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Complete Integrability of Relativistic Calogero-Moser Systems and Elliptic Function Identities

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Abstract. Poincaré-invariant generalizations of the Galilei-invariant Calogero-Moser N-particle systems are studied. A quantization of the classical integrals $S_1, \ldots, S_N$ is presented such that the operators $\hat{S}_1, \ldots, \hat{S}_N$ mutually commute. As a corollary it follows that $S_1, \ldots, S_N$ Poisson commute. These results hinge on functional equations satisfied by the Weierstrass σ- and $\mathscr{P}$-functions. A generalized Cauchy identity involving the σ-function leads to an $N \times N$ matrix L whose symmetric functions are proportional to $S_1, \ldots, S_N$.

1. Introduction

Recently, new integrable classical N-particle systems have been discovered [1] that may be viewed as relativistic generalizations of the well-known nonrelativistic Calogero-Moser systems [2]. The time translation, space translation, and boost generators of these systems are given by

$$H = mc^2 \sum_{i=1}^{N} \operatorname{ch}\theta_i \prod_{j \neq i} f(q_i - q_j), \tag{1.1}$$

$$P = mc \sum_{i=1}^{N} \operatorname{sh}\theta_i \prod_{j \neq i} f(q_i - q_j), \tag{1.2}$$

$$B = -\frac{1}{c} \sum_{i=1}^{N} q_i. \tag{1.3}$$

Here, m denotes the particle mass, c the speed of light, θ the particle rapidity, and q the canonically conjugate generalized position. Moreover, the potential energy function $f(q)$ reads

$$f(q) = (a + b\mathscr{P}(q))^{1/2}, \tag{1.4}$$

where a and b are arbitrary constants and where $\mathscr{P}$ is the Weierstrass $\mathscr{P}$-function. This choice of f not only guarantees Poincaré invariance, but also the existence of N independent integrals for the H flow, given by

$$S_k = \sum_{\substack{I \subset \{1, \ldots, N\} \\ |I| = k}} \exp\left(\sum_{i \in I} \theta_i\right) \prod_{\substack{i \in I \\ j \notin I}} f(q_i - q_j), \qquad k = 1, \ldots, N. \tag{1.5}$$

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The special case $f^2=1+\text{sh}^{-2}$ turns out to be intimately related to several soliton equations, the sine-Gordon equation being a prime example.

These results from [1] form the starting-point for the present paper, whose principal result is a solution to the problem of quantizing the classical systems in such a fashion that they remain completely integrable. The term "quantum integrability" is used here in the customary loose sense of there existing N independent mutually commuting *formal* operators $\hat{S}_1,\ldots,\hat{S}_N$. We regard our demonstration that this state of affairs obtains as a first step towards the goal of making rigorous sense of these operators as pull-backs of real-valued multiplication operators under a unitary eigenfunction transform (a point of view described in more detail in [3]). Elsewhere [4] we shall return to this problem, and present arguments to the effect that the equivalence of the N-particle systems with the N-soliton/antisoliton sectors of the sine-Gordon theory persists for the quantization described in this paper (the soliton-antisoliton interaction being described by the "crossed channel" potential $f^2=1-\text{ch}^{-2}$).

We shall now sketch the plan of the paper and describe its results in more detail. We begin by showing how the integrals $S_1,\ldots,S_N$ can be quantized in such a fashion that they mutually commute. The vanishing of the quantum commutators hinges on functional equations satisfied by the Weierstrass σ-function. These identities [cf. (2.4), (2.10) below] are new, as far as we know. As a corollary it follows that the $\mathscr{P}$-function satisfies functional equations (2.5), (2.8) entailing that the Poisson brackets $\{S_k,S_l\}$ vanish. Thus, classical integrability follows from quantum integrability. In Sect. 2 these results are detailed in a discursive fashion; the technicalities are relegated to Appendix A.

In Sect. 3 we generalize the Lax matrix found in [1] for the hyperbolic case to the elliptic case, cf. (3.13)–(3.14). The fact that the above S_k are proportional to the symmetric functions Σ_k of L [cf. (3.16)] follows from an explicit formula for the determinant of an $N\times N$ matrix whose elements are expressed in terms of σ-functions. This formula, (3.18) below, may be viewed as a generalization of Cauchy's identity. We prove it in Appendix B, where we also consider special cases of interest.

Our conventions concerning elliptic functions are those of Erdélyi [5]. In the appendices we assume some familiarity with the results and arguments to be found there and (in more detail) in Whittaker and Watson [6, Chap. XX]. However, to render the main text more self-contained, it may be in order to add some remarks and formulas, most of which we have occasion to use.

First, we should mention that the term "elliptic function" is often reserved for doubly periodic meromorphic functions, like the $\mathscr{P}$-function. Here, the term includes the ζ- and σ-function, which are meromorphic and entire, respectively, but not doubly periodic. They are, however, quasi-periodic, in the sense that

$$\zeta(q+2\omega_k)=\zeta(q)+2\eta_k\,, \tag{1.6}$$

$$\sigma(q+2\omega_k)=-\sigma(q)\exp[2\eta_k(q+\omega_k)]\,. \tag{1.7}$$

Here, k takes the values $1,2,3$, and one has

$$\omega_1=\omega\,,\quad \omega_2=-\omega-\omega'\,,\quad \omega_3=\omega'\,,\quad \eta_k=\zeta(\omega_k)\,. \tag{1.8}$$

Moreover, 2ω and $2\omega'$ denote a pair of primitive periods of the $\mathscr{P}$-function.

The σ-function is odd and has simple zeros at the points of the period lattice $2m\omega+2n\omega'$, $n,m\in\mathbb{Z}$. Furthermore, it satisfies the scaling relation

$$\sigma(\lambda q;\omega,\omega')=\lambda\sigma(q;\omega/\lambda,\omega'/\lambda). \tag{1.9}$$

Corresponding properties of ζ and $\mathscr{P}$ can be read off from the relations

$$\zeta(q)=\sigma'(q)/\sigma(q), \qquad \mathscr{P}(q)=-\zeta'(q). \tag{1.10}$$

In particular, $\mathscr{P}$ is even and has second-order poles at the lattice points.

As a rule, we shall choose $\omega, -i\omega'\in(0,\infty]$. With this convention $\mathscr{P}(q)$ decreases monotonically from ∞ to $e_1>0$, $e_2,e_3<0$, $-\infty$ as q varies along the rectangle 0, ω_1, $-\omega_2$, ω_3, 0. Also, σ is real on the real axis and purely imaginary on the imaginary axis. Most of what follows does not depend on this choice of periods. The main reason for our convention is the ensuing positivity of $\mathscr{P}(q)$ on the real axis. By choosing appropriate coupling constants a and b in (1.4), we can then ensure that the S_k are real-valued at the classical and formally hermitian at the quantum level.

Let us finish this introduction by specifying $\mathscr{P}(q)$ and $\sigma(q)$ for the degenerate cases $\omega=\infty$ or $\omega'=i\infty$.

A. *Hyperbolic case* ($\omega=\infty$, $\omega'=i\pi/2\nu$),

$$\mathscr{P}(q)=\frac{\nu^2}{3}+\frac{\nu^2}{\operatorname{sh}^2\nu q}, \qquad \sigma(q)=\frac{\operatorname{sh}\nu q}{\nu}\exp\left[-\frac{\nu^2}{6}q^2\right]. \tag{1.11}$$

B. *Trigonometric case* ($\omega=\pi/2\nu$, $\omega'=i\infty$),

$$\mathscr{P}(q)=-\frac{\nu^2}{3}+\frac{\nu^2}{\sin^2\nu q}, \qquad \sigma(q)=\frac{\sin\nu q}{\nu}\exp\left[\frac{\nu^2}{6}q^2\right]. \tag{1.12}$$

C. *Rational case* ($\omega=\infty$, $\omega'=i\infty$),

$$\mathscr{P}(q)=\frac{1}{q^2}, \qquad \sigma(q)=q. \tag{1.13}$$

2. Quantum and Classical Integrability

We begin by discussing the quantization of $S_1,\dots,S_N$ (denoted $\hat{S}_1,\dots,\hat{S}_N$) in the free case $f(q)=1$. From (1.1) and (1.2) one sees that the rapidity variable θ is dimensionless. The canonically conjugate variable q is related to the customary position x by $q=mcx\operatorname{ch}\theta$, and hence has the dimension of action. Thus, the obvious quantization procedure reads

$$\theta_j\to\hat{\theta}_j\equiv\frac{\hbar}{i}\partial_{q_j}, \qquad j=1,\dots,N. \tag{2.1}$$

We shall put $\hbar=1$ henceforth.

Clearly, this prescription yields commuting operators $\hat{S}_1,\dots,\hat{S}_N$ which are all diagonalized by Fourier transformation. Their action can be exemplified by the formula

$$(e^{\hat{\theta}_1}\psi)(q_1,\dots,q_N)=\psi(q_1-i,\dots,q_N). \tag{2.2}$$

Here, $\psi(z)$, $z\in\mathbb{C}^N$, denotes a function that is supposed to be "sufficiently analytic" for (2.2) to make sense. Specifically, ψ should at least be analytic in the strip $-1<\operatorname{Im} q_1<0$ and have reasonable boundary values.

