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Complete Convergence and Complete Moment Convergence for Negatively Dependent Random Variables Under Sub-Linear Expectations

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Abstract. This paper we study and establish the complete convergence and complete moment convergence theorems under a sub-linear expectation space. As applications, the complete convergence and complete moment convergence for negatively dependent random variables with $C_{\mathbb{V}}\left(\exp\left(\ln^{\alpha}|X|\right)\right) < \infty$, $\alpha > 1$ have been generalized to the sub-linear expectation space context. We extend some complete convergence and complete moment convergence theorems for the traditional probability space to the sub-linear expectation space. Our results generalize corresponding results obtained by Gut and Stadtmüller (2011), Qiu and Chen (2014) and Wu and Jiang (2016). There is no report on the complete moment convergence under sub-linear expectation, and we provide the method to study this subject.

1. Introduction

The sub-linear expectation space have advantages of modelling the uncertainty of probability and distribution. Therefore, limit theorems for sub-linear expectations have raised a large number of issues of interest recently. Limit theorems are important research topics in probability and statistics. They were widely used in finance and other fields. Classical limit theorems only hold in the case of model certainty. However, in practice, such model certainty assumption is not realistic in many areas of applications because the uncertainty phenomena cannot be modeled using model certainty. Motivated by modeling uncertainty in practice, Peng (2006 [14]) introduced a new notion of sub-linear expectation. As an alternative to the traditional probability/expectation, capacity/sub-linear expectation has been studied in many fields such as statistics, finance, economics, and measures of risk (see Denis and Martini (2006 [5]), Gilboa (1987 [6]), Marinacci (1999 [11]), Peng (1997 [12], 1999 [13], 2006 [14], 2008 [15]) etc). The general framework of the sub-linear expectation in a general function space was introduced by Peng (2006 [14], 2008 [15], 2009 [16]), and sub-linear expectation is a natural extension of the classical linear expectation.

Because the sub-linear expectation provides a very flexible framework to model sub-linear probability problems, the limit theorems of the sub-linear expectation have received more and more attention and research recently. A series of useful results have been established. Peng (2006 [14], 2008 [15], 2009

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[16]) constructed the basic framework, basic properties and the central limit theorem under sub-linear expectations, Zhang (2016a [26], 2016b [27], 2016c [28]) established the exponential inequalities, Rosenthal's inequalities, strong law of larger numbers and law of iterated logarithm, Cheng (2016 [3]), Chen (2016 [2]) and Wu and Jiang (2017 [24]) also studied strong law of larger numbers, and so on. In general, extending the limit properties of conventional probability space to the cases of sub-linear expectation is highly desirable and of considerably significance in the theory and application. Because sub-linear expectation and capacity are not additive, many powerful tools and common methods for linear expectations and probabilities are no longer valid, so that the study of the limit theorems under sub-linear expectation becomes much more complex and difficult.

Complete convergence and complete moment convergence are the most important problems in probability theory. Many of their related results have been obtained in the probabilistic space. However, the complete moment convergence under sub-linear expectation has not been reported. In this paper, we establish the complete convergence and complete moment convergence for negatively dependent random variables under sub-linear expectation. As a result, the corresponding results obtained by Gut and Stadtmüller (2011 [8]), Qiu and Chen (2014 [17]) and Wu and Jiang (2016 [23]) have been generalized to the sub-linear expectation space context.

In the next section, we summarize some basic notations and concepts, related properties under the sub-linear expectations and present the preliminary lemmas that are useful to prove the main results. In Section 3, complete convergence and complete moment convergence theorems for negatively dependence random variables under sub-linear expectation space are established.

2. Preliminaries

We use the framework and notations of Peng (2009 [16]). Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \ldots, X_n \in \mathcal{H}$ then $\varphi(X_1, \ldots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,Lip}(\mathbb{R}_n)$, where $C_{l,Lip}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \le c(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_n,$$

for some $c > 0, m \in \mathbb{N}$ depending on φ. $\mathcal{H}$ is considered as a space of random variables . In this case we denote $X \in \mathcal{H}$.

Definition 2.1. *A sub-linear expectation* $\hat{\mathbb{E}}$ *on* $\mathcal{H}$ *is a function* $\hat{\mathbb{E}} : \mathcal{H} \to \bar{\mathbb{R}}$ *satisfying the following properties: for all* $X, Y \in \mathcal{H}$, *we have*
(a) Monotonicity: If $X \ge Y$ *then* $\hat{\mathbb{E}}X \ge \hat{\mathbb{E}}Y$;
(b) Constant preserving: $\hat{\mathbb{E}}c = c$;
(c) Sub-additivity: $\hat{\mathbb{E}}(X + Y) \le \hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$ *whenever* $\hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$ *is not of the form* $+\infty - \infty$ *or* $-\infty + \infty$;
(d) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda\hat{\mathbb{E}}X, \lambda \ge 0$.

Here $\bar{\mathbb{R}} = [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sub-linear expectation space.
Give a sub-linear expectation $\hat{\mathbb{E}}$, let us denote the conjugate expectation $\hat{\varepsilon}$ of $\hat{\mathbb{E}}$ by

$$\hat{\varepsilon}X := -\hat{\mathbb{E}}(-X), \quad \forall X \in \mathcal{H}.$$

From the definition, it is easily shown that for all $X, Y \in \mathcal{H}$

$$\hat{\varepsilon}X \le \hat{\mathbb{E}}X, \quad \hat{\mathbb{E}}(X + c) = \hat{\mathbb{E}}X + c, \quad |\hat{\mathbb{E}}(X - Y)| \le \hat{\mathbb{E}}|X - Y| \text{ and } \hat{\mathbb{E}}(X - Y) \ge \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y.$$

If $\hat{\mathbb{E}}Y = \hat{\varepsilon}Y$, then $\hat{\mathbb{E}}(X + aY) = \hat{\mathbb{E}}X + a\hat{\mathbb{E}}Y$ for any $a \in \mathbb{R}$.

