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Complete interpolating sequences for Fourier transforms supported by convex symmetric polygons

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1. Introduction

We study sampling and interpolation for two-dimensional Fourier transforms. Given a convex domain $M\subset\mathbf{R}^2$, consider the corresponding Paley–Wiener space

$$PW_M=\left\{f\,;f(z)=\int_M e^{i\langle z,\xi\rangle}\phi(\xi)\,dm_\xi,\ \phi\in L^2(M)\right\} \tag{1}$$

endowed with the $L^2(\mathbf{R}^2)$-norm. Here $\mathbf{R}^2$ is considered as the real plane in $\mathbf{C}^2$, dm stands for the plane Lebesgue measure, and $\langle\cdot,\cdot\rangle$ is the $\mathbf{C}^2$-scalar product: $\langle z,\zeta\rangle=z_1\bar\zeta_1+z_2\bar\zeta_2$ for $z=(z_1,z_2)$, $\zeta=(\zeta_1,\zeta_2)\in\mathbf{C}^2$.

Following [2] and [8] we say that a sequence $\Omega=\{\omega\}\subset\mathbf{R}^2$ is *interpolating* for PW_M if for each $a=\{a_\omega\}_{\omega\in\Omega}\in l^2(\Omega)$ there exists $f\in PW_M$ solving the interpolation problem

$$f(\omega)=a_\omega,\quad \omega\in\Omega. \tag{2}$$

If the solution to this problem is always unique we say that Ω is a *complete interpolating sequence* for PW_M. It follows from the Banach inverse operator theorem that in this case Ω is also a *sampling* sequence, i.e.

$$A\|\{f(\omega)\}\|_{l^2}\le\|f\|_{PW_M}\le B\|\{f(\omega)\}\|_{l^2},\quad f\in PW_M,$$

for some $A,B>0$ independent of f. These notions admit a natural image processing interpretation. The sampling property implies stability of the reconstruction of an image (two-dimensional signal) with spectra located in M via its sample values $\{f(\omega)\}_{\omega\in\Omega}$, and the interpolation property implies non-redundancy of the set Ω.

Thus a complete interpolating sequence provides both stable and non-redundant sampling of images with spectra in M. What makes the problem interesting besides the image processing interpretation is the connection with multiple Fourier series. A simple duality reasoning (see e.g. [7] in the one-dimensional case) shows that, if Ω is a complete interpolating sequence, the corresponding system of exponential functions

$$\mathcal{E}(\Omega)=\{e^{i\langle\,\cdot\,,\omega\rangle}\,;\omega\in\Omega\}$$

is a Riesz basis in $L^2(M)$. This means that functions from $L^2(M)$ can be expanded in multiple Fourier series similar to the classical ones.

The problem of describing complete interpolating sequences for one-dimensional Paley–Wiener spaces has been studied very intensively, starting from the classical works [17], [3], [2]. A full description of real complete interpolating sequences (as zero sets of certain entire functions) was obtained in [22]. See [15], [7], [14], [13] for further developments as well as for a historic survey. We also refer the reader to [6], [1] for references concerning applications to signal analysis.

Interpolation from discrete subsets of $\mathbf{C}^n$ was studied in [5], [11], [25] for entire functions f of given exponential type $\sigma>0$, $\|f\|_{L^p(\mathbf{R}^n)}<\infty$. For spaces of entire functions of finite order with growth controlled by a given indicator, a similar interpolation problem was considered in [24], [18], [19], [20]. The topology of those spaces is generated by a countable family of norms and interpolation is possible only from sets of non-uniqueness. The notion of complete interpolating sequences does not make sense for those spaces. The only examples of complete interpolating sequences in several variables known to the authors concern the spaces PW_M with rectangular (or parallelogram) domains M. For such domains a complete interpolating sequence may be realized as, say, a lattice, the proof of this fact is just a "direct product" of the corresponding one-dimensional proofs.