When $f=1$ there is no problem in being more precise than this. However, the situation is drastically different when $f(q)$ is not constant. Elsewhere we will return to the difficulties associated with a rigorous definition [4]. Our present purpose is to show that there exists a formal quantization of $S_1,\ldots,S_N$ such that the resulting operators are (formally) hermitian and commute. It should be emphasized that there exist (to date) no general principles guaranteeing that such a quantization is possible, even within the "formal algebra" framework adopted here.

The solution we have found is most likely unique. A description of its relevant features is facilitated by first considering a seemingly different question. Let us start with a meromorphic function $h(q)$ and set

$$\hat{S}_k \equiv \sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=k}} \prod_{\substack{i\in I\\ j\notin I}} h(q_j-q_i)^{1/2} \exp\Big(\beta\sum_{i\in I}\hat{\theta}_i\Big) \prod_{\substack{i\in I\\ j\notin I}} h(q_i-q_j)^{1/2},$$
$$k=1,\ldots,N, \tag{2.3}$$

where β is an arbitrary positive number. Now we ask: What condition on h ensures that all $\hat{S}_k$ commute pairwise for any N? As proved in Appendix A, the answer is: If and only if h satisfies the functional equations

$$\sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=k}} \Big(\prod_{\substack{i\in I\\ j\notin I}} h(q_j-q_i)h(q_i-q_j-i\beta) - \prod_{\substack{i\in I\\ j\notin I}} h(q_i-q_j)h(q_j-q_i-i\beta)\Big)$$
$$=0, \quad \forall N>1, \quad \forall k\in\{1,\ldots,N\}. \tag{2.4}$$

Next, *assume* that h satisfies (2.4). Dividing by β and sending β to 0 yields

$$\sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=k}} \Big(\sum_{i\in I}\partial_{q_i}\Big) \prod_{\substack{i\in I\\ j\notin I}} F(q_i-q_j)=0, \quad \forall N>1, \quad \forall k\in\{1,\ldots,N\}, \tag{2.5}$$

where

$$F(q)\equiv h(q)h(-q). \tag{2.6}$$

For $N=3$ and $k=1$ (2.5) reduces to the functional equation

$$\begin{vmatrix} F(d_1) & F'(d_1) & 1 \\ F(d_2) & F'(d_2) & 1 \\ F(d_1+d_2) & -F'(d_1+d_2) & 1 \end{vmatrix} = 0, \tag{2.7}$$

which is known to be satisfied if and only if

$$F(q)=a+b\mathscr{P}(q), \tag{2.8}$$

where a and b are arbitrary constants. Thus we are led back to our potential (1.4): A necessary condition for commutativity of $\hat{S}_1,\ldots,\hat{S}_N$ with arbitrary β is that h be related to the $\mathscr{P}$-function by

$$h(q)h(-q)=a+b\mathscr{P}(q). \tag{2.9}$$

In [1] it is proved that the identities (2.5) are equivalent to involutivity of the classical functions $S_1, \ldots, S_N$ with $f^2 = F$. Moreover, it is shown there that the function $a + b\mathscr{P}(q)$ satisfies (2.5) for $k=1$. However, the proof could not be adapted to cover the case $k>1$, so that complete integrability for $N>4$ was left open. From the above it transpires that involutivity of $S_1, \ldots, S_N$ will follow, once one finds a factorization $a + b\mathscr{P}(q) = h(q)h(-q)$ such that h satisfies (2.4), or, equivalently, such that $\hat{S}_1, \ldots, \hat{S}_N$ commute.

It remains to prove that a function h with these properties exists. Let us first note that (2.9) does not determine h uniquely: If a meromorphic function h satisfies (2.9), then this is also true for the function $\tilde{h} = he^E$, where E is an arbitrary entire odd function. However, if h satisfies (2.4), then there is no reason why $\tilde{h}$ would also satisfy (2.4), except in the trivial case where E is proportional to q. At any rate, we consider it plausible that the solution we have found, viz.,

$$h(q) = \sigma(q+\mu)/\sigma(q), \qquad \mu \in \mathbb{C}, \tag{2.10}$$

is unique up to multiplication by $c_1 e^{c_2 q}$, with c_1, c_2 arbitrary constants. [As concerns replacing q by $c_3 q$, recall the scaling relation (1.9).]

The proof that h satisfies (2.4) can be found in Appendix A. As explained above, it follows from this that h satisfies (2.9). Of course, (2.9) is also obvious from the well-known relation

$$\frac{\sigma(q+\mu)\sigma(q-\mu)}{\sigma^2(q)\sigma^2(\mu)} = \mathscr{P}(\mu) - \mathscr{P}(q), \tag{2.11}$$

which suggested (2.10) as a candidate.

To finish this section, we tie up some loose ends and add various remarks.

(i) *(Hermiticity)* Ensuring hermiticity amounts to ensuring that $h(-q)$ equal $\bar{h}(q)$ for $q \in \mathbb{R}$, and this can be attained by picking μ on the imaginary axis (cf. our remarks at the end of Sect. 1). Note, however, that this leads to a restriction on the coupling constants in the classical potential $(a + b\mathscr{P}(q))^{1/2}$: One must have

$$b/a \in [0, -1/e_3] \tag{2.12}$$

for a *hermitian* quantization of the form (2.3), (2.10) to exist. Indeed, from (2.11) one has

$$h(-q)h(q) = \sigma^2(\mu)\mathscr{P}(\mu)[1 - \mathscr{P}(q)/\mathscr{P}(\mu)] \tag{2.13}$$

and $1/\mathscr{P}(\mu)$ takes values in $[1/e_3, 0]$ as μ varies over the imaginary axis. Note also that one may as well restrict μ to vary between $-\omega'$ and ω', since multiplicative constants are irrelevant.

(ii) *(Degenerate cases)* From (1.11) we see that we may take

$$h(q) = \frac{\operatorname{sh} \nu(q+\mu)}{\operatorname{sh} \nu q}, \qquad \mu \in i\left[-\frac{\pi}{2\nu}, \frac{\pi}{2\nu}\right] \tag{2.14}$$

in the hyperbolic case. The exponential factor has been omitted, since the exponent is linear in q, and hence only gives rise to multiplicative constants in the operators $\hat{S}_1, \ldots, \hat{S}_N$. The "critical points" $\mu = \pm i\pi/2\nu$ correspond to the sine-Gordon theory [4]. Similarly, in the trigonometric case one gets

$$h(q) = \frac{\sin \nu(q+\mu)}{\sin \nu q}, \qquad \mu \in i\mathbb{R} \tag{2.15}$$

from (1.12), and in the rational case

$$h(q)=1+\frac{\mu}{q}, \qquad \mu\in i\mathbb{R} \tag{2.16}$$

from (1.13).

(iii) *(Relativistic invariance)* Let us take $\beta=1$ in (2.3) [with h given by (2.10)] and set

$$\hat{H}\equiv\frac{mc^2}{2}(\hat{S}_1+\hat{S}_{-1}), \qquad \hat{P}\equiv\frac{mc}{2}(\hat{S}_1-\hat{S}_{-1}), \qquad \hat{B}=-\frac{1}{c}\sum_{i=1}^{N}q_i, \tag{2.17}$$

where $\hat{S}_{-1}\equiv\hat{S}_N^{-1}\hat{S}_{N-1}$ [cf. also (A 11)]. Then it is clear from the above that $\hat{H},\hat{P},\hat{B}$ are hermitian when μ is purely imaginary, and that

$$[\hat{H},\hat{P}]=0, \qquad [\hat{H},\hat{B}]=i\hat{P}, \qquad [\hat{P},\hat{B}]=iH/c^2. \tag{2.18}$$

Thus, $\hat{H}$, $\hat{P}$, and $\hat{B}$ represent the Lie algebra of the Poincaré group. We also point out that for imaginary μ $\hat{H}$ has the physically desirable property of being positive.

(iv) *(Nonrelativistic limit)* So far, we have treated θ as a dimensionless variable and q as having the dimension of action. This is in agreement with the relations $p=mc\,\mathrm{sh}\,\theta$, $x=q/mc\,\mathrm{ch}\,\theta$, the first of which is the standard one defining the rapidity variable. If one takes this point of view, one can only hope to get a sensible nonrelativistic limit by transforming $\hat{H},\hat{P},\hat{B}$ to x-space and then sending c to ∞. However, this is an awkward enterprise at the quantum level. Even at the classical level, where no ordering problems occur, one must work harder to obtain the nonrelativistic Calogero-Moser systems in this way than when one takes a suitable limit directly on the (q,θ) phase space, cf. [1, Chap. 4]. The latter limit (which amounts to exploiting the parameter β) can be readily taken at the quantum level as well. However, though this limit is mathematically unimpeachable, it is physically unsatisfactory: It does not respect the dimensions of the quantities involved and cannot be viewed as a nonrelativistic limit in the usual sense.