Next, we consider the capacities corresponding to the sub-linear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V : \mathcal{G} \to [0, 1]$ is called a capacity if

$$V(\emptyset) = 0, \quad V(\Omega) = 1 \text{ and } V(A) \le V(B) \text{ for } \forall A \subseteq B, A, B \in \mathcal{G}.$$

It is called to be sub-additive if $V(A\bigcup B)\leq V(A)+V(B)$ for all $A,B\in\mathcal{G}$ with $A\bigcup B\in\mathcal{G}$. In the sub-linear space $(\Omega,\mathcal{H},\hat{\mathbb{E}})$, we denote a pair $(\mathbb{V},\nu)$ of capacities by

$$\mathbb{V}(A):=\inf\{\hat{\mathbb{E}}\xi;I(A)\leq\xi,\xi\in\mathcal{H}\},\ \ \nu(A):=1-\mathbb{V}(A^c),\forall A\in\mathcal{F},$$

where A^c is the complement set of A. By definition of $\mathbb{V}$ and ν, it is obvious that $\mathbb{V}$ is sub-additive, and

$$\nu(A)\leq\mathbb{V}(A),\ \ \forall A\in\mathcal{F};\ \ \mathbb{V}(A)=\hat{\mathbb{E}}(I(A)),\ \ \nu(A)=\hat{\varepsilon}(I(A)),\ \text{ if }\ I(A)\in\mathcal{H},$$

$$\hat{\mathbb{E}}f\leq\mathbb{V}(A)\leq\hat{\mathbb{E}}g,\hat{\varepsilon}f\leq\nu(A)\leq\hat{\varepsilon}g,\ \text{ if }\ f\leq I(A)\leq g,\ \ f,g\in\mathcal{H}. \tag{1}$$

This implies Markov inequality: $\forall X\in\mathcal{H}$,

$$\mathbb{V}(|X|\geq x)\leq\hat{\mathbb{E}}(|X|^p)/x^p,\ \ \forall\ \ x>0,p>0$$

from $I(|X|\geq x)\leq|X|^p/x^p\in\mathcal{H}$. By Lemma 4.1 in Zhang (2016b [27]), we have Hölder inequality: $\forall X,Y\in\mathcal{H},p,q>1$ satisfying $p^{-1}+q^{-1}=1$,

$$\hat{\mathbb{E}}(|XY|)\leq\left(\hat{\mathbb{E}}(|X|^p)\right)^{1/p}\left(\hat{\mathbb{E}}(|Y|^q)\right)^{1/q},$$

particularly, Jensen inequality: $\forall X\in\mathcal{H}$,

$$\left(\hat{\mathbb{E}}(|X|^r)\right)^{1/r}\leq\left(\hat{\mathbb{E}}(|X|^s)\right)^{1/s}\quad\text{for}\quad 0<r\leq s.$$

Also, we define the Choquet integrals/expecations $(C_{\mathbb{V}},C_\nu)$ by

$$C_V(X):=\int_0^\infty V(X>x)\mathrm{d}x+\int_{-\infty}^0(V(X>x)-1)\mathrm{d}x$$

with V being replaced by $\mathbb{V}$ and ν respectively.

Definition 2.2. *(Peng 2006 [13], Zhang 2016a [26])*
(i) (Identical distribution) Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two n-dimensional random vectors defined respectively in sub-linear expectation spaces $(\Omega_1,\mathcal{H}_1,\hat{\mathbb{E}}_1)$ and $(\Omega_2,\mathcal{H}_2,\hat{\mathbb{E}}_2)$. They are called identically distributed if

$$\hat{\mathbb{E}}_1(\varphi(\mathbf{X}_1))=\hat{\mathbb{E}}_2(\varphi(\mathbf{X}_2)),\ \ \forall\varphi\in C_{l,Lip}(\mathbb{R}_n),$$

whenever the sub-expectations are finite. A sequence $\{X_n;n\geq1\}$ of random variables is said to be identically distributed if for each $i\geq1$, X_i and X_1 are identically distributed.

(ii) (Negative dependence) In a sub-linear expectation space $(\Omega,\mathcal{H},\hat{\mathbb{E}})$, a random vector $\mathbf{Y}=(Y_1,\ldots,Y_n)$, $Y_i\in\mathcal{H}$ is said to be negatively dependent (ND) to another random vector $\mathbf{X}=(X_1,\ldots,X_m),X_i\in\mathcal{H}$ under $\hat{\mathbb{E}}$ if for each pair of test functions $\varphi_1\in C_{l,Lip}(\mathbb{R}_m)$ and $\varphi_2\in C_{l,Lip}(\mathbb{R}_n)$ we have $\hat{\mathbb{E}}(\varphi_1(\mathbf{X})\varphi_2(\mathbf{Y}))\leq\hat{\mathbb{E}}(\varphi_1(\mathbf{X})\hat{\mathbb{E}}(\varphi_2(\mathbf{Y}))$, whenever φ_1,φ_2 are coordinatewise nondecreasing or φ_1,φ_2 are coordinatewise non-increasing with $\varphi_1(\mathbf{X})\geq0,\hat{\mathbb{E}}\varphi_2(\mathbf{Y})\geq0$, $\hat{\mathbb{E}}|\varphi_1(\mathbf{X})\varphi_2(\mathbf{Y})|<\infty,\hat{\mathbb{E}}|\varphi_1(\mathbf{X})|<\infty,\hat{\mathbb{E}}|\varphi_2(\mathbf{Y})|<\infty$.

