_1(1)$.

If M is a parallel surface of $S^4_1(1) \cap \mathcal{E}_1$, we may assume $\mathcal{E}_1$ is defined by $x_5 = 0$. Then $S^4_1(1) \cap \mathcal{E}_1 = S^3_1(1)$. Thus, by Theorem 5.2, we obtain case (1) or (2).

If M is minimal and parallel, this reduces to the first case according to Lemma 8.1. □

THEOREM 8.3. *If M is a Lorentzian parallel surface in $H^4_1(-1) \subset \mathbb{E}^5_2$, then M is congruent to an open part of one of the following twelve types of surfaces*:

(1) *a totally umbilical de Sitter space S^2_1 given by*

$$L = (c, a \sinh u, a \cosh u \cos v, a \cosh u \sin v, b)\,, \quad c^2 - a^2 - b^2 = 1\,;$$

(2) *a totally umbilical anti-de Sitter space H^2_1 given by*

$$L = (a \sin u, a \cos u \cosh v, a \cos u \sinh v, 0, b)\,, \quad a^2 - b^2 = 1\,;$$

(3) *a flat surface $S^1_1 \times H^1$ given by*

$$L = (a \sinh u, b \cosh v, a \cosh u, b \sinh v, c)\,, \quad a^2 - b^2 + c^2 = -1\,;$$

(4) *a flat surface $H^1_1 \times S^1$ given by*

$$L = (a \cos u, a \sin u, b \cos v, b \sin v, c)\,, \quad a^2 + b^2 - c^2 = 1\,;$$

(5) *a flat surface $S^1_1 \times S^1$ given by*

$$L = (a, b \sinh u, b \cosh u, c \cos v, c \sin v)\,, \quad a^2 - b^2 - c^2 = 1\,;$$

(6) *a totally umbilical flat surface defined by*

$$L = \left(\left(u^2 - v^2 - \frac{5}{4}\right), au, av, a\left(u^2 - v^2 - \frac{3}{4}\right), b\right), a^2 - b^2 = 1\,;$$

(7) *a flat surface defined by*

$$L = \left(a\cos v - \frac{a(u-v)}{2}\sin v, a\sin v + \frac{a(u-v)}{2}\cos v, \frac{a(u-v)}{2}\sin v\,,\right.$$
$$\left.\frac{a(u-v)}{2}\cos v, b\right), \quad a^2 - b^2 = 1\,;$$

(8) *a flat surface defined by*

$$L = \left(a\cosh v - \frac{a(u+v)}{2}\sinh v, \frac{a(u+v)}{2}\cosh v, a\sinh v - \frac{a(u+v)}{2}\cosh v\,,\right.$$
$$\left.\frac{a(u+v)}{2}\sinh v, b\right), \quad a^2 - b^2 = 1\,;$$

(9) *a surface defined by*

$$L = (a\cos u\cosh v - a\tan k\sin u\sinh v, a\sec k\sin u\cosh v,$$
$$a\cos u\sinh v - a\tan k\sin u\cosh v, a\sec k\sin u\sinh v, b)\,, \quad a^2 - b^2 = 1, \cos k \neq 0\,;$$

(10) *a flat surface defined by*

$$L = \left(\frac{b^2(u^2 - k^2 - 1) - 1}{2b^2k}, u, \frac{\cos bv}{b}, \frac{\sin bv}{b}, \frac{b^2(u^2 + k^2 - 1) - 1}{2b^2k}\right), \quad b, k \neq 0\,;$$

(11) *a flat surface defined by*

$$L = \left(-\frac{a^2(v^2 + k^2 + 1) + 1}{2a^2k}, \frac{\sinh au}{a}, \frac{\cosh au}{a}, v, \frac{a^2(k^2 - v^2 - 1) - 1}{2a^2k}\right), \quad a, k \neq 0\,;$$

(12) *a flat surface defined by*

$$L = \left(\frac{(u-v)^4}{24k} + \frac{u^2 - v^2 - k^2 - 1}{2k}, \frac{1}{6}(u-v)^3 + u, \frac{1}{2}(u-v)^2,\right.$$
$$\left.\frac{1}{6}(u-v)^3 + v, \frac{(u-v)^4}{24k} + \frac{u^2 - v^2 + k^2 - 1}{2k}\right), \quad k \neq 0\,.$$

Conversely, each surface defined above is a Lorentzian parallel surface in $H^4_1(-1)$.