These problems can be cured in a simple way: One needs only replace q,θ by $mcq,\theta/mc$. Then the dimensions of q and θ change to position and momentum, respectively. Let us write out the Poincaré group generators (2.17) with these new conventions:

$$\hat{H}=\frac{mc^2}{2}\sum_{i=1}^{N}[\bar{W}_i(q)\exp(\hat{\theta}_i/mc)W_i(q)$$
$$+W_i(q)\exp(-\hat{\theta}_i/mc)\bar{W}_i(q)], \tag{2.19}$$

$$\hat{P}=\frac{mc}{2}\sum_{i=1}^{N}[\bar{W}_i(q)\exp(\hat{\theta}_i/mc)W_i(q)$$
$$-W_i(q)\exp(-\hat{\theta}_i/mc)\bar{W}_i(q)], \tag{2.20}$$

$$\hat{B}=-m\sum_{i=1}^{N}q_i, \tag{2.21}$$

where

$$W_i(q)\equiv\prod_{j\neq i}\left(\frac{\sigma(q_i-q_j+ig/mc)}{\sigma(q_i-q_j)}\right)^{1/2}, \qquad g\in-i\omega'mc[-1,1]. \tag{2.22}$$

We have used (1.9) to scale out the factor mc. However, we continue denoting the scaled periods by ω, ω', since they have to be kept fixed when $c \to \infty$ for the Galilei-invariant Calogero-Moser systems to result. Indeed, using (1.10) one gets

$$\hat{H}_{\mathrm{nr}} = \lim_{c\to\infty} (\hat{H} - Nmc^2) = \sum_{i=1}^{N} \frac{\hat{\theta}_i^2}{2m} + \frac{g(g-1)}{m} \sum_{1 \leq i < j \leq N} \mathscr{P}(q_i - q_j), \tag{2.23}$$

$$\hat{P}_{\mathrm{nr}} = \lim_{c\to\infty} \hat{P} = \sum_{i=1}^{N} \hat{\theta}_i, \tag{2.24}$$

$$\hat{B}_{\mathrm{nr}} = \lim_{c\to\infty} \hat{B} = -m \sum_{i=1}^{N} q_i. \tag{2.25}$$

Note the change $g^2 \to g(g-1)$ as compared to the nonrelativistic limit at the classical level [cf. (3.25) below].

Probably, a more general result holds true: We expect that, just as at the classical level, suitable linear combinations of $\hat{S}_1, \ldots, \hat{S}_N$ converge to the usual $\hat{S}_{1,\mathrm{nr}}, \ldots, \hat{S}_{N,\mathrm{nr}}$ as $c \to \infty$. Note that one would recover the quantum integrability of the nonrelativistic Calogero-Moser systems (2.23) (cf. [7–9]) from such a convergence result. We shall briefly return to this question at the end of Sect. 3.

(v) *(Classical limit)* It is of interest to note that the parameter β in (2.3) may be interpreted as Planck's constant, cf. (2.1). Thus, the implication "quantum integrability $\Rightarrow$ classical integrability" established above agrees with the physicist's expectation that quantum mechanics reduces to classical mechanics in the limit $\hbar \to 0$.

3. The Elliptic Lax Matrix

The customary approach [2] to the classical nonrelativistic Calogero-Moser systems is based on the existence of a pair of $N \times N$ matrices $L_{\mathrm{nr}}, M_{\mathrm{nr}}$ depending on the canonical variables of the N-particle system, which are such that $\{L_{\mathrm{nr}}, H_{\mathrm{nr}}\} = [L_{\mathrm{nr}}, M_{\mathrm{nr}}]$. Here, $H_{\mathrm{nr}} = \frac{1}{2}\mathrm{Tr}\, L_{\mathrm{nr}}^2$ denotes the nonrelativistic energy function. This leads to the existence of N independent integrals $S_{k,\mathrm{nr}}$ for the H_{nr} flow, defined by

$$|L_{\mathrm{nr}} + \alpha \mathbb{1}| = \sum_{\ell=0}^{N} \alpha^{\ell} S_{N-\ell,\mathrm{nr}}. \tag{3.1}$$

Subsequently, one shows that the eigenvalues of L_{nr} and hence the $S_{k,\mathrm{nr}}$, too, are in involution. There is no simple "closed form" formula for the $S_{k,\mathrm{nr}}$; in particular, it is not obvious that they can be expressed solely in terms of the momenta and the potentials.

Let us now compare this state of affairs to the relativistic case. There, one has the explicit formulas (1.4)–(1.5), expressing the N independent Poisson commuting Hamiltonians $S_1, \ldots, S_N$ in terms of $e^{\theta_1}, \ldots, e^{\theta_N}$ and the potentials. Neither the proof of [1, Theorem A4] to the effect that the S_k are integrals for H, nor the more general proof that the S_k are in involution (cf. Sect. 2) involves matrices. Therefore, it might appear irrelevant to look for an $N\times N$ matrix L such that

$$|L+\alpha\mathbb{1}|=\sum_{\ell=0}^{N}\alpha^{\ell}S_{N-\ell}. \tag{3.2}$$

In a way, this is indeed the case: If this were the only objective, one could take e.g.

$$L=\begin{pmatrix} S_1 & -S_2 & \cdots & \cdots & (-)^{N+1}S_N \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{3.3}$$

Indeed, (3.2) is then readily verified. [We mention in passing that a matrix of the form (3.3) can be used to show that it is not generally true that isospectrality implies involutivity of the eigenvalues: If one picks

$$S_1=\sum_{i=1}^{N}p_i^2, \qquad S_k=x_{k-1}p_k-x_kp_{k-1}, \qquad k=2,\dots,N, \tag{3.4}$$

then L is isospectral under the S_1 flow, yet its symmetric functions do not commute.]

However, in the nonrelativistic case it has turned out that the known Lax matrices $L_{\rm nr}$ play a much more fundamental role than just yielding the commuting Hamiltonians: They can be used to construct explicit solutions and the action-angle map [2]. The Lax matrix found in [1] for the relativistic hyperbolic case also has these properties [1, 10].

We are not aware of any general arguments entailing that such a matrix should exist in the relativistic elliptic case. However, it is natural to believe that this case (which contains all other cases) is not going to be an exception. As explained in [1, Chap. 4], the structure of the S_k [cf. (1.5)] suggests the Ansatz

$$L_{ij}=e^{\theta_i}\prod_{l\neq i}f(q_i-q_l)C_{ij}(q). \tag{3.5}$$

For (3.2) to follow, the 2×2 principal minor $C(i,j)$ should equal $1/f^2(q_i-q_j)$, while the general principal minor should be the product of all 2×2 principal minors contained in it. For the hyperbolic case $f^2-1\sim \mathrm{sh}^{-2}$ this can be attained by substituting

$$x_i=\varphi(q_i), \qquad y_i=\chi(q_i) \tag{3.6}$$

(with φ,χ exponential functions) in Cauchy's identity [1]. Thus, an obvious guess is that the elliptic case can be handled by making a more general substitution.

To study this, let us recall that Cauchy's identity is equivalent to

$$|C|=\prod_{i<j}C(i,j). \tag{3.7}$$

Here, C is defined by

$$C_{ij}\equiv(x_i-y_i)^{1/2}(x_i-y_j)^{-1}(x_j-y_j)^{1/2}, \tag{3.8}$$

so that

$$C(i,j)=\frac{(x_i-x_j)(y_i-y_j)}{(x_i-y_j)(y_i-y_j)}. \tag{3.9}$$

Let us, therefore, ask the general question: Can one characterize pairs of functions φ, χ such that the principal minor $C(i,j)$ is a function of $q_i - q_j$ after the substitution (3.6)? Acting with $\partial_{q_i} + \partial_{q_j}$ on $C(i,j)$, one gets as the necessary and sufficient condition the functional equation

$$\begin{vmatrix} \varphi'(q_1) & \varphi^2(q_1) & \varphi(q_1) & 1 \\ \varphi'(q_2) & \varphi^2(q_2) & \varphi(q_2) & 1 \\ \chi'(q_1) & \chi^2(q_1) & \chi(q_1) & 1 \\ \chi'(q_2) & \chi^2(q_2) & \chi(q_2) & 1 \end{vmatrix} = 0 . \tag{3.10}$$

Equivalently, there should be a, b, c, d such that

$$a\psi'(q_i) + b\psi^2(q_i) + c\psi(q_i) + d = 0 , \quad \psi = \varphi, \chi , \quad i = 1, 2 . \tag{3.11}$$

If one assumes that the coefficients are constant, then one is led back to the hyperbolic Lax matrix of [10], in essence. This follows from a straightforward analysis we shall skip. We do not know whether (3.11) also admits elliptic solutions with non-constant a, b, c, and/or d, leading to the elliptic S_k.

However, even if such solutions would exist, they cannot yield the matrix that is undoubtedly the "right" elliptic Lax matrix and to whose description we now turn. The point is that this matrix L is of the form (3.5), but with a matrix C that is not a Cauchy matrix, i.e., C does *not* arise from (3.8) by appropriate substitution. Specifically, its principal minor $C(I)$ is not *equal* to the product of all principal minors $C(i, j)$ with $i, j \in I$, but only *proportional* to this product. Thus one gets

$$|L + \alpha \mathbb{1}| = \sum_{\ell=0}^{N} \alpha^\ell \Sigma_{N-\ell} , \quad \Sigma_k = c_k S_k \tag{3.12}$$

instead of (3.2).