A sequence of random variables $\{X_n;n\geq1\}$ is said to be negatively dependent if X_{i+1} is negatively dependent to $(X_1,\ldots,X_i)$ for each $i\geq1$.

It is obvious that, if $\{X_n;n\geq1\}$ is a sequence of negatively dependent random variables and functions $f_1(x),f_2(x),\ldots\in C_{l,Lip}(\mathbb{R})$ are all non-decreasing (resp. all non-increasing), then $\{f_n(X_n);n\geq1\}$ is also a sequence of negatively dependent random variables.

In the following, let $\{X_n;n\geq1\}$ be a sequence of random variables in $(\Omega,\mathcal{H},\hat{\mathbb{E}})$, and $S_n=\sum_{i=1}^n X_i$. The symbol c stands for a generic positive constant which may differ from one place to another. Let $a_x\sim b_x$ denote $\lim_{x\to\infty}a_x/b_x=1$, $a_x\ll b_x$ denote that there exists a constant $c>0$ such that $a_x\leq cb_x$ for sufficiently large x, and $I(\cdot)$ denote an indicator function.

To prove our results, we need the following two lemmas.

Lemma 2.3. *(Zhang 2016b, Theorem 3.1 [27]) Let $\{X_k; k \geq 1\}$ be a sequence of negatively dependent random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $\hat{\mathbb{E}}X_k \leq 0$. Then for any $x, y > 0$*

$$\mathbb{V}(S_n \geq y) \leq \mathbb{V}\left(\max_{1\leq k\leq n} X_k > x\right) + \exp\left(-\frac{y^2}{2(xy+B_n)}\left\{1+\frac{2}{3}\ln\left(1+\frac{xy}{B_n}\right)\right\}\right),$$

where $B_n = \sum_{k=1}^{n} \hat{\mathbb{E}}X_k^2$.

Lemma 2.4. *Suppose $X \in \mathcal{H}, \alpha > 1$.*
(i) *Then*

$$C_{\mathbb{V}}\left(\exp\left(\ln^{\alpha}|X|\right)\right) < \infty \Leftrightarrow \sum_{n=1}^{\infty} \exp\left(\ln^{\alpha} n\right) \frac{\ln^{\alpha-1} n}{n} \mathbb{V}\left(|X| > n\right) < \infty. \tag{2}$$

(ii) *If $C_{\mathbb{V}}\left(\exp\left(\ln^{\alpha}|X|\right)\right) < \infty$, and $f(x) := \frac{cx}{\ln^{\alpha} x}$ for any $c > 0$ then*

$$\lim_{n\to\infty} \exp\left(\ln^{\alpha} f(n)\right) \mathbb{V}(|X| > f(n)) = 0. \tag{3}$$

Proof (i) Note that

$$\begin{aligned} C_{\mathbb{V}}\left(\exp\left(\ln^{\alpha}|X|\right)\right) &\sim \int_1^{\infty} \mathbb{V}\left(\exp(\ln^{\alpha}|X|) > x\right) \mathrm{d}x \\ &= \int_1^{\infty} \mathbb{V}\left(|X| > \exp(\ln^{1/\alpha} x)\right) \mathrm{d}x \qquad (\text{let } \exp(\ln^{1/\alpha} x) = y) \\ &= \int_1^{\infty} \frac{\alpha \ln^{\alpha-1} y \exp(\ln^{\alpha} y)}{y} \mathbb{V}(|X| > y) \mathrm{d}y, \end{aligned} \tag{4}$$

and

$$\begin{aligned} \frac{\exp(\ln^{\alpha}(n+1))}{\exp(\ln^{\alpha} n)} &= \exp\left(\ln^{\alpha} n\left[\left(\frac{\ln(n+1)}{\ln n}\right)^{\alpha} - 1\right]\right) \\ &= \exp\left(\ln^{\alpha} n\left[\left(1+\frac{\ln(1+1/n)}{\ln n}\right)^{\alpha} - 1\right]\right) \\ &\sim \exp\left(\ln^{\alpha} n \frac{\alpha\ln(1+1/n)}{\ln n}\right) \sim \exp\left(\frac{\alpha \ln^{\alpha-1} n}{n}\right) \\ &\to 1. \end{aligned}$$

Therefore, (2) follows from (4) and $n^{-1}\ln^{\alpha-1} n \exp(\ln^{\alpha} n) \sim (n+1)^{-1}\ln^{\alpha-1}(n+1)\exp(\ln^{\alpha}(n+1))$.

(ii) By $\int_1^{\infty} \mathbb{V}\left(|X| > \exp(\ln^{1/\alpha} x)\right) \mathrm{d}x < \infty$ and $\mathbb{V}\left(|X| > \exp(\ln^{1/\alpha} x)\right) \downarrow$, we get

$$\lim_{x\to\infty} x\mathbb{V}\left(|X| > \exp(\ln^{1/\alpha} x)\right) = 0.$$

This is equivalent to

$$\lim_{y\to\infty} \exp(\ln^{\alpha} f(y)) \mathbb{V}(|X| > f(y)) = 0.$$

Therefore, (3) holds.