PROOF. Since M is a parallel Lorentzian surface, we have $DH = 0$. Hence, Theorem 4.3 implies that M is one of the following:

(a) a parallel surface of $H^4_1(-1) \cap \mathcal{E}_1$, where $\mathcal{E}_1$ is a Lorentzian hyperplane in $\mathbb{E}^5_2$;
(b) a parallel surface of $H^4_1(-1) \cap \mathcal{E}_2$, where $\mathcal{E}_2$ is a hyperplane of index 2 in $\mathbb{E}^5_2$;
(c) a parallel surface of $H^4_1(-1) \cap \mathcal{H}$, where $\mathcal{H}$ is a degenerate hyperplane in $\mathbb{E}^5_2$;
(d) a minimal parallel surface of $H^4_1(-1)$.

Case (a): *M is a parallel surface of $H^4_1(-1) \cap \mathcal{E}_1$.* We may assume $\mathcal{E}_1$ is defined by $x_1 = a \geq 0$. Then the intersection $H^4_1(-1) \cap \mathcal{E}_1$ is given by

$$\{(a, x_2, \dots, x_5) \in \mathbb{E}^5_2 \,;\; -x_2^2 + x_3^2 + x_4^2 + x_5^2 = a^2 - 1\}\,.$$

If $a < 1$, M is Lorentzian and lies in a hyperbolic 3-space, which is impossible.

If $a > 1$, M lies in $H^4_1(-1) \cap \mathcal{E}_1 = S^3_1(1/(a^2-1))$. Since M is also parallel in this de Sitter space, Theorem 5.2 implies that M is congruent to case (1) or (5).

If $a = 1$, then M lies in a light cone in an $\mathbb{E}^4_1$, which is impossible (Theorem 6.1).

Case (b): *M is a parallel surface of $H^4_1(-1) \cap \mathcal{E}_2$.* We may assume $\mathcal{E}_2$ is given by $x_5 = b \geq 0$. Thus, $H^4_1(-1) \cap \mathcal{E}_2 = H^3_1(-1/(1+b^2)) \subset \mathbb{E}^4_1$ is defined by

$$-x_1 - x_2^2 + x_3^2 + x_4^2 = -(1+b^2) < 0\,.$$

Hence, we may applying Theorem 5.3 to obtain special cases of (1), (2), (3), (4), (6), (7), (8) or (9).

Case (c): *M is a parallel surface of $H^4_1(-1) \cap \mathcal{H}$.* We may assume that $\mathcal{H}$ is the hyperplane $\mathcal{G}_k = \{(x_1, \dots, x_5) \in \mathbb{E}^5_2; x_5 = x_1 + k\}$.

Case (c.1): $k = 0$. In this case, we have $x_5 = x_1$. If $e_3 = (1, 0, 0, 0, 1)$, then $\langle L, e_3\rangle = \langle L_x, e_3\rangle = \langle L_y, e_3\rangle = 0$, which implies that e_3 is a light-like normal vector of M orthogonal to L. But this is impossible since M is Lorentzian.

Case (c.2): $k \neq 0$. Since M lies in $\mathcal{G}_k$, we get $x_5 = x_1 + k$. So, if we put $e_3 = L + k^{-1}(1, 0, 0, 0, 1)$, then by applying $\langle L, L\rangle = -1$ we know that e_3 is a space-like unit normal vector field of M in $H^4_1(-1)$ satisfying $De_3 = 0$. Let e_4 be another unit space-like normal vector field of M in $H^4_1(-1)$ with $\langle e_3, e_4\rangle = 0$. Then, from the definition of e_3, we have $A_{e_3} = -I$.