Explicitly, this matrix reads

$$L_{ij} \equiv e^{\theta_i} \prod_{\ell \neq i} f(q_i - q_\ell) C_{ij}(q) , \quad f^2(q) \equiv N(\mu)[\mathscr{P}(\mu) - \mathscr{P}(q)] , \tag{3.13}$$

$$C_{ij} \equiv \frac{\sigma(q_i - q_j + \lambda)}{\sigma(\lambda)} \frac{\sigma(\mu)}{\sigma(q_i - q_j + \mu)} , \quad i, j = 1, \ldots, N . \tag{3.14}$$

Here, $N(\mu)$ is a normalization constant, whose choice is to a large extent arbitrary. We shall set

$$N(\mu) \equiv \sigma^2(\mu) , \tag{3.15}$$

since this is the simplest choice of potential in the quantum case [cf. (2.13)], and since it yields the simplest formulas for the symmetric functions of L: with (3.15) in force they are given by

$$\Sigma_k = \frac{\sigma(\lambda - \mu)^{k-1} \sigma(\lambda + (k-1)\mu)}{\sigma(\lambda)^k} \sum_{\substack{I \subset \{1, \ldots, N\} \\ |I| = k}} \exp\left(\sum_{i \in I} \theta_i\right)$$
$$\times \prod_{i \in I,\ j \notin I} (\sigma^2(\mu)[\mathscr{P}(\mu) - \mathscr{P}(q_i - q_j)])^{1/2} . \tag{3.16}$$

These formulas readily follow from (3.13)–(3.15) by using the identity

$$|C| = \frac{\sigma(\lambda - \mu)^{N-1} \sigma(\lambda + (N-1)\mu)}{\sigma(\lambda)^N \sigma(\mu)^{N(N-1)}} \prod_{1 \leq i < j \leq N} \frac{1}{\mathscr{P}(\mu) - \mathscr{P}(q_i - q_j)} \tag{3.17}$$

(from which the above-announced "principal minor property" of C can also be read off). This identity is a consequence of the more general identity

$$\left|\left(\frac{\sigma(q_i-r_j+v)}{\sigma(q_i-r_j)}\right)_{N\times N}\right| = \sigma(v)^{N-1}\sigma\left(v+\sum_{i=1}^{N}(q_i-r_i)\right)$$
$$\times \prod_{1\leqq i<j\leqq N}\sigma(q_i-q_j)\sigma(r_j-r_i)\prod_{i,j=1}^{N}\frac{1}{\sigma(q_i-r_j)}. \quad (3.18)$$

Indeed, (3.17) follows by setting $r=q-\mu$, $v=\lambda-\mu$ in (3.18) and then using (2.11). The proof of the latter identity is relegated to Appendix B, where we also consider various other specializations of interest.

Let us now complete the picture by considering classical analogs of the issues (i)–(iv) in Sect. 2 and, last but not least, by discussing the "correctness" of the above elliptic Lax matrix.

(i) *(Reality)* Due to our standing assumption that ω and $-i\omega'$ are positive (cf. Sect. 1), we can ensure real-valuedness of the functions $\Sigma_1, \ldots, \Sigma_N$ for $q, \theta \in \mathbb{R}^N$ by choosing λ and μ on the imaginary axis. This choice also entails that C is self-adjoint, cf. (3.14). Thus, one can get a self-adjoint Lax matrix by taking $D^{1/2}CD^{1/2}$ instead of DC, cf. (3.13). Recall that the choice amounts to the restriction (2.12) on the parameters a and b. Of course, the restriction can be considerably relaxed if one only demands reality of the functions $S_1, \ldots, S_N$: They are real (in fact, positive) whenever $a+b\mathscr{P}(q)$ is positive on $\mathbb{R}$, i.e., when $b\geqq 0$ and $a\geqq -be_1$, cf. (1.4)–(1.5).

(ii) *(Degenerate cases)* As the generalized hyperbolic Lax matrix we can take

$$L_{ij}=e^{\theta_i}\prod_{\ell\neq i}\left(1-\frac{\operatorname{sh}^2 v\mu}{\operatorname{sh}^2 v(q_i-q_\ell)}\right)^{1/2}\frac{\operatorname{sh} v(q_i-q_j+\lambda)}{\operatorname{sh} v\lambda}\,\frac{\operatorname{sh} v\mu}{\operatorname{sh} v(q_i-q_j+\mu)}. \quad (3.19)$$

Its symmetric functions are given by

$$\Sigma_k=\frac{(\operatorname{sh} v(\lambda-\mu))^{k-1}\operatorname{sh} v(\lambda+(k-1)\mu)}{(\operatorname{sh} v\lambda)^k}\sum_{\substack{I\subset\{1,\ldots,N\}\\|I|=k}}\exp\left(\sum_{i\in I}\theta_i\right)$$
$$\times\prod_{i\in I,\ j\notin I}\left(1-\frac{\operatorname{sh}^2 v\mu}{\operatorname{sh}^2 v(q_i-q_j)}\right)^{1/2}, \quad (3.20)$$

cf. (B28). For the trigonometric and rational cases one needs only replace sh by sin, and send v to 0, respectively.

(iii) *(Relativistic invariance)* As the classical analogs of (2.19)–(2.21) we may take

$$H=mc^2\sum_{i=1}^{N}\operatorname{ch}(\theta_i/mc)V_i(q), \quad (3.21)$$

$$P=mc\sum_{i=1}^{N}\operatorname{sh}(\theta_i/mc)V_i(q), \quad (3.22)$$

$$B=-m\sum_{i=1}^{N}q_i, \quad (3.23)$$

where

$$V_i(q)\equiv\prod_{j\neq i}\left(\sigma^2\left(\frac{ig}{mc}\right)\left[\mathscr{P}\left(\frac{ig}{mc}\right)-\mathscr{P}(q_i-q_j)\right]\right)^{1/2}. \quad (3.24)$$

Note that the difference with Eqs. (1.1)–(1.4) consists only in the adoption of position and momentum dimensions for q and θ, and in the parametrization of the potential. As in Sect. 2, these conventions simplify the nonrelativistic limit, which we consider next.

iv) *(Nonrelativistic limit)* Using the fact that the functions $\varepsilon^{-2}\sigma(\varepsilon)^2-1$ and $\varepsilon^2\mathscr{P}(\varepsilon)-1$ are $O(\varepsilon^4)$ for $\varepsilon\to 0$, one gets

$$H_{\mathrm{nr}}=\lim_{c\to\infty}(H-Nmc^2)$$
$$=\sum_{i=1}^{N}\frac{\theta_i^2}{2m}+\frac{g^2}{m}\sum_{1\leq i<j\leq N}\mathscr{P}(q_i-q_j), \tag{3.25}$$

$$P_{\mathrm{nr}}=\lim_{c\to\infty}P=\sum_{i=1}^{N}\theta_i, \tag{3.26}$$

$$B_{\mathrm{nr}}=\lim_{c\to\infty}B=-m\sum_{i=1}^{N}q_i. \tag{3.27}$$

Note that one can take g in H_{nr} purely imaginary without violating reality, whereas this choice would lead to a loss of generality in (3.21) due to the branch points at $q_i-q_j=\pm ig/mc$ [cf. also (i)].

Taking $m=1$ henceforth, let us set

$$\theta_i\to\theta_i/c,\qquad \mu\to ig/c \tag{3.28}$$

in the elliptic Lax matrix (3.13)–(3.15). Denoting the result by $L(c)$, one readily verifies that

$$L(c)=\mathbb{1}+L_{\mathrm{nr}}/c+O(1/c^2),\qquad c\to\infty, \tag{3.29}$$

where

$$(L_{\mathrm{nr}})_{ij}\equiv\delta_{ij}\theta_j+ig(1-\delta_{ij})\frac{\sigma(q_i-q_j+\lambda)}{\sigma(\lambda)\sigma(q_i-q_j)}. \tag{3.30}$$

The matrix L_{nr} is in essence Krichever's [11] Lax matrix for the nonrelativistic elliptic case, special cases of which were first found by Calogero [12]. From (3.29) it follows as in [1, Chap. 4] that its symmetric functions are given by

$$S_{k,\mathrm{nr}}=\lim_{c\to\infty}G_k(c), \tag{3.31}$$

where

$$G_k(c)\equiv c^k\sum_{\ell=0}^{k}(-)^{k+\ell}\binom{N-\ell}{N-k}\Sigma_\ell(c). \tag{3.32}$$

Here, $\Sigma_\ell(c)$ denotes the symmetric functions of $L(c)$, explicitly given by (3.16) with the substitutions (3.28). It follows that $S_{1,\mathrm{nr}},\ldots,S_{N,\mathrm{nr}}$ are in involution and depend only on the momenta θ_i and the potentials $\mathscr{P}(q_i-q_j)$, something which is far from obvious from (3.30).