3. Complete Convergence and Complete Moment Convergence Theorems

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins (1947 [9]). Chow (1988 [4]) first investigated the complete moment convergence, which is more exact than complete convergence. Complete convergence and complete moment convergence are two of the most important problems in probability theory. Their recent results can be found in Wu (2015 [22]), Wu and Jiang (2016 [23]), Xu and Tang (2014 [25]), Guo et al. (2014 [7]), Gut and Stadtmüller (2011 [8]), Qiu and Chen (2014 [17]), Wang and Hu (2014 [20]), Wang et al (2015 [21]), Liu et al (2015 [10]), Chen and Sung (2016 [1]), Tan et al (2016 [19]), and Shen et al (2017 [18]). In sub-linear expectations, due to the uncertainty of expectation and capacity, the complete convergence is essentially different from the ordinary probability space. The study of complete convergence and complete moment convergence for sub-linear expectations are much more complex and difficult. The purpose of this paper is to extend corresponding results obtained by Gut and Stadtmüller (2011 [8]), Qiu and Chen (2014 [17]), and and Wu and Jiang (2016 [23]) from the probabilistic space to sub-linear expectation space. Our results are as follows.

Theorem 3.1. *Let $\alpha > 1$, $\{X, X_n; n \ge 1\}$ be a sequence of negatively dependent and identically distributed random variables with $\hat{\mathbb{E}}(X^2) < \infty$. Suppose that*

$$C_{\mathbb{V}}\left(\exp(\ln^{\alpha}|X|)\right) < \infty, \tag{5}$$

then

$$\sum_{n=1}^{\infty} \exp\left(\ln^{\alpha} n\right) \frac{\ln^{\alpha-1} n}{n^2} \mathbb{V}\left(\sum_{i=1}^{n}(X_i - \hat{\mathbb{E}}X_i) > n\beta\right) < \infty \quad \text{for all} \quad \beta > 1, \tag{6}$$

and

$$\sum_{n=1}^{\infty} \exp\left(\ln^{\alpha} n\right) \frac{\ln^{\alpha-1} n}{n^2} \mathbb{V}\left(\sum_{i=1}^{n}(X_i - \hat{\varepsilon}X_i) < -n\beta\right) < \infty \quad \text{for all} \quad \beta > 1. \tag{7}$$

In particular, if $\hat{\mathbb{E}}X_i = \hat{\varepsilon}X_i$, then

$$\sum_{n=1}^{\infty} \exp\left(\ln^{\alpha} n\right) \frac{\ln^{\alpha-1} n}{n^2} \mathbb{V}\left(\left|\sum_{i=1}^{n}(X_i - \hat{\mathbb{E}}X_i)\right| > n\beta\right) < \infty \quad \text{for all} \quad \beta > 1, \tag{8}$$

Theorem 3.2. *Assume that the conditions of Theorem 3.1 hold and $\hat{\mathbb{E}}X_i = \hat{\varepsilon}X_i$. Then*

$$\sum_{n=1}^{\infty} \exp\left(\ln^{\alpha} n\right) \frac{\ln^{\alpha-1} n}{n^{2+q}} C_{\mathbb{V}}\left\{\left|\sum_{i=1}^{n}(X_i - \hat{\mathbb{E}}X_i)\right| - \beta n\right\}_+^q < \infty \quad \text{for all} \quad \beta > 1 \quad \text{and all} \quad q > 0. \tag{9}$$

Remark 3.3. *Theorems 3.1-3.2 extend the corresponding results obtained by Gut and Stadtmüller (2011 [8]), Qiu and Chen (2014 [17]), and Wu and Jiang (2016 [23]) from the probability space to sub-linear expectation space.*

Proof of Theorem 3.1. Without loss of generality, we can assume that $\hat{\mathbb{E}}X_1 = 0$.

For negatively dependent random variables $\{X_n; n \ge 1\}$, in order to ensure that the truncated random variables are also negatively dependent, we need that truncated functions belong to $C_{l,Lip}$ and are non-decreasing. Let $\beta > 1$ be arbitrary, set, $\beta^{-1} < \mu < 1$, for $n \ge 1$, $b_n = \beta n/(8\ln^{\alpha} n)$, define, $f_c(x) = xI(x \le c) + cI(x > c)$ for any $c > 0$, for $1 \le k \le n$,

$$X'_k := f_{b_n}(X_k) = X_k I(X_k \le b_n) + b_n I(X_k > b_n), \quad S'_n := \sum_{k=1}^{n} X'_k,$$

$$X''_k := (X_k - b_n)I(b_n < X_k \le n/\mu), \quad X'''_k := (X_k - b_n)I(X_k > n/\mu).$$

Obviously, $X_k = X'_k + X''_k + X'''_k$ and $\{X'_k; k \geq 1\}$ is also a sequence of negatively dependent random variables by $f_c(x) \in C_{l,Lip}$ and $f_c(x)$ being non-decreasing. Note that

$$\begin{aligned}
& \{S_n > n\beta\} \\
\subseteq\ & \{S_n > n\beta \text{ and } X_k \leq b_n \text{ for all } k \leq n\} \\
& \cup \left\{S_n > n\beta \text{ and } b_n < X_{k_0} \leq n/\mu \text{ for exactly one } k_0 \leq n \text{ and } X_j \leq b_n \text{ for all } j \neq k_0\right\} \\
& \cup \{X''_k \neq 0 \text{ for at least two } k \leq n\} \cup \{X'''_k \neq 0 \text{ for at least one } k \leq n\} \\
:=\ & A_n \cup B_n \cup C_n \cup D_n.
\end{aligned}$$