Now, by applying Lemma 4.1, we know that there exists a basis $\{e_1, e_2\}$ satisfying $\langle e_2, e_2\rangle = -\langle e_1, e_1\rangle = 1, \langle e_1, e_2\rangle = 0$ such that A_{e_4} takes one of forms given by (4.1), (4.2) or (4.3) with respect to $\{e_1, e_2\}$.

Case (c.2.i): *A_{e_4} takes the form* (4.1). In this case we have

$$h(e_1, e_1) = e_3 - \alpha e_4\,, \quad h(e_1, e_2) = 0\,, \quad h(e_2, e_2) = -e_3 + \beta e_4\,. \tag{8.2}$$

Thus, it follows from $De_3 = De_4 = \overline{\nabla} h = 0$ that α, β are constant and $\omega_1^2 = 0$. Thus, M is flat, and hence, the equation of Gauss gives $\alpha\beta = 0$.

Case (c.2.i.1): $\alpha = 0$ *and* $\beta = b \neq 0$. If we choose coordinates (u, v) with $\partial_u = e_1, \partial_v = e_2$, then we obtain from (8.2) that

$$\begin{gathered} L_{uu} = e_3 - L\,, \quad L_{uv} = 0\,, \quad L_{vv} = -e_3 + be_4 + L\,, \\ \tilde{\nabla}_{\partial_u} e_3 = L_u\,, \quad \tilde{\nabla}_{\partial_v} e_3 = L_v\,, \quad \tilde{\nabla}_{\partial_u} e_4 = 0\,, \quad \tilde{\nabla}_{\partial_v} e_4 = -bL_v\,. \end{gathered} \tag{8.3}$$

After solving this system and choosing suitable Minkowskian coordinates, we obtain case (10).

Case (c.2.i.2): $\alpha = a \neq 0$ *and* $\beta = 0$. In this case we obtain case (11) in a similar way as (c.2.i.1).

Case (c.2.ii): *A_{e_4} takes the form* (4.2), $\beta \neq 0$. In this case we have

$$h(e_1, e_1) = e_3 - \alpha e_4\,, \quad h(e_1, e_2) = \beta e_4\,, \quad h(e_2, e_2) = -e_3 + \alpha e_4\,. \tag{8.4}$$

Thus, it follows from $De_3 = De_4 = \overline{\nabla} h = 0$ that α, β are constant and $\omega_1^2 = 0$. Hence, the equation of Gauss gives $0 = K = \alpha^2 + \beta^2$ which is impossible.

Case (c.2.iii): *A_{e_4} takes one of the forms* (4.3). In this case we have

$$h(e_1, e_1) = e_3 - \alpha e_4\,, \quad h(e_1, e_2) = -e_4\,, \quad h(e_2, e_2) = -e_3 + (\alpha \pm 2) e_4\,. \tag{8.5}$$

Thus, it follows from $De_3 = De_4 = \overline{\nabla} h = 0$ that α is a constant and $\omega_1^2 = 0$. Hence, M is flat. Moreover, we may choose coordinates (u, v) such that $\partial_u = e_1$, $\partial_v = e_2$. Also, it follows from (8.5) and the equation of Gauss that $0 = K = (1 \pm \alpha)^2$. Thus, A_{e_4} takes one of the following two forms:

$$A_{e_4} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \text{ (with } \alpha = -1) \quad \text{or} \quad A_{e_4} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \text{ (with } \alpha = 1)\,. \tag{8.6}$$

In the first case, we obtain

$$L_{uu} = e_3 + e_4 - L\,, \quad L_{uv} = -e_4\,, \quad L_{vv} = -e_3 + e_4 + L\,,$$
$$\tilde{\nabla}_{\partial_u} e_3 = L_u\,, \quad \tilde{\nabla}_{\partial_v} e_3 = L_v\,, \quad \tilde{\nabla}_{\partial_u} e_4 = L_u + L_v\,, \quad \tilde{\nabla}_{\partial_v} e_4 = -L_u - L_v\,.$$

Solving this system and choosing suitable Minkowskian coordinates yield case (12).