As promised below (2.25), let us briefly return to the question whether an analog of (3.31)–(3.32) holds true on the quantum level. As is well known (and easily verified), the quantization prescription (2.1) yields unambiguous, formally hermitian operators $\hat{S}_{1,\mathrm{nr}}, \ldots, \hat{S}_{N,\mathrm{nr}}$. The point is, that no ordering problems occur in the nonrelativistic case. If one *ignores* the ordering in $\hat{\Sigma}_1(c), \ldots, \hat{\Sigma}_N(c)$, then (3.31) *would* hold at the quantum level. However, since $\hat{\theta}_i$ and $h(q_i - q_j)^{1/2}$ do not commute, one gets additional terms, as we have already seen [recall (2.23) and (3.25)]. A priori, these terms might spoil the existence of $\lim_{c\to\infty} \hat{G}_k(c)$, but this turns out not to happen for small k. Indeed, it is straightforward to verify that for $k \leqq 3$ the limit exists, yielding the "renormalized" operators

$$\hat{S}^{\mathrm{ren}}_{0,\mathrm{nr}} = 1\,, \qquad \hat{S}^{\mathrm{ren}}_{1,\mathrm{nr}} = \sum_i \hat{\theta}_i\,, \tag{3.33}$$

$$\hat{S}^{\mathrm{ren}}_{2,\mathrm{nr}} = \sum_{i<j} \hat{\theta}_i \hat{\theta}_j - g(g-1) \sum_{i<j} \mathscr{P}(q_i - q_j) + \frac{g^2}{2} N(N-1)\mathscr{P}(\lambda)\,, \tag{3.34}$$

$$\hat{S}^{\mathrm{ren}}_{3,\mathrm{nr}} = \sum_{i<j<k} \hat{\theta}_i \hat{\theta}_j \hat{\theta}_k - g(g-1) \sum_{\substack{i\neq j,k \\ j<k}} \mathscr{P}(q_j - q_k)\hat{\theta}_i + \frac{g^2}{2}(N-1)(N-2)\mathscr{P}(\lambda)\sum_i \hat{\theta}_i + \frac{ig^3}{6} N(N-1)(N-2)\mathscr{P}'(\lambda)\,. \tag{3.35}$$

The only change compared to the symmetric functions of $\hat{L}_{\mathrm{nr}}$ is the replacement of g^2 by $g(g-1)$ in the second term at the right-hand side of (3.34) and (3.35). We expect a similar behavior for arbitrary k, but have not found a proof.

(v) *(The role of the Lax matrix)* As evinced by the above developments, our elliptic Lax matrix L leads to several useful insights that would be hard to obtain from a direct consideration of the Poisson commuting functions $S_1, \ldots, S_N$. However, the above does not answer the question whether L can be used to give an explicit construction of the action-angle map (whose *existence*, it should be recalled, follows from the Liouville-Arnold theorem). In this connection the obvious guess is that the "extra" parameter λ plays the same role as in Krichever's [11] treatment of the nonrelativistic elliptic case.

To study this, let us consider (following Krichever) the transcendental curve Γ_N, defined by setting

$$R(\alpha, \lambda) = 0\,, \tag{3.36}$$

where

$$R(\alpha, \lambda) \equiv |L(\lambda) + \alpha \mathbb{1}| = \sum_{\ell=0}^{N} \alpha^{\ell} \Sigma_{N-\ell}(\lambda)\,. \tag{3.37}$$

We may view Γ_N as an N-fold cover of the torus T^2, since the functions $\Sigma_k(\lambda)$ are meromorphic on T^2, cf. (3.16), (1.7). From (3.16) we also infer that near $\lambda = 0$ one has

$$R(\alpha, \lambda) \sim \lambda^{-N}[(\alpha\lambda)^N + (\alpha\lambda)^{N-2} c_2(\mu) S_2 + \ldots + c_N(\mu) S_N]\,. \tag{3.38}$$

Putting $z \equiv \alpha\lambda$, the polynomial in brackets has N non-zero distinct roots $z_1, \ldots, z_N$ for μ, q, θ in general position. Consequently, Γ_N has no branch points over $\lambda = 0$ and one must have (generically)

$$R(\alpha, \lambda) = \prod_{\ell=1}^{N} \left(\alpha - \frac{z_\ell}{\lambda} + f_\ell(\lambda) \right), \qquad z_\ell \neq 0\,, \qquad z_i \neq z_j \tag{3.39}$$

near $\lambda=0$, where $f_1,\ldots,f_N$ are regular. It follows from this that the functions $\partial_\alpha R$ and $\partial_\lambda R$ have poles of order $N-1$ and $N+1$, respectively, at each of the N points lying over $\lambda=0$.

Let us now assume that μ, q, θ are such that Γ_N is irreducible. (For $N=2$ this is easily seen to be the generic situation; for arbitrary N this is probably true as well.) Then we may regard Γ_N as a closed Riemann surface. The functions $p_1:(\alpha,\lambda)\to\alpha$, $p_2:(\alpha,\lambda)\to\lambda$ are holomorphic from Γ_N onto $\mathbb{C}\cup\{\infty\}$ and T^2, respectively, and both have degree N. Assuming that $\nabla R\neq 0$ on Γ_N (which, again, should be true in general), the total branch number of p_1 and p_2 equals the degree of $\partial_\lambda R$ and $\partial_\alpha R$, respectively, viewed as holomorphic functions from Γ_N onto the Riemann sphere. Thus, by virtue of the above pole count these numbers equal $N(N+1)$ and $N(N-1)$, respectively. Applying now the Riemann-Hurwitz relation to p_1 or p_2 one concludes that the genus of Γ_N equals $N(N-1)/2+1$.

For $N=2$ this is the same result as in the nonrelativistic case; in fact, Γ_2 is not essentially different from $(\Gamma_2)_{\text{nr}}$, as is readily seen. Also, the motion of q_1-q_2 under the S_1 flow is in essence the same as under the H_{nr} flow, cf. [1, Eq. (2.14)]. Thus, we expect that for $N=2$ the Jacobian variety $J(\Gamma_N)$ gives rise to an explicit model of the invariant tori, as in the nonrelativistic case. However, for $N>2$ the genus of Γ_N is greater than N, whereas it equals N for $(\Gamma_N)_{\text{nr}}$ [11]. Thus, although the flows generated by $S_1,\ldots,S_N$ might still linearize on $J(\Gamma_N)$ under an appropriate map, one cannot fill out the Jacobian as in the nonrelativistic case.

Unfortunately, these somewhat sketchy remarks are all we have to offer concerning the connection between our Lax matrix and the action-angle map in the general case. In support of our conjecture that such a connection should exist, let us point out once more that for special parameter values L reduces to Lax matrices that are *known* to yield action-angle maps [10, 11]. Moreover, there appear to be no examples of Lax matrices [but for our contrived example (3.3) above] that are unrelated to the action-angle map; in fact, for many other integrable systems with compact level sets (e.g. the generalized periodic Toda systems) the flows linearize on the Jacobian of a curve whose relation to the Lax matrix is defined via (analogs of) (3.36)–(3.37) [13].

Appendix A. Commutativity and Functional Equations

In this appendix we first prove that the operators $\hat{S}_1,\ldots,\hat{S}_N$ defined by (2.3) commute for any N if and only if the function h satisfies the functional equations (2.4). This is the content of Theorem A1. In Theorem A2 we show that $h=\sigma(q+\mu)/\sigma(q)$ obeys these identities. As a consequence it follows that the function $a+b\mathcal{P}(q)$ satisfies the identities expressing classical commutativity (Corollary A3).

It is convenient to employ the following notation. Let I, J be disjoint subsets of $\{1,\ldots,N\}$ and let h be a meromorphic function. Then we set

$$(IJ)\equiv \prod_{\substack{i\in I\\ j\in J}} h(q_i-q_j)^{1/2}, \tag{A1}$$

$$\hat{\theta}_I\equiv \sum_{i\in I}\hat{\theta}_i, \tag{A2}$$

so that the operators (2.3) can be written

$$\hat{S}_k = \sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=k}} (I^cI)e^{\beta\hat{\theta}_I}(II^c), \qquad k=1,\ldots,N. \tag{A3}$$

We also introduce

$$\hat{S}_{-k} \equiv \sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=k}} (II^c)e^{-\beta\hat{\theta}_I}(I^cI), \qquad k=1,\ldots,N, \tag{A4}$$

Finally, we set

$$(I_\varepsilon J_{\varepsilon'}) \equiv \exp[-\beta(\varepsilon\hat{\theta}_I+\varepsilon'\hat{\theta}_J)](IJ)\exp[\beta(\varepsilon\hat{\theta}_I+\varepsilon'\hat{\theta}_J)], \qquad \varepsilon,\varepsilon'=+,-. \tag{A5}$$

Thus, a $-/+$ on a set denotes shifting down/up all q in the set by $i\beta$, cf. (2.2). Note also that

$$(I_\varepsilon J_\varepsilon) = (IJ), \tag{A6}$$

$$(I_+J) = (IJ_-). \tag{A7}$$

We are now prepared for Theorem A1, whose proof is patterned after [1, Theorem A1].