Therefore,

$$\mathbb{V}(S_n > n\beta) \leq \mathbb{V}(A_n) + \mathbb{V}(B_n) + \mathbb{V}(C_n) + \mathbb{V}(D_n). \tag{10}$$

It shall be noted that, in the probability space, there is an equality: $EI(|X| \leq a) = P(|X| \leq a)$, however, in the sub-linear expectation space, $\hat{\mathbb{E}}$ is defined through continuous functions in $C_{l,Lip}$ and the indicator function $I(|x| \leq a)$ is not continuous. Therefore, the expression $\hat{\mathbb{E}}I(|X| \leq a)$ does not exist. This needs to modify the indicator function by functions in $C_{l,Lip}$. To this end, we define the function $g(x) \in C_{l,Lip}(\mathbb{R})$ as follows.

For $0 < \mu < 1$, let $g(x) \in C_{l,Lip}(\mathbb{R})$ be a non-increasing function such that $0 \leq g(x) \leq 1$ for all x and $g(x) = 1$ if $x \leq \mu$, $g(x) = 0$ if $x > 1$. Then

$$I(x \leq \mu) \leq g(x) \leq I(x \leq 1), \quad I(x > 1) \leq 1 - g(x) \leq I(x > \mu). \tag{11}$$

Note that

$$|X_1 - X'_1| = (X_1 - b_n)I(X_1 > b_n) \leq |X_1|I(|X_1| > b_n) \leq \frac{X_1^2}{b_n}.$$

By $\hat{\mathbb{E}}X_1 = 0, \hat{\mathbb{E}}X_1^2 < \infty$,

$$\left|\sum_{k=1}^{n} \hat{\mathbb{E}}X'_k\right| = n|\hat{\mathbb{E}}X'_1| = n|\hat{\mathbb{E}}X_1 - \hat{\mathbb{E}}X'_1| \leq n\hat{\mathbb{E}}|X_1 - X'_1| \leq \frac{n}{b_n}\hat{\mathbb{E}}X^2 = c\beta \ln^\alpha n, \tag{12}$$

where $c = 8\beta^{-2}\hat{\mathbb{E}}X^2 > 0$, so that, taking $y = (n - c\ln^\alpha n)\beta, x = 2b_n$ in Lemma 2.3, noting that $X'_k - \hat{\mathbb{E}}X'_k \leq 2b_n, k \leq n$, for sufficiently large n, we get

$$\begin{aligned}
\mathbb{V}(A_n) &= \mathbb{V}(S'_n > n\beta) \leq \mathbb{V}\left(\sum_{k=1}^{n}(X'_k - \hat{\mathbb{E}}X'_k) > (n - c\ln^\alpha n)\beta\right) \\
&\leq \exp\left(-\frac{(n - c\ln^\alpha n)^2\beta^2}{2(\frac{\beta^2 n(n-c\ln^\alpha n)}{4\ln^\alpha n} + n\hat{\mathbb{E}}X^2)}\right) \\
&= \exp\left(-\frac{2\beta^2\left(1 - \frac{c\ln^\alpha n}{n}\right)^2}{\beta^2\left(1 - \frac{c\ln^\alpha n}{n}\right) + \frac{4\hat{\mathbb{E}}X^2\ln^\alpha n}{n}}\ln^\alpha n\right) \\
&\leq \exp(-\ln^\alpha n),
\end{aligned} \tag{13}$$

from $\frac{2\beta^2\left(1-\frac{c\ln^\alpha n}{n}\right)^2}{\beta^2\left(1-\frac{c\ln^\alpha n}{n}\right)+\frac{4\hat{\mathbb{E}}X^2\ln^\alpha n}{n}} \to 2 > 1$ as $n \to \infty$.

By $\sum_{1\le i\le n, i\ne k_0} X_i'$ and X_{k_0} are negatively dependent random variable, and set, $0<\delta := 1-(\mu\beta)^{-1}<1$, we get from (1) and (11)

$$\begin{aligned}
\mathbb{V}(B_n) &\le \mathbb{V}\left(\exists\ 1\le k_0\le n \text{ such that } \sum_{1\le i\le n, i\ne k_0} X_i' > \beta n - n/\mu, X_{k_0} > b_n\right)\\
&\le \sum_{k_0=1}^{n} \mathbb{V}\left(\sum_{1\le i\le n, i\ne k_0} X_i' > \beta n - n/\mu = \beta\delta n, X_{k_0} > b_n\right)\\
&\le \sum_{k_0=1}^{n} \hat{\mathbb{E}}\left[\left(1-g\left(\frac{\sum_{1\le i\le n, i\ne k_0} X_i'}{\beta\delta n}\right)\right)\left(1-g\left(\frac{X_{k_0}}{b_n}\right)\right)\right]\\
&\le \sum_{k_0=1}^{n} \hat{\mathbb{E}}\left(1-g\left(\frac{\sum_{1\le i\le n, i\ne k_0} X_i'}{\beta\delta n}\right)\right)\hat{\mathbb{E}}\left(1-g\left(\frac{X}{b_n}\right)\right)\\
&\le \sum_{k_0=1}^{n} \mathbb{V}\left(\sum_{1\le i\le n, i\ne k_0} X_i' > \mu\beta\delta n\right)\mathbb{V}\left(|X|>\mu b_n\right). \qquad (14)
\end{aligned}$$