If the second case, we also obtain case (12) after a reparametrization.

Case (d): *M minimal and parallel.* This reduces to case (b) by Lemma 8.1. □

REFERENCES

[1] C. BLOMSTROM, Symmetric immersions in pseudo-Riemannian space forms, Global Differential Geometry and Global Analysis, 1984 (Berlin, 1984), 30–45, Lecture Notes in Math. 1156, Springer, Berlin, 1985.

[2] B.-Y. CHEN, Geometry of Submanifolds, Pure and Applied Mathematics, No. 22, Marcel Dekker, New York, 1973.

[3] B.-Y. CHEN, Total mean curvature and submanifolds of finite type, Ser. Pure Math. 1, World Scientific, Singapore, 1984.

[4] B.-Y. CHEN and J. VAN DER VEKEN, Marginally trapped surfaces in Lorentzian space forms with positive relative nullity, Classical Quantum Gravity 24 (2007), 551–563.

[5] B.-Y. CHEN and J. VAN DER VEKEN, Spatial and Lorentzian surfaces in Robertson-Walker space times, J. Math. Phys. 48 (2007), no. 7, 073509, 12 pp.

[6] B.-Y. CHEN and J. VAN DER VEKEN, Classification of marginally trapped surfaces with parallel mean curvature vector in Lorentzian space forms, Houston J. Math. (to appear).

[7] M. DAJCZER and K. NOMIZU, On flat surfaces in S_1^3 and H_1^3, Manifolds and Lie Groups (Notre Dame, Ind., 1980), pp. 71–108, Progr. Math. 14, Birkhäuser, Boston, Mass., 1981.

[8] D. FERUS, Immersions with parallel second fundamental form, Math. Z. 140 (1974), 87–93.

[9] L. K. GRAVES, On codimension one isometric immersions between indefinite space forms, Tsukuba J. Math. 3 (1979), 17–29.

[10] L. K. GRAVES, Codimension one isometric immersions between Lorentz spaces, Trans. Amer. Math. Soc. 252 (1979), 367–392.

[11] J. C. LARSEN, Complex analysis, maximal immersions and metric singularities, Monatsh. Math. 122 (1996), 105–156.

[12] M. A. MAGID, Isometric immersions of Lorentz space with parallel second fundamental forms, Tsukuba J. Math. 8 (1984), 31–54.

[13] P. MIRA and J. A. PASTOR, Helicoidal maximal surfaces in Lorentz-Minkowski space, Monatsh. Math. 140 (2003), 315–334.

[14] H. NAITOH, Parallel submanifolds of complex space forms. I, Nagoya Math. J. 90 (1983), 85–117.

[15] H. NAITOH, Parallel submanifolds of complex space forms. II, Nagoya Math. J. 91 (1983), 119–149.

[16] B. O'NEILL, Semi-Riemannian geometry with applications to relativity, Academic Press, New York, 1982.

[17] R. PENROSE, Gravitational collapse and space-time singularities, Phys. Rev. Lett. 14 (1965), 57–59.

[18] M. TAKEUCHI, Parallel submanifolds of space forms, Manifolds and Lie Groups (Notre Dame, Ind., 1980), pp. 429–447, Progr. Math. 14, Birkhäuser, Boston, Mass., 1981.

DEPARTMENT OF MATHEMATICS
MICHIGAN STATE UNIVERSITY
EAST LANSING, MICHIGAN 48824–1027
U.S.A.

E-mail address: bychen@math.msu.edu

KATHOLIEKE UNIVERSITEIT LEUVEN
DEPARTEMENT WISKUNDE
CELESTIJNENLAAN 200B, B-3001 LEUVEN
BELGIUM

E-mail address: joeri.vanderveken@wis.kuleuven.be