For more general M's the question is to find at least some complete interpolating sequences for a given M, rather than to obtain their complete description. In the present article we answer this question for the case when M is a convex polygon, which is symmetric with respect to the origin. The problem of constructing complete interpolating sequences for the spaces PW_M with polygonal domains M was mentioned to the first author by V. P. Palamodov during a conference at the University of Bordeaux I in 1995. Another feature of polygonal M is that in this case PW_M consists of functions of completely regular growth with respect to the supporting function of M [26]. We restrict ourselves to the two-dimensional case only; in higher dimensions additional symmetries for each side of the supporting domain are needed for our approach. The objectives are to show that this case can still be handled with in essence one-dimensional machinery. The key words are *entire* *functions with plane zeros*, i.e. functions whose zero sets are unions of hyperplanes. Such functions were studied in [4]. They were applied to interpolation problems in [20], [25] (interpolation with indicator control), and then in [21] (interpolation from unions of hyperplanes). Being of a quite simple nature, these functions make it possible to avoid "standard" difficulties related to division of analytic functions of several variables.

We construct entire functions with plane zeros in $\mathbf{C}^2$, generating complete interpolating sequences for the space PW_M. Being of complex dimension 1, the zero set Z of an entire function in $\mathbf{C}^2$ itself cannot form such a sequence (in contrast to the one-dimensional case), however it produces a discrete set $\Omega \subset Z$ which fits our needs, namely the collection of all pairwise intersections of the zero hyperplanes. In our construction $\Omega \subset \mathbf{R}^2$ and, if being uniformly separated (i.e. the distance between each pair of distinct points is uniformly bounded off zero), it forms the desired complete interpolating sequence. Such an idea was exploited for example in [25].

If Ω is not uniformly separated, the interpolation problem (2) can have no solution. One has to modify it by introducing a corresponding block interpolation procedure with simultaneous interpolation at bunches of points which are located close to each other. In this procedure the assumption $\{a_\omega\} \in l^2$ should be replaced by a more complicated one (see below, Section 6).

The article is organized as follows. The next section contains preliminary information on spaces of analytic functions and also on Riesz bases. Construction of a complete interpolating sequence Ω is given in Section 3. In Section 4 we prove that Ω is a set of uniqueness. If being uniformly separated this set is a complete interpolating sequence for PW_M. This is proved in Section 5, which completes "the simpler part" of the article. For the case of non-separated Ω (which is considered in Section 6) we need some heavier machinery. In particular we obtain a special integral representation for rational functions (see Lemma 6.3) which seems to be of some independent interest.

2. Preliminary information

We start with a multi-dimensional analog of the classical Paley–Wiener theorem.

Let

$$H_M(y) = \sup_{\xi \in M} \langle y, \xi \rangle$$

be the support function of M.

Theorem A. (See e.g. [23].) *In order that an entire function $F(z)$, $z\in\mathbf{C}^2$, belong to PW_M it is necessary and sufficient that $F\in L^2(\mathbf{R}^2)$ and*

$$\sup_{x\in\mathbf{R}^2}\limsup_{R\to\infty}\frac{\log|F(x+iRy)|}{R}\le H_M(y),\quad y\in\mathbf{R}^2.$$

The following statements are straightforward consequences of the representation (1).

Proposition 2.1. *Let $\varkappa\in\mathbf{R}^2$. Then, for each $F\in PW_M$, we have*(1)

$$F(\,\cdot\,+i\varkappa)\in PW_M\quad\text{and}\quad \|F(\,\cdot\,+i\varkappa)\|\asymp\|F\|,$$

the constants in the equivalence relation depend only upon $|\varkappa|$.

In particular, for any $H>0$ and $f\in PW_M$,

$$\int_{-H}^{H}\int_{-H}^{H}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(x_1+iy_1,x_2+iy_2)|^2\,dx_1\,dx_2\,dy_1\,dy_2\le \mathrm{Const}\,\|f\|^2. \tag{3}$$

Recall that, given a number $\sigma>0$, we denote by PW_σ the classical Paley–Wiener space in one variable. This space consists of all entire functions of the form

$$f(\zeta)=\int_{-\sigma}^{\sigma}e^{i\tau\zeta}\phi(\tau)\,d\tau,\quad \phi\in L^2(-\sigma,\sigma).$$

Proposition 2.2. *Let vectors $b,c\in\mathbf{R}^2$ be given. Then, for each $F\in PW_M$ the function $F(b+c\zeta)$, $\zeta\in\mathbf{C}$, belongs to the one-dimensional Paley–Wiener space $PW_{H_M(c)}$.*