Similarly to the proof of (12), we have $\left|\sum_{1\le i\le n, i\ne k_0} \hat{\mathbb{E}}X_i'\right| \le \mu\beta c\ln^\alpha n$, where $c = 8\beta^{-1}\mu^{-1}\hat{\mathbb{E}}X^2>0$, so that, taking $y=\mu\beta(\delta n - c\ln^\alpha n), x = 2b_n$ in Lemma 2.3, using the fact that $\frac{2\beta^2\left(\delta-\frac{c\ln^\alpha n}{n}\right)^2}{\beta^2\left(\delta-\frac{c\ln^\alpha n}{n}\right)+\frac{4\hat{\mathbb{E}}X^2(n-1)\ln^\alpha n}{n^2}} \to 2>1$ as $n\to\infty$, for sufficiently large n, we get

$$\begin{aligned}
&\mathbb{V}\left(\sum_{1\le i\le n, i\ne k_0} X_i' > \mu\beta\delta n\right)\\
\le\ &\mathbb{V}\left(\sum_{1\le i\le n, i\ne k_0} (X_i' - \hat{\mathbb{E}}X_i') > \mu\beta(\delta n - c\ln^\alpha n)\right)\\
\le\ &\exp\left(-\frac{\mu^2\beta^2(\delta n - c\ln^\alpha n)^2}{2\left(\frac{\mu\beta^2 n(\delta n - c\ln^\alpha n)}{4\ln^\alpha n} + (n-1)\hat{\mathbb{E}}X^2\right)}\right)\\
=\ &\exp\left(-\frac{2\mu^2\beta^2\left(\delta-\frac{c\ln^\alpha n}{n}\right)^2}{\mu\beta^2\left(\delta-\frac{c\ln^\alpha n}{n}\right)+\frac{4\hat{\mathbb{E}}X^2(n-1)\ln^\alpha n}{n^2}}\ln^\alpha n\right)\\
\le\ &\exp(-\mu\delta\ln^\alpha n). \qquad (15)
\end{aligned}$$

By (3), and for any $\theta>0$, $(\ln n + \ln(\mu\beta/8) - \alpha\ln\ln n)^\alpha/\ln^\alpha n \to 1 > 1-\theta$ as $n\to\infty$, for sufficiently large n, $\ln^\alpha(\mu b_n) = (\ln n + \ln(\mu\beta/8) - \alpha\ln\ln n)^\alpha \ge (1-\theta)\ln^\alpha n$, thus,

$$\mathbb{V}(|X|>\mu b_n) \ll \exp(-\ln^\alpha(\mu b_n)) \le \exp(-(1-\theta)\ln^\alpha n) \text{ for any } \theta>0, \qquad (16)$$

Substituting (15) and (16) in (14), we obtain

$$\mathbb{V}(B_n) \ll n\exp(-\mu\delta\ln^\alpha n)\exp(-(1-\mu\delta/2)\ln^\alpha n) \le \exp(-\ln^\alpha n). \qquad (17)$$

By (1), (11), and (16),

$$\begin{aligned}
\mathbb{V}(C_n) &= \mathbb{V}\Big(\exists\ 1 \le k_1 < k_2 \le n \ \text{ such that } \ X''_{k_1} \neq 0, X''_{k_2} \neq 0\Big) \\
&\le \sum_{1\le k_1<k_2\le n} \mathbb{V}(X_{k_1} > b_n, X_{k_2} > b_n) \\
&\le \sum_{1\le k_1<k_2\le n} \hat{\mathbb{E}}\left[\left(1 - g\left(\frac{X_{k_1}}{b_n}\right)\right)\left(1 - g\left(\frac{X_{k_2}}{b_n}\right)\right)\right] \\
&\le \sum_{1\le k_1<k_2\le n} \hat{\mathbb{E}}\left(1 - g\left(\frac{X_{k_1}}{b_n}\right)\right)\hat{\mathbb{E}}\left(1 - g\left(\frac{X_{k_2}}{b_n}\right)\right) \\
&\le n^2\hat{\mathbb{E}}^2\left(1 - g\left(\frac{X}{b_n}\right)\right) \le n^2\mathbb{V}^2(|X| > \mu b_n) \\
&\quad \le n^2 \exp(-2(1-\delta/2)\ln^\alpha n) = n^2 \exp(-1-(1-\delta)\ln^\alpha n) \\
&= \exp(-\ln^\alpha n)\frac{n^2}{(\mathrm{e}^{\ln n})^{(1-\delta)\ln^{\alpha-1} n}} \\
&\le \exp(-\ln^\alpha n). \qquad (18)
\end{aligned}$$

From (1) and (11),

$$\begin{aligned}
\mathbb{V}(D_n) &\le \sum_{k=1}^{n} \mathbb{V}(X_k > n/\mu) \le \sum_{k=1}^{n} \hat{\mathbb{E}}\left(1 - g\left(\frac{\mu X_k}{n}\right)\right) \\
&= n\hat{\mathbb{E}}\left(1 - g\left(\frac{\mu X}{n}\right)\right) \le n\mathbb{V}(|X| > n). \qquad (19)
\end{aligned}$$

This, together with (2), (5), (10), (13), (17), and (18), shows

$$\begin{aligned}
&\sum_{n=1}^{\infty} \exp\left(\ln^\alpha n\right) \frac{\ln^{\alpha-1} n}{n^2}\mathbb{V}\left(S_n > \beta n\right) \\
\ll\ & \sum_{n=1}^{\infty} \frac{\ln^{\alpha-1} n}{n^2} + \sum_{n=1}^{\infty} \exp\left(\ln^\alpha n\right) \frac{\ln^{\alpha-1} n}{n}\mathbb{V}(|X| > n) \\
<\ & \infty.
\end{aligned}$$

That is, (6) holds.
Obviously, $\{-X, -X_k; k \ge 1\}$ also satisfies the conditions of Theorem 3.1. Considering $\{-X_n; n \ge 1\}$ instead of $\{X_n; n \ge 1\}$ in (6), we can obtain (7).