Theorem A1. *One has*

$$[\hat{S}_k, \hat{S}_\ell] = 0, \qquad \forall (k,\ell)\in\{1,\ldots,N\}^2, \qquad \forall N>1, \tag{A8}$$

if and only if

$$\sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=k}} ((I^cI)^2(I_-I^c)^2-(I^c_-I)^2(II^c)^2) = 0, \qquad \forall k\in\{1,\ldots,N\}, \qquad \forall N>1. \tag{A9}$$

Proof. Due to (A6) the operators

$$\hat{S}_{\pm N} = \exp(\pm\beta\hat{\theta}_{\{1,\ldots,N\}}) \tag{A10}$$

commute with $\hat{S}_k$, $k=\pm 1,\ldots,\pm(N-1)$. It is also readily verified that

$$\hat{S}_{k-N} = \hat{S}_k\hat{S}_{-N}, \qquad k=1,\ldots,N-1. \tag{A11}$$

Hence, (A8) is equivalent to

$$[\hat{S}_k, \hat{S}_{-\ell}] = 0, \qquad \forall (k,\ell)\in\{1,\ldots,N-1\}^2, \qquad \forall N>1. \tag{A12}$$

Next, we use (A3), (A4) to obtain

$$[\hat{S}_k, \hat{S}_{-\ell}] = \sum_{\substack{|I|=k\\ |J|=\ell}} ((I^cI)e^{\beta\hat{\theta}_I}(II^c)(JJ^c)e^{-\beta\hat{\theta}_J}(J^cJ) - (JJ^c)e^{-\beta\hat{\theta}_J}(J^cJ)(I^cI)e^{\beta\hat{\theta}_I}(II^c)). \tag{A13}$$

Introducing the pairwise disjoint sets

$$A\equiv I\backslash J, \qquad B\equiv J\backslash I, \qquad C\equiv I\cap J, \qquad D\equiv(I\cup J)^c, \tag{A14}$$

one gets after a straightforward calculation using (A6) and (A7),

$$[\hat{S}_k, \hat{S}_{-\ell}] = \sum_{\substack{|I|=k \\ |J|=\ell}} (B, C\cup D)(C\cup D, A)(BA)(A_-B)(BA_-) \times ((DC)^2(C_-D)^2 - (D_-C)^2(CD)^2) \times e^{\beta(\hat{\theta}_A - \hat{\theta}_B)}(AB)(C\cup D, B)(A, C\cup D). \quad \text{(A15)}$$

Clearly, the terms in the sum involving $\exp\beta(\hat{\theta}_A - \hat{\theta}_B)$ sum to zero if and only if the bracketed expression vanishes when summed over all disjoint C, D with $C\cup D = (A\cup B)^c$, $|C| = k - |A| = \ell - |B|$, $|D| = N - k - |B| = N - \ell - |A|$. This entails the equivalence of (A12) and (A9). □

This theorem has a corollary whose statement and proof amount to making some obvious changes in Corollary A2 of [1], so that we shall not spell it out. However, there appears to be no quantum analog of Lemma A3 in [1]. For instance, $h(q) = 1/\mathrm{sh}\, q$ obeys (A9) in view of the following theorem, whereas $h_c(q) = 1/\mathrm{sh}\, q + c$ violates (A9) with $k=1$ and $N=3$, as is readily verified. This holds true in spite of the fact that $h_c(q)h_c(-q)$ gives rise to an integrable classical potential.

Theorem A2. *The function*

$$h(q) = c_1 e^{c_2 q} \frac{\sigma(q+\mu)}{\sigma(q)}, \qquad c_1, c_2, \mu \in \mathbb{C} \quad \text{(A16)}$$

(where σ is the Weierstrass sigma-function) satisfies the functional equations (A9).

Proof. We need only consider the case $c_1 = 1$, $c_2 = 0$.

Let us introduce

$$E_{k,N}(q_1, \ldots, q_N) \equiv \sum_{\substack{I\subset\{1,\ldots,N\} \\ |I|=k}} \left[\prod_{i\in I,\ j\notin I} \frac{\sigma(q_j - q_i + \mu)\sigma(q_j - q_i - \mu - \lambda)}{\sigma(q_j - q_i)\sigma(q_j - q_i - \lambda)} - (q \to -q) \right]. \quad \text{(A17)}$$

Then our claim is equivalent to the assertion that E vanishes for arbitrary $N > 1$, $k \in \{1, \ldots, N\}$, $q \in \mathbb{C}^N$, $\lambda, \mu \in \mathbb{C}$. To prove this assertion, we begin by noting that E is doubly periodic in each q_j in view of (1.7). Since E is a symmetric function of $q_1, \ldots, q_N$ satisfying

$$E(-q) = -E(q), \quad \text{(A18)}$$

we need only show that E, viewed as a function of q_1, is pole-free. Indeed, Liouville's theorem then entails that E does not depend on q, and zero is the only constant satisfying (A18).

To prove absence of poles in the variable q_1, we fix the remaining variables in general position. Specifically, we choose the points $\lambda, 2\lambda, q_j - q_i, q_j - q_i \pm \lambda, j > i > 1$, incongruent to 0. Note that this ensures that the terms in the sum have at most simple poles in q_1. By double periodicity, symmetry and oddness we need only show that the residue sum at the two points 1) $q_1 = q_2$ and 2) $q_1 = q_2 - \lambda$ vanishes. To this end we pair off the singular I in (A17), i.e., we consider $I = \{1\}\cup J$, $I = \{2\}\cup J$ with $1, 2 \notin J$. For such a pair the residues at 1) of the two "left" products cancel.

Indeed, if one omits the singular factors $\sigma(q_2-q_1)^{-1}$ and $\sigma(q_1-q_2)^{-1} = -\sigma(q_2-q_1)^{-1}$ in the first and second case, respectively, and then puts $q_1=q_2$ in the remaining products, then these products are manifestly equal. Similarly, the residues coming from the "right" products cancel at 1).

To handle the residue sum at 2), we use induction on k. First, let $k=1$. For $I=\{1\}$ only the left product has a pole, whereas for $I=\{2\}$ only the right product does (recall $2\lambda \neq 0$). Hence the residue sum equals

$$
\begin{aligned}
&-\frac{\sigma(q_2-q_1+\mu)\sigma(q_2-q_1-\mu-\lambda)}{\sigma(q_2-q_1)} \prod_{j>2} \frac{\sigma(q_j-q_1+\mu)\sigma(q_j-q_1-\mu-\lambda)}{\sigma(q_j-q_1)\sigma(q_j-q_1-\lambda)} \\
&+\frac{\sigma(-q_1+q_2+\mu)\sigma(-q_1+q_2-\mu-\lambda)}{\sigma(-q_1+q_2)} \prod_{j>2} \frac{\sigma(-q_j+q_2+\mu)\sigma(-q_j+q_2-\mu-\lambda)}{\sigma(-q_j+q_2)\sigma(-q_j+q_2-\lambda)}
\end{aligned}
\tag{A19}
$$

evaluated at $q_1=q_2-\lambda$, which indeed vanishes.

Now assume

$$
E_{k-1,N}=0, \qquad \forall N \geqq k-1, \tag{A20}
$$

and consider $E_{k,N}$. The residue sum at $q_1=q_2-\lambda$ is then equal to

$$
\begin{aligned}
&\sum_{\substack{J\subset\{3,\dots,N\}\\ |J|=k-1}} \Bigg[-\frac{\sigma(\lambda+\mu)\sigma(-\mu)}{\sigma(\lambda)} \prod_{\substack{j\notin J\\ j>2}} \frac{\sigma(q_j-q_2+\lambda+\mu)\sigma(q_j-q_2-\mu)}{\sigma(q_j-q_2+\lambda)\sigma(q_j-q_2)} \\
&\quad \times \prod_{i\in J} \frac{\sigma(q_2-q_i+\mu)\sigma(q_2-q_i-\mu-\lambda)}{\sigma(q_2-q_i)\sigma(q_2-q_i-\lambda)} \prod_{\substack{i\in J,\ j\notin J\\ j>2}} \frac{\sigma(q_j-q_i+\mu)\sigma(q_j-q_i-\mu-\lambda)}{\sigma(q_j-q_i)\sigma(q_j-q_i-\lambda)} \\
&\quad + \frac{\sigma(\lambda+\mu)\sigma(-\mu)}{\sigma(\lambda)} \prod_{\substack{j\notin J\\ j>2}} \frac{\sigma(-q_j+q_2+\mu)\sigma(-q_j+q_2-\mu-\lambda)}{\sigma(-q_j+q_2)\sigma(-q_j+q_2-\lambda)} \\
&\quad \times \prod_{i\in J} \frac{\sigma(-q_2+\lambda+q_i+\mu)\sigma(-q_2+q_i-\mu)}{\sigma(-q_2+\lambda+q_i)\sigma(-q_2+q_i)} \\
&\quad \times \prod_{\substack{i\in J,\ j\notin J\\ j>2}} \frac{\sigma(-q_j+q_i+\mu)\sigma(-q_j+q_i-\mu-\lambda)}{\sigma(-q_j+q_i)\sigma(-q_j+q_i-\lambda)} \Bigg] \\
&= \frac{\sigma(\lambda+\mu)}{\sigma(\lambda)} \prod_{\ell>2} \frac{\sigma(q_\ell-q_2+\lambda+\mu)\sigma(q_\ell-q_2-\mu)}{\sigma(q_\ell-q_2+\lambda)\sigma(q_\ell-q_2)} \\
&\quad \times \sum_{\substack{J\subset\{3,\dots,N\}\\ |J|=k-1}} \Bigg[\prod_{\substack{i\in J,\ j\notin J\\ j>2}} \frac{\sigma(q_j-q_i+\mu)\sigma(q_j-q_i-\mu-\lambda)}{\sigma(q_j-q_i)\sigma(q_j-q_i-\lambda)} - (q_\ell \to -q_\ell,\ \ell>2) \Bigg] .
\end{aligned}
\tag{A21}
$$

However, by virtue of the induction hypothesis (A20) the last sum vanishes, completing the proof. □

Corollary A3. *Let*

$$
F(q) \equiv a + b\mathscr{P}(q), \qquad a,b \in \mathbb{C}, \tag{A22}
$$

where $\mathscr{P}$ is the Weierstrass $\mathscr{P}$-function. Then one has

$$\sum_{\substack{I\subset\{1,\ldots,N\}\\ |I|=k}}\left(\sum_{i\in I}\partial_i\right)\prod_{\substack{i\in I\\ j\notin I}}F(q_i-q_j)=0\,,\qquad \forall k\in\{1,\ldots,N\}\,,\qquad \forall N>1\,. \tag{A23}$$

Proof. We have just seen that the function E given by (A17) vanishes identically. Dividing E by λ, sending λ to 0 and using (2.11) one arrives at (A23). □

Appendix B. Generalized Cauchy Identities

In this appendix we prove the identity (3.18) and then derive various special cases of interest. An ingredient of the proof is the following fact, which is a special case of the Weinstein-Aronszajn formula [14]. For completeness we include a proof.