Proposition 2.3. *Let γ_1 and γ_2 be two smooth closed curves located in the strip $\{\zeta;|\operatorname{Im}\zeta|<H\}$. Then, for each $F\in PW_M$,*

$$\int_{\gamma_2}\int_{\gamma_1}|F(\zeta_1,\zeta_2)|^2\,|d\zeta_1|\,|d\zeta_2|\le \mathrm{Const}\,|\gamma_1|\,|\gamma_2|\,\|F\|^2_{PW_M}. \tag{4}$$

The constant is independent of $F\in PW_M$.

Proof. Set

$$\chi_M(z)=\int_M e^{i\langle z,\xi\rangle}\,dm_\xi.$$

(1) Here and in the sequel the sign $\asymp$ means that the ratio of the two sides lies between two positive constants.

The Plancherel equality yields

$$F(z)=\int_{\mathbf{R}^2} F(\xi)\chi_M(z-\xi)\,dm_\xi.$$

Therefore

$$|F(\zeta)|\le \|F\|_{PW_M}\|\chi_M(\zeta-\cdot\,)\|_{PW_M}\le \text{Const}\,\|F\|_{PW_M},$$

where the constant is independent of F. Inequality (4) is now straightforward.

Now we recall some basic facts concerning complete interpolating sequences in one-dimensional Paley–Wiener spaces. We refer the reader to [7], [10], [13] for complete proofs.

A sequence of points $\{\zeta_k\}\subset\mathbf{R}$ is called a complete interpolation sequence for PW_σ if the interpolation problem

$$f(\zeta_k)=a_k,\quad k\in\mathbf{Z},$$

has a unique solution $f\in PW_\sigma$ for each $\{a_k\}\in l^2$. A complete interpolating sequence is a sampling set, i.e. there exists $K>0$ such that

$$\frac{1}{K}\|\{f(\zeta_k)\}\|_{l^2}\le \|f\|_{PW_\sigma}\le K\|\{f(\zeta_k)\}\|_{l^2} \tag{5}$$

for all $f\in PW_\sigma$.

A special case of complete interpolating sequences is due to Levin [9].

Definition 1. An entire function $S(\zeta)$ is called a *sine-type function of type* σ if all its zeros $\{\zeta_k\}$ are simple and real(2), and also

(i) the zero set $\{\zeta_k\}$ is uniformly separated:

$$\inf_{k\neq l}|\zeta_k-\zeta_l|=\delta>0; \tag{6}$$

(ii) there exists a constant $C>0$ such that

$$\frac{1}{C}<|S(\zeta)|e^{-\sigma|\operatorname{Im}\zeta|}<C\quad\text{for }\operatorname{dist}(\zeta,\{\zeta_k\})>\tfrac{1}{10}\delta. \tag{7}$$

(2) Both in [22] and [9], a more general class consisting of sequences located in a horizontal strip is considered. For our purposes it suffices to consider the real sequences only.

Theorem B. *Let $S(\zeta)$ be a sine-type function of type σ and $\{\zeta_k\}$ be its zero set. Then $\{\zeta_k\}$ is a complete interpolating sequence for PW_σ and the constant K in* (5) *depends only upon δ and C in* (6) *and* (7).

We also need some basic information concerning Riesz bases from subspaces.

Let a Hilbert space $\mathcal{H}$ and a sequence of its subspaces $\{X_k\}$ be given. Denote by $l^2(\{X_k\})$ the space of sequences $\{x_k\}$, $x_k \in X_k$, such that

$$\|\{x_k\}\|^2_{l^2(\{X_k\})} := \sum_k \|x_k\|^2 < \infty.$$

Definition 2. The sequence $\{X_k\}$ forms a Riesz basis in $\mathcal{H}$ if each $h \in \mathcal{H}$ admits a unique representation of the form

$$h = \sum_k x_k, \quad x_k \in X_k,$$

and

$$\|h\|^2 \asymp \|\{x_k\}\|^2_{l^2(\{X_k\})}.$$

We shall use the following criteria for a sequence of subspaces to be a Riesz basis (see [16, Lecture VI]).