In particular, if $\hat{\mathbb{E}}X_k = \hat{\varepsilon}X_k$, then (8) follows from (6), (7), and

$$\begin{aligned}
&\mathbb{V}\left(\left|\sum_{i=1}^{n}(X_i - \hat{\mathbb{E}}X_i)\right| > n\beta\right) \\
\le\ & \mathbb{V}\left(\sum_{i=1}^{n}(X_i - \hat{\mathbb{E}}X_i) > n\beta\right) + \mathbb{V}\left(\sum_{i=1}^{n}(X_i - \hat{\mathbb{E}}X_i) < -n\beta\right).
\end{aligned}$$

This completes the proof of Theorem 3.1.

Proof of Theorem 3.2. Without loss of generality, we still assume that $\hat{\mathbb{E}}X_1 = 0$. Note that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \exp(\ln^{\alpha} n) \frac{\ln^{\alpha-1} n}{n^{2+q}} C_{\mathbb{V}} (|S_n| - \beta n)_+^q \\
= \quad & \beta^q \sum_{n=1}^{\infty} \exp(\ln^{\alpha} n) \frac{\ln^{\alpha-1} n}{n^{2+q}} \int_0^n q x^{q-1} \mathbb{V}(|S_n| - \beta n > \beta x)\, \mathrm{d}x \\
& + \beta^q \sum_{n=1}^{\infty} \exp(\ln^{\alpha} n) \frac{\ln^{\alpha-1} n}{n^{2+q}} \int_n^{\infty} q x^{q-1} \mathbb{V}(|S_n| - \beta n > \beta x)\, \mathrm{d}x \\
\ll \quad & \sum_{n=1}^{\infty} \exp(\ln^{\alpha} n) \frac{\ln^{\alpha-1} n}{n^{2}} \mathbb{V}(|S_n| > \beta n) \\
& + \sum_{n=1}^{\infty} \exp(\ln^{\alpha} n) \frac{\ln^{\alpha-1} n}{n^{2+q}} \int_n^{\infty} x^{q-1} \mathbb{V}(|S_n| > \beta x)\, \mathrm{d}x.
\end{aligned}$$

Hence, by (8), in order to establish (9), it suffices to prove that

$$\sum_{n=1}^{\infty} \exp(\ln^{\alpha} n) \frac{\ln^{\alpha-1} n}{n^{2+q}} \int_n^{\infty} x^{q-1} \mathbb{V}(|S_n| > \beta x)\, \mathrm{d}x < \infty. \tag{20}$$

Let $\beta > 1$ be an arbitrary, and $\beta^{-1} < \mu < 1$, set, for $x \geq n$, $b_x = \beta x/(8 \ln^{\alpha} x)$, define, for $1 \leq k \leq n$,

$$Y'_k := X_k I(X_k \leq b_x) + b_x I(X_k > b_x), \quad U'_n := \sum_{k=1}^{n} Y'_k,$$

$$Y''_k := (X_k - b_x) I(b_x < X_k \leq x/\mu), \quad Y'''_k := (X_k - b_x) I(X_k > x/\mu).$$

By similar methods to the proof of (10), we have

$$\mathbb{V}(S_n > x\beta) \leq \mathbb{V}(A_x) + \mathbb{V}(B_x) + \mathbb{V}(C_x) + \mathbb{V}(D_x), \tag{21}$$

which leads to

$$A_x = \{U'_n > x\beta\},$$

$$B_x = \left\{ S_n > x\beta \text{ and } b_x < X_{k_0} \leq x/\mu \text{ for exactly one } k_0 \leq n \text{ and } X_j \leq b_x \text{ for all } j \neq k_0 \right\},$$

$$C_x = \{Y''_k \neq 0 \text{ for at least two } k \leq n\},$$

$$D_x = \{Y'''_k \neq 0 \text{ for at least one } k \leq n\}.$$

Using similar methods to those used in the proof of (12), (13), and (16)-(19), for $0 < \delta := 1 - \beta^{-1} < 1$ and $x \geq n$, we have $|\sum_{k=1}^{n} \hat{\mathbb{E}} Y'_k| \leq c\beta \ln^{\alpha} x$, where $c = 8\beta^{-2} \hat{\mathbb{E}} X^2$, and

$$\mathbb{V}(A_x) \ll \exp(-\ln^{\alpha} x),$$

$$\mathbb{V}(|X| > \mu b_x) \ll \exp(-(1-\theta) \ln^{\alpha} x) \text{ for any } \theta > 0,$$