Lemma B1. *Let M be a regular $N\times N$ matrix and let*

$$\tilde{M}\equiv M+u\otimes v\,, \tag{B1}$$

where $u,v\in\mathbb{C}^N$. Then one has

$$|\tilde{M}|=|M|[1+(v,M^{-1}u)]\,. \tag{B2}$$

Proof. Since

$$M^{-1}\tilde{M}=\mathbb{1}+(M^{-1}u)\otimes v\,, \tag{B3}$$

we need only show

$$|\mathbb{1}+w\otimes v|=1+(v,w)\,. \tag{B4}$$

This is clear when v and w are proportional, so let us assume w and v are linearly independent. With respect to a base $u_1\equiv w$, $u_2\equiv v$, $u_j\perp v$, $j=3,\ldots,N$ of $\mathbb{C}^N$ the matrix of $\mathbb{1}+w\otimes v$ is triangular with diagonal elements $1+(v,w),1,\ldots,1$. Hence, (B4) follows. □

Theorem B2. *Let $q_1,\ldots,q_N,r_1,\ldots,r_N,\lambda,\mu\in\mathbb{C}$ and let σ denote the Weierstrass sigma-function. Let C denote the matrix with elements*

$$C_{ij}=\frac{\sigma(\mu)}{\sigma(\lambda)}\,\frac{\sigma(q_i-r_j+\lambda)}{\sigma(q_i-r_j+\mu)}\,,\qquad i,j=1,\ldots,N\,. \tag{B5}$$

Then one has

$$\begin{aligned}|C|=&\left(\frac{\sigma(\mu)}{\sigma(\lambda)}\right)^N\sigma(\lambda-\mu)^{N-1}\sigma(\lambda+(N-1)\mu+\Sigma)\\ &\times\prod_{i<j}\sigma(q_i-q_j)\sigma(r_j-r_i)\prod_{i,j}\frac{1}{\sigma(q_i-r_j+\mu)}\,,\end{aligned} \tag{B6}$$

where

$$\Sigma\equiv\sum_{i=1}^N(q_i-r_i)\,. \tag{B7}$$

Proof. Let us introduce the auxiliary function

$$A(v) \equiv \frac{\sigma(v)}{\sigma(v+\Sigma)} \left| \left(\frac{\sigma(q_i - r_j + v)}{\sigma(v)\sigma(q_i - r_j)} \right) \right|, \qquad v \in \mathbb{C}. \tag{B8}$$

Using the quasi-periodicity relation (1.7) one infers that

$$\begin{aligned} A(v+2\omega_k) &= \frac{\sigma(v)}{\sigma(v+\Sigma)} \exp[-2\eta_k \Sigma] \left| \left(\frac{\exp[2\eta_k(q_i - r_j)]\sigma(q_i - r_j + v)}{\sigma(v)\sigma(q_i - r_j)} \right) \right| \\ &= A(v). \end{aligned} \tag{B9}$$

Hence, A is doubly periodic. It is clear from (B8) that poles of A can only occur for $v \equiv 0$ and $v \equiv -\Sigma$.

We now assert that A is actually regular at $v=0$ for q, r in general position. To prove this assertion, we write

$$A = \frac{\sigma(v)}{\sigma(v+\Sigma)} \left| M + \frac{1}{\sigma(v)} e \otimes e \right|, \tag{B10}$$

where

$$e \equiv (1, 1, \ldots, 1), \tag{B11}$$

$$M_{ij} \equiv \frac{1}{\sigma(v)} \left[\frac{\sigma(q_i - r_j + v)}{\sigma(q_i - r_j)} - 1 \right], \tag{B12}$$

and note that due to (1.10) one has

$$\lim_{v \to 0} M = (\zeta(q_i - r_j)) \equiv M_0 . \tag{B13}$$

In order to invoke Lemma B1, we now show that $|M_0|$ (and hence $|M|$, too) does not vanish identically. Indeed, let us set

$$q_k = \frac{k\omega}{N}, \qquad r_k = \frac{k\omega}{N} - \delta\omega, \qquad k = 1, \ldots, N. \tag{B14}$$

If we expand $|M_0|$, then the term coming from the product of all diagonal elements equals $\zeta(\delta\omega)^N$. Thus it blows up like δ^{-N} for $\delta \to 0$. Since the off-diagonal elements have finite limits for $\delta \to 0$, the other terms in the expansion cannot cancel it for $\delta \to 0$. Thus, M_0 and M are generically regular, as claimed.

By virtue of Lemma B1, it then follows that

$$A = |M| [\sigma(v) + (e, M^{-1} e)] / \sigma(v+\Sigma). \tag{B15}$$

Therefore, one obtains

$$\lim_{v \to 0} A = |M_0| (e, M_0^{-1} e) / \sigma(\Sigma). \tag{B16}$$

Since $|M_0|(e, M_0^{-1} e)$ is equal to the sum of all cofactors of M_0 [recall (B11)], it follows that $A(v)$ has no pole at 0 whenever $\Sigma \not\equiv 0$ and $q_i - r_j \not\equiv 0$ [recall (B13)]. Thus the above assertion is proved.

It follows that A can only have poles when $v \equiv -\Sigma$. However, it is obvious from the definition (B8) that the order of these poles is at most one. Since A is doubly periodic, it follows that the residue at these poles vanishes. Thus A is everywhere regular, so that $A = K(q, r)$ by Liouville's theorem. We have, therefore, proved that

$$\left|\left(\frac{\sigma(q_i - r_j + v)}{\sigma(v)\sigma(q_i - r_j)}\right)\right| = \frac{\sigma(v + \Sigma)}{\sigma(v)} K(q, r). \tag{B17}$$

We proceed by determining the q_1-dependence of $K(q, r)$. To this end we introduce

$$\varphi(q_1) \equiv \frac{\sigma(q_1 - v)}{\sigma(q_1)} \left|\left(\frac{\sigma(q_i - r_j + v)}{\sigma(v)\sigma(q_i - r_j)}\right)\right|. \tag{B18}$$

This function is doubly periodic in view of (1.7). Picking $v \neq 0$, one easily verifies that φ has zeros at

$$q_1 = \sum_{j=1}^{N} r_j - \sum_{j=2}^{N} q_j - v, \quad q_1 = v, \quad q_1 = q_j, \quad j = 2, \ldots, N, \tag{B19}$$

the first zero being a consequence of (B17). Generically, these points are incongruent and $\sigma(q_i - r_j)$ does not vanish. Hence, the order of $\varphi(q_1)$ is at least $N+1$ for $q_2, \ldots, q_N, r_1, \ldots, r_N$ in general position.

On the other hand, φ has at most $N+1$ incongruent simple poles, e.g. at

$$q_1 = 0, \quad q_1 = r_j, \quad j = 1, \ldots, N. \tag{B20}$$

Thus it follows that φ is generically of order $N+1$. Moreover, since the sum of the zeros (B19) equals the sum of the poles (B20) one must have

$$\varphi(q_1) = K(v, q_2, \ldots, q_N, r) \frac{\sigma(v + \Sigma)\sigma(q_1 - v) \prod_{j>1} \sigma(q_1 - q_j)}{\sigma(q_1) \prod_j \sigma(q_1 - r_j)}. \tag{B21}$$

Comparing this with (B18) and (B17), we conclude that

$$K(q, r) = K(q_2, \ldots, q_N, r) \prod_{j>1} \sigma(q_1 - q_j) \prod_j \frac{1}{\sigma(q_1 - r_j)}. \tag{B22}$$

Evidently, the dependence on $q_2, \ldots, q_N$ and $r_1, \ldots, r_N$ can be determined analogously, yielding

$$K(q, r) = K \prod_{i<j} \sigma(q_i - q_j)\sigma(r_j - r_i) \prod_{i,j} \frac{1}{\sigma(q_i - r_j)}. \tag{B23}$$

Next, we substitute this in (B17), after which we replace v by $\lambda - \mu$ and r by $r - \mu$. Then the result can be rewritten as (B 6), but for an extra constant K at the right-hand side. Thus it remains to prove that $K = 1$. To show this, we first set $q = r$, which implies $C_{jj} = 1$. Taking then μ to 0, the off-diagonal elements of C go to 0, so that $|C| \to 1$ for $\mu \to 0$. However, if we set $q = r$ at the right-hand side of (B 6) and then take μ to 0, we also get 1 as limit. Hence it follows that $K = 1$, so that the proof is complete. □