Theorem C. *Let* $\operatorname{Span}(\{X_k\})$ *be dense in $\mathcal{H}$. In order that $\{X_k\}$ be a Riesz basis in $\mathcal{H}$ it is necessary and sufficient that*

(i) *for each sequence $\{x_k\} \in l^2(\{X_k\})$ the series $\sum_k x_k$ converges in $\mathcal{H}$ and*

$$\left\|\sum_k x_k\right\|^2 \le \mathrm{Const}\, \|\{x_k\}\|^2_{l^2(\{X_k\})},$$

where the constant is independent of the choice of $\{x_k\}$;

(ii) *there exist projectors $\mathcal{P}^{(k)}: \mathcal{H} \to X_k$, $\mathcal{P}^{(k)} X_l = 0$, $l \ne k$, $\mathcal{P}^{(k)}|_{X_k} = \mathrm{id}$ and, for each $h \in \mathcal{H}$, $\{\mathcal{P}^{(k)} h\} \in l^2(\{X_k\})$.*

We also need some facts about Hardy spaces in two variables. Let

$$\mathbf{R}^2_\pm = \{u = (u_1, u_2) \in \mathbf{R}^2 ; \pm u_1, \pm u_2 > 0\},$$

and consider the spaces

$$\mathcal{H}^\pm = \left\{\phi ; \phi(\zeta) = \int_{\mathbf{R}^2_\pm} e^{i\langle \zeta, u\rangle} s(u)\, dm_u,\ s \in L^2(\mathbf{R}^2_\pm)\right\}.$$

Standard arguments show that $\phi\in\mathcal{H}^{\pm}$ if and only if it is analytic in $\mathbf{R}^2+i\mathbf{R}^2_{\pm}$ and

$$\sup_{\eta\in\mathbf{R}^2_{\pm}}\left\{\int_{\mathbf{R}^2}|\phi(\xi+i\eta)|^2\,dm_\xi\right\}<\infty. \tag{8}$$

One can also see that the dual space of $\mathcal{H}^+$ can be realized as $\mathcal{H}^-$ with the form of functional

$$\mathcal{L}_\psi(\phi)=\int_{\mathbf{R}^2}\phi(\xi)\psi(\xi)\,dm_\xi$$

and $\|\mathcal{L}_\psi\|\asymp\|\psi\|_{\mathcal{H}^-}$.

An important example of functions in $\mathcal{H}^+$ comes from rational functions. Let vectors $b^{(1)},b^{(2)}\in\mathbf{R}^2$, $0\le\arg b^{(1)}<\arg b^{(2)}\le\frac{1}{2}\pi$ and numbers $s_1,s_2\in\mathbf{R}$, $\beta_1,\beta_2>0$ be given. Then functions of the form

$$\phi(\zeta)=\frac{1}{(\langle\zeta,b^{(1)}\rangle-s_1+i\beta_1)(\langle\zeta,b^{(2)}\rangle-s_2+i\beta_2)}$$

(as well as their linear combinations) belong to $\mathcal{H}^+$.

We also need several versions of embedding theorems for spaces of analytic functions.

Proposition 2.4. *Let a sequence $\{\zeta^{(k)}=\xi^{(k)}+i\eta^{(k)}\}\subset\mathbf{R}^2+i\mathbf{R}^2_+$, with $\eta^{(k)}=(\eta_1^{(k)},\eta_2^{(k)})$, be such that*

$$\delta<\eta_1^{(k)},\eta_2^{(k)}<\Delta \tag{9}$$

for some $0<\delta<\Delta<\infty$ independent of k, and also

$$d=\inf_{k\neq l}|\zeta^{(k)}-\zeta^{(l)}|>0. \tag{10}$$

Then, for each $h\in\mathcal{H}^+$,

$$\sum_k|h(\zeta^{(k)})|^2\le\text{Const}\,\|h\|^2_{\mathcal{H}^+},$$

where the constant is independent of h.