$$\begin{aligned}
\mathbb{V}(B_x) \quad & \ll \quad n \exp(-\mu\delta \ln^{\alpha} x - (1 - \mu\delta/2) \ln^{\alpha} x) \\
& = \quad \exp(-\ln^{\alpha} x) \frac{n}{\exp(\mu\delta \ln^{\alpha} x/2)} \leq \exp(-\ln^{\alpha} x),
\end{aligned}$$

$$\begin{aligned}
\mathbb{V}(C_x) \quad & \leq \quad n^2 \mathbb{V}^2(|X| > \mu b_x) \ll n^2 \exp(-2(1 - \delta/2) \ln^{\alpha} x) \\
& = \quad \exp(-\ln^{\alpha} x) n^2 \exp(-(1-\delta) \ln^{\alpha} x) \leq \exp(-\ln^{\alpha} x),
\end{aligned}$$

$$\mathbb{V}(D_x) \le n\mathbb{V}(|X| > x),$$

which, combining with (21), shows

$$\mathbb{V}\left(S_n > x\beta\right) \ll \exp(-\ln^\alpha x) + n\mathbb{V}(|X| > x).$$

Because $\{-X, -X_k; k \ge 1\}$ is also a sequence of negatively dependent random variables. Obviously, $\{-X, -X_k; k \ge 1\}$ also satisfies the condition (5) and $\hat{\mathbb{E}}(-X_k) = -\hat{\varepsilon}X_k = -\hat{\mathbb{E}}(X_k) = 0$ from the assumption: $\hat{\mathbb{E}}(X_k) = \hat{\varepsilon}X_k$. Therefore, replacing X_k by $-X_k$ in the above inequality, we get

$$\mathbb{V}\left(S_n < -x\beta\right) = \mathbb{V}\left(-S_n > x\beta\right) \ll \exp(-\ln^\alpha x) + n\mathbb{V}(|X| > x).$$

Therefore,

$$\begin{aligned}\mathbb{V}\left(|S_n| > x\beta\right) &\le \mathbb{V}\left(S_n > x\beta\right) + \mathbb{V}\left(S_n < -x\beta\right)\\ &\ll \exp(-\ln^\alpha x) + n\mathbb{V}(|X| > x).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_n^\infty x^{q-1}\mathbb{V}\left(|S_n| > x\beta\right)\mathrm{d}x\\ \ll\ &\int_n^\infty x^{q-1}\exp\left(-\ln^\alpha x\right)\mathrm{d}x + \int_n^\infty x^{q-1} n\mathbb{V}(|X| > x)\mathrm{d}x\\ :=\ &I_1 + I_2.\end{aligned} \tag{22}$$

By the fact that $(a+b)^\alpha \ge a^\alpha + b^\alpha$ for any $a, b > 0$ and $\alpha > 1$, let $t = x/n$, we get

$$\begin{aligned}\sum_{n=1}^\infty \exp\left(\ln^\alpha n\right)\frac{\ln^{\alpha-1} n}{n^{2+q}} I_1 &= \sum_{n=1}^\infty \exp\left(\ln^\alpha n\right)\frac{\ln^{\alpha-1} n}{n^{2+q}}\int_1^\infty n^q t^{q-1}\exp\left(-(\ln n + \ln t)^\alpha\right)\mathrm{d}t\\ &\le \sum_{n=1}^\infty \exp\left(\ln^\alpha n\right)\frac{\ln^{\alpha-1} n}{n^{2+q}} n^q \exp(-\ln^\alpha n)\int_1^\infty t^{q-1}\exp\left(-\ln^\alpha t\right)\mathrm{d}t\\ &\ll \sum_{n=1}^\infty \exp\left(\ln^\alpha n\right)\frac{\ln^{\alpha-1} n}{n^{2+q}} n^q \exp(-\ln^\alpha n)\\ &= \sum_{n=1}^\infty \frac{\ln^{\alpha-1} n}{n^2} < \infty.\end{aligned} \tag{23}$$

By (5) and (2),

$$\begin{aligned}
&\sum_{n=1}^{\infty} \exp\left(\ln^{\alpha} n\right) \frac{\ln^{\alpha-1} n}{n^{2+q}} I_2 \\
= & \sum_{n=1}^{\infty} \exp\left(\ln^{\alpha} n\right) \frac{\ln^{\alpha-1} n}{n^{1+q}} \int_{n}^{\infty} x^{q-1} \mathbb{V}(|X| > x) \mathrm{d}x \\
= & \sum_{n=1}^{\infty} \exp\left(\ln^{\alpha} n\right) \frac{\ln^{\alpha-1} n}{n^{1+q}} \sum_{j=n}^{\infty} \int_{j}^{j+1} x^{q-1} \mathbb{V}(|X| > x) \mathrm{d}x \\
\ll & \sum_{n=1}^{\infty} \exp\left(\ln^{\alpha} n\right) \frac{\ln^{\alpha-1} n}{n^{1+q}} \sum_{j=n}^{\infty} \mathbb{V}(|X| > j) j^{q-1} \\
= & \sum_{j=1}^{\infty} \mathbb{V}(|X| > j) j^{q-1} \sum_{n=1}^{j} \exp\left(\ln^{\alpha} n\right) \frac{\ln^{\alpha-1} n}{n^{1+q}} \\
\ll & \sum_{j=1}^{\infty} \exp\left(\ln^{\alpha} j\right) \frac{\ln^{\alpha-1} j}{j} \mathbb{V}(|X| > j) \\
< & \infty,
\end{aligned}$$

from which, combining with (22) and (23), we see that (20) holds. This completes the proof of Theorem 3.2.

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