We proceed by pointing out some interesting special cases of the identity (B6). First, if we multiply by $\sigma(\lambda)^N$ or $\sigma(\mu)^{-N}$ and then send λ or μ, respectively, to 0, we obtain

$$\left|\left(\frac{\sigma(\mu)\sigma(q_i-r_j)}{\sigma(q_i-r_j+\mu)}\right)\right| = (-)^{N-1}\sigma(\mu)^{2N-1}\sigma((N-1)\mu+\Sigma) \times \prod_{i<j}\sigma(q_i-q_j)\sigma(r_j-r_i)\prod_{i,j}\frac{1}{\sigma(q_i-r_j+\mu)}, \tag{B24}$$

$$\left|\left(\frac{\sigma(q_i-r_j+\lambda)}{\sigma(\lambda)\sigma(q_i-r_j)}\right)\right| = \frac{\sigma(\lambda+\Sigma)}{\sigma(\lambda)}\prod_{i<j}\sigma(q_i-q_j)\sigma(r_j-r_i)\prod_{i,j}\frac{1}{\sigma(q_i-r_j)}. \tag{B25}$$

Let us now recall that

$$\frac{\sigma(v)\sigma(x)}{\sigma(x+v)} = \begin{cases} e^{-\eta_3 x} snax/a, & v=\omega_3, \\ e^{-\eta_2 x} snax/adnax, & v=\omega_2, \\ e^{-\eta_1 x} snax/acnax, & v=\omega_1. \end{cases} \tag{B.26}$$

where $a=(e_1-e_3)^{1/2}$. Thus we can obtain explicit formulas for the determinants of the matrices $(J(q_i-r_j))$, with J one of the six odd Jacobian functions *sn*, *sd*, *sc*, *ns*, *ds*, *cs* by setting $\mu=\omega_k$, $\lambda=\omega_k$; formulas for the even ones follow by shifting r. The formula for $J=ds$ can also be deduced from recent work by Carey and Hannabuss, who study temperature states on loop groups [15].

In (B24) one can set in addition $q=r$. Then one gets (a similarity transform of) an antisymmetric matrix, and the factor $\sigma((N-1)\omega_k)$ at the right-hand side ensures that its determinant vanishes for N odd, as required. For $N=2M$ and $\mu=\omega_3$ we recover the formula

$$|(sn(u_i-u_j))_{2M\times 2M}| = k^{M(2M-1)}\prod_{1\leq i<j\leq 2M} sn^2(u_i-u_j) \tag{B27}$$

obtained first by Palmer and Tracy in their study of the Ising model correlation functions (cf. [15, pp. 376–377]). [To verify that (B27) follows from (B24), one needs the little known relation $\sigma(\omega_3)^{-4}=(e_1-e_3)(e_2-e_3)\exp(-2\eta_3\omega_3)$.]

Let us now derive various identities for the hyperbolic case $\omega=\infty$. Combining (B6) with (1.11), it is straightforward to verify that all exponentials cancel. Thus one concludes that

$$\left|\left(\frac{\operatorname{sh}\mu}{\operatorname{sh}\lambda}\frac{\operatorname{sh}(q_i-r_j+\lambda)}{\operatorname{sh}(q_i-r_j+\mu)}\right)\right| = \left(\frac{\operatorname{sh}\mu}{\operatorname{sh}\lambda}\right)^N \operatorname{sh}(\lambda-\mu)^{N-1}\operatorname{sh}(\lambda+(N-1)\mu+\Sigma) \times \prod_{i<j}\operatorname{sh}(q_i-q_j)\operatorname{sh}(r_j-r_i)\prod_{i,j}\frac{1}{\operatorname{sh}(q_i-r_j+\mu)}. \tag{B28}$$

To simplify special cases of this identity, we introduce the products

$$P_s \equiv \prod_{i<j}\operatorname{sh}(q_i-q_j)\operatorname{sh}(r_j-r_i)\prod_{i,j}\frac{1}{\operatorname{sh}(q_i-r_j)}, \tag{B29}$$

$$P_c \equiv \prod_{i<j}\operatorname{sh}(q_i-q_j)\operatorname{sh}(r_i-r_j)\prod_{i,j}\frac{1}{\operatorname{ch}(q_i-r_j)}, \tag{B30}$$

which are related by

$$P_s(r-i\pi/2)=i^{-N}P_c(r) \tag{B31}$$

(note that one has r_j-r_i in P_s and r_i-r_j in P_c).

If we take $\lambda\to\mu+i\pi/2$, $r\to r+\mu$ in (B28) we get

$$|(\operatorname{cth}(q_i-r_j))|=\operatorname{ch}(\Sigma)P_s. \tag{B32}$$

Taking $r\to r-i\pi/2$ in this and using (B31) yields

$$|(\operatorname{th}(q_i-r_j))|=\begin{cases}\operatorname{ch}(\Sigma)P_c, & N \text{ even},\\ \operatorname{sh}(\Sigma)P_c, & N \text{ odd}.\end{cases} \tag{B33}$$

Letting $\lambda\to\infty$ in (B28) one obtains, shifting r by μ,

$$\left|\left(\frac{1}{\operatorname{sh}(q_i-r_j)}\right)\right|=P_s, \tag{B34}$$

and shifting r by $-i\pi/2$ yields

$$\left|\left(\frac{1}{\operatorname{ch}(q_i-r_j)}\right)\right|=P_c. \tag{B35}$$

Also, letting $\mu\to\infty$ in (B28) and shifting r by λ results in

$$|(\operatorname{sh}(q_i-r_j))|=\begin{cases}\operatorname{sh}(q_1-q_2)\operatorname{sh}(r_1-r_2), & N=2,\\ 0, & N>2,\end{cases} \tag{B36}$$

and shifting r by $i\pi/2$ this yields

$$|(\operatorname{ch}(q_i-r_j))|=\begin{cases}\operatorname{sh}(q_1-q_2)\operatorname{sh}(r_2-r_1), & N=2,\\ 0, & N>2.\end{cases} \tag{B37}$$

Next, we observe that Lemma B1 can be used to infer that

$$\sum_{co}(\operatorname{cth}(q_i-r_j))=\operatorname{sh}(\Sigma)P_s, \tag{B38}$$

$$\sum_{co}(\operatorname{th}(q_i-r_j))=\begin{cases}\operatorname{sh}(\Sigma)P_c, & N \text{ even},\\ \operatorname{ch}(\Sigma)P_c, & N \text{ odd},\end{cases} \tag{B39}$$

where $\sum_{co}$ denotes the sum over all cofactors. Indeed, setting $M\equiv(\operatorname{cth}(q_i-r_j))$ we have, using the notation (B11),

$$\begin{aligned}\left|\left(\frac{1}{\operatorname{sh}\lambda}\frac{\operatorname{sh}(q_i-r_j+\lambda)}{\operatorname{sh}(q_i-r_j)}\right)\right|&=|M|(1+\operatorname{cth}\lambda(e,M^{-1}e))\\ &=|M|+\operatorname{cth}\lambda\sum_{co}M.\end{aligned} \tag{B40}$$

Combining this with (B32) and (B28) yields (B38), and (B39) then follows upon shifting r by $-i\pi/2$.

A particularly striking special case of (B39) is obtained by setting $q=r$:

$$\sum_{co}(\operatorname{th}(q_i-q_j))_{(2M+1)\times(2M+1)}=\prod_{1\leq i<j\leq 2M+1}\operatorname{th}^2(q_i-q_j). \tag{B41}$$

This should be compared with the identity

$$|(\mathrm{th}(q_i-q_j))_{2M\times 2M}|=\prod_{1\leq i<j\leq 2M}\mathrm{th}^2(q_i-q_j), \tag{B42}$$

which follows similarly from (B33) [or alternatively, by taking $k\to 1$ in the Palmer/Tracy identity (B27)].

Let us also note the formulas

$$\sum_{k,\ell=1}^{N}(\mathrm{cth}(q_i-r_j))^{-1}{}_{k\ell}=\mathrm{th}(\Sigma), \tag{B43}$$

$$\sum_{k,\ell=1}^{N}(\mathrm{th}(q_i-r_j))^{-1}{}_{k\ell}=\begin{cases}\mathrm{th}(\Sigma), & N \text{ even},\\ \mathrm{cth}(\Sigma), & N \text{ odd}.\end{cases} \tag{B44}$$

These follow upon combining (B32), (B38) and (B33), (B39), respectively.

Of course, trigonometric analogs of the above hyperbolic identities follow in the same way by using (1.12) or by taking $q, r\to iq, ir$. Moreover, replacing q, r, λ, μ by εq, εr, $\varepsilon\lambda$, $\varepsilon\mu$ in (B28) and sending $\varepsilon\to 0$ yields the rational identity

$$\left|\left(\frac{\mu}{\lambda}\frac{q_i-r_j+\lambda}{q_i-r_j+\mu}\right)\right|=\left(\frac{\mu}{\lambda}\right)^N(\lambda-\mu)^{N-1}(\lambda+(N-1)\mu+\Sigma)$$
$$\times\prod_{i<j}(q_i-q_j)(r_j-r_i)\prod_{i,j}\frac{1}{q_i-r_j+\mu}. \tag{B45}$$

Taking $\mu\to\infty$ and shifting r by λ yields

$$|(q_i-r_j)|=\begin{cases}(q_1-q_2)(r_1-r_2), & N=2,\\ 0, & N>2.\end{cases} \tag{B46}$$

Finally, letting $\lambda\to\infty$ and $r\to r+\mu$, we obtain Cauchy's identity,

$$\left|\left(\frac{1}{q_i-r_j}\right)\right|=\prod_{i<j}(q_i-q_j)(r_j-r_i)\prod_{i,j}\frac{1}{q_i-r_j}. \tag{B47}$$

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