Proof. It follows from the hypothesis that there exists a sequence of disjoint balls $\{B_k\}$ centered at $\zeta^{(k)}$ and of fixed radii r (depending upon δ, Δ, and d only), located in $\mathbf{R}^2+i\mathbf{R}^2_+$. Subharmonicity of $|h|$ yields

$$|h(\zeta^{(k)})|^2\le\frac{\text{Const}}{r^4}\int_{B^{(k)}}|h(\zeta)|^2\,dm_\zeta$$

(here m stands for the Lebesgue measure in $\mathbf{C}^2$). Therefore

$$\sum_k |h(\zeta^{(k)})|^2 \le \sum_k \frac{\mathrm{Const}}{r^4} \int_{B^{(k)}} |h(\zeta)|^2 \, dm_\zeta \le \mathrm{Const} \int_{\mathbf{R}^2+i[0,\Delta+r]} |h(\zeta)|^2 \, dm_\zeta.$$

Together with (8) this yields the proposition.

A similar inequality holds for functions in PW_M.

Proposition 2.5. *Let a sequence of points $\{\zeta^{(k)}\}$ satisfy* (9) *and* (10). *Then, for each $F \in PW_M$,*

$$\sum_k |F(\zeta^{(k)})|^2 \le \mathrm{Const}\, \|F\|^2_{PW_M},$$

where the constant is independent of F.

Proposition 2.6. *Let numbers $a, b \in \mathbf{R}$, $ab \neq 0$ and $\delta_1, \delta_2 > 0$ be given. Then, for each $h \in \mathcal{H}^-$,*

$$\sum_n \int_{-\infty}^{\infty} |h(a\xi+bn-i\delta_1, \xi-i\delta_2)|^2 \, d\xi \le \mathrm{Const}\, \|h\|^2_{\mathcal{H}^-},$$

where the constant is independent of h.

Proof. Let, for definiteness, $b>0$. Fix a number s with $0<s<\frac{1}{10}\min\{b, \delta_1, \delta_2\}$. Using subharmonicity we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} |h(a\xi+bn-i\delta_1, \xi-i\delta_2)|^2 \, d\xi \\
&\qquad \le \mathrm{Const} \int_{-\infty}^{\infty} \int_{|z_1|<s} |h(a\xi+z_1+bn-i\delta_1, \xi-i\delta_2)|^2 \, dm_{z_1} \, d\xi \\
&\qquad \le \mathrm{Const} \int_{-\infty}^{\infty} \int_0^{2\pi} \int_{|z_1|<s} |h(a\xi+z_1+bn-i\delta_1, \xi+se^{i\theta}-i\delta_2)|^2 \, dm_{z_1} \, d\theta \, d\xi.
\end{aligned}$$

Consider the closed domain

$$\Pi_n = \{(a\xi+z_1+bn-i\delta_1, \xi+se^{i\theta}-i\delta_2) \,;\, |z_1| \le s,\ \theta \in [0, 2\pi],\ \xi \in \mathbf{R}\}.$$

It belongs to the tube domain $\{(\zeta_1, \zeta_2) \in \mathbf{C}^2 ; |\operatorname{Im} \zeta_j + \delta_j| < s,\ j=1,\ 2\}$, each point of Π_n can be obtained by at most two choices of the parameters z_1, θ, ξ, and also

$$\left| \frac{D(\zeta_1, \bar{\zeta}_1, \theta, \xi)}{D(\zeta_1, \bar{\zeta}_1, \zeta_2, \bar{\zeta}_2} \right| \le \mathrm{Const}\,.$$

Besides $\Pi_n \cap \Pi_m = \emptyset$, $m \neq n$. Therefore

$$\sum_n \int_{-\infty}^{\infty} |h(a\xi+bn-i\delta_1, \xi-i\delta_2)|^2 \, d\xi \le \mathrm{Const} \iint_{\substack{|\operatorname{Im} \zeta_1+\delta_1|<s \\ |\operatorname{Im} \zeta_2+\delta_2|<s}} |h(\zeta_1, \zeta_2)| \, dm_\zeta$$

and it remains to apply (8).

The lemma below will be used in Section 6 for the block interpolation.

Lemma 2.1. *Let a sequence of points* $\{\zeta_n\}\subset\mathbf{C}_+$ *be given satisfying*

$$0<\delta<\inf(\operatorname{Im}\zeta_n)<\Delta<\infty,$$

and

$$d=\inf|\zeta_n-\zeta_m|>0.$$

Let also $\{\psi_n\}$ *be rational functions of degree at most* s *(*s *does not depend on* n*) vanishing at infinity and such that all poles of* ψ_n *are located in* ε*-neighbourhoods of* ζ_n*,* $\varepsilon<\frac{1}{10}\min(\delta,d)$*. Then*

$$\left(\int_{-\infty}^{\infty}\left|\sum_n\psi_n(\xi)\right|^2d\xi\right)^{1/2}\le\mathrm{Const}\left(\sum_n\int_{-\infty}^{\infty}|\psi_n(\xi)|^2\,d\xi\right)^{1/2}.$$

Proof. Without loss of generality we may assume the sequence $\{\psi_n\}$ to be finite. Let $p_n(\zeta)$ be polynomials with leading coefficients 1, whose zeros coincide with the poles of ψ_n. For simplicity we assume $\deg p_n=s$ for all n. Clearly

$$0<a|\zeta-\zeta_n|^s<|p_n(\zeta)|<A|\zeta-\zeta_n|^s,\quad\text{if }|\zeta-\zeta_n|>2\varepsilon,$$

where a and A are independent of n. Letting $c_n=\psi_n(\zeta)p_n(\zeta)$ we see that

$$\|\psi_n\|_{L^2(\mathbf{R})}\asymp|c_n|. \tag{11}$$

Let $\gamma_n=\{\zeta;|\zeta-\zeta_n|=3\varepsilon\}$. We have

$$|\psi_n(\zeta)|\asymp c_n,\quad\zeta\in\Gamma_n,$$

and, besides,

$$\psi_n(\xi)=\frac{1}{2i\pi}\int_{\Gamma_n}\psi_n(\zeta)\,\frac{d\zeta}{\xi-\zeta},\quad\xi\in\mathbf{R}.$$

Now since $\psi_n\in H^2(\mathbf{C}_-)$,

$$\left(\int_{-\infty}^{\infty}\left|\sum_n\psi_n(\xi)\right|^2d\xi\right)^{1/2}=\sup\left\{\left|\int_{-\infty}^{\infty}h(\xi)\sum_n\psi_n(\xi)\,d\xi\right|;h\in H^2(\mathbf{C}_+),\ \|h\|\le1\right\}.$$

Evaluation of the latter integral is straightforward:

$$\begin{aligned}I=\left|\int_{-\infty}^{\infty}h(\xi)\sum_n\psi_n(\xi)\,d\xi\right|&=\frac{1}{2i\pi}\sum_n\int_{-\infty}^{\infty}h(\xi)\int_{\Gamma_n}\psi_n(\zeta)\frac{d\zeta}{\xi-\zeta}\,d\xi\\&=\sum_n\int_{\Gamma_n}\psi_n(\zeta)\frac{1}{2i\pi}\int_{-\infty}^{\infty}h(\xi)\,\frac{d\xi}{\xi-\zeta}\,d\zeta=\sum_n\int_{\Gamma_n}\psi_n(\zeta)h(\zeta)\,d\zeta.\end{aligned}$$

Therefore

$$|I| \leq \sum_n \left(\int_{\Gamma_n} |h(\zeta)|^2 \, d\zeta \right)^{1/2} \left(\int_{\Gamma_n} |\psi_n(\zeta)|^2 \, d\zeta \right)^{1/2}$$
$$\leq \mathrm{Const} \left(\sum_n \int_{\Gamma_n} |h(\zeta)|^2 \, d\zeta \right)^{1/2} \left(\sum_n \int_{\Gamma_n} |\psi_n(\zeta)|^2 \, d\zeta \right)^{1/2}$$

and it remains to use (11) and also the fact that the linear measure concentrated on $\bigcup \Gamma_n$ is a Carleson measure.

3. Construction of a complete interpolating sequence

Let $M \subset \mathbf{R}^2$ be a convex polygon which is symmetric with respect to the origin. It has an even number of vertices. Denote them $a^{(k)}$, $k=1, 2, \dots, 2N$ according to the counterclockwise order in which they appear on the boundary of M. We have $a^{(k)} = -a^{(k+N)}$, $k=1, \dots, N$. Define the vectors $b^{(k)} \in \mathbf{R}^2$ by

$$2\pi b^{(k)} = a^{(k+1)} - a^{(k)}, \quad k = 1, \dots, N.$$

Then by induction on N,

$$M = \sum_{k=1}^{N} [-\pi b^{(k)}, \pi b^{(k)}] = \{t_1 b^{(1)} + \dots + t_N b^{(N)} \, ; t_k \in [-\pi, \pi]\}.$$

The supporting function of M is now explicit

$$H_M(x) = \sup_{-\pi \leq t_k \leq \pi} \left\{ \sum_{k=1}^{N} t_k \langle b^{(k)}, x \rangle \right\} = \pi \sum_{k=1}^{N} |\langle b^{(k)}, x \rangle|. \tag{12}$$

Set

$$S(z) = \prod_{k=1}^{N} \sin(\pi(\langle z, b^{(k)} \rangle - \alpha_k)), \quad z \in \mathbf{C}^2, \tag{13}$$

where the real constants α_k will be specified later. This function will generate the desired complete interpolating sequence for PW_M, and we summarize here some of its properties.

Proposition 3.1.

(i) *Given $\delta>0$, the function S satisfies*

$$|S(z)|\asymp\exp(H_M(y)) \quad \textit{for } \operatorname{dist}(\langle z,b^{(k)}\rangle-\alpha_k,\mathbf{Z})>\delta,\ k=1,\dots,N; \tag{14}$$

(ii) *the zero set Z of the function S is the union of the hyperplanes*

$$P^{(k,n)}=\{z\in\mathbf{C}^2\,;\langle z,b^{(k)}\rangle=n+\alpha_k\},\qquad n\in\mathbf{Z},\ k=1,\dots,N;$$

(iii) *the set Ω of the points $\omega^{(k_1,n_1)(k_2,n_2)}$ which are pairwise intersections of the hyperplanes $P^{(k_1,n_1)}$ and $P^{(k_2,n_2)}$, $n_1,n_2\in\mathbf{Z}$, $k_1\neq k_2$, is a subset of $\mathbf{R}^2$;*

(iv) *Ω contains no multiple points (i.e. no triple of zero hyperplanes $P^{(k,n)}$ has nonempty intersection) for all $\alpha=(\alpha_1,\dots,\alpha_N)\in\mathbf{R}^N\setminus E_M$, E_M being the union of a denumerable collection of (real) hyperplanes in the parameter space $\mathbf{R}^N_{(\alpha)}$ and hence of zero Lebesgue measure;*

(v) *Ω is uniformly separated if and only if $s_l^{(k)}/s_m^{(k)}\in\mathbf{Q}$ for all $k,l,m\in\mathbf{Z}$; here*

$$s_j^{(k)}=\frac{1}{|\langle b^{(j)},c^{(k)}\rangle|} \tag{15}$$

and $c^{(k)}\in\mathbf{R}^2$ is the unit normal vector to $b^{(k)}$ which turns into $b^{(k)}/|b^{(k)}|$ if being rotated by $\frac{1}{2}\pi$ in the clockwise direction.

Proof. Relation (14) follows from (12) and a direct estimate of each factor in (13).

The zero set of the k-th factor $\sin(\pi(\langle z,b^{(k)}\rangle-\alpha_k))$ in (13) is the union of the disjoint hyperplanes $P^{(k,n)}=\{z\in\mathbf{C}^2;\langle z,b^{(k)}\rangle=n+\alpha_k\}$, $n\in\mathbf{Z}$. Each such plane admits the representation

$$P^{(k,n)}=\{z=b^{(k,n)}+c^{(k)}\zeta\,;\zeta\in\mathbf{C}\}.$$

Here

$$b^{(k,n)}=(n+\alpha_k)\frac{b^{(k)}}{|b^{(k)}|^2}$$

is the normal vector to $P^{(k,n)}$ dropped from the origin. We shall also consider the "real" and "imaginary" parts of $P^{(k,n)}$:

$$\mathbf{R}^{(k,n)}=\{z=b^{(k,n)}+c^{(k)}x\,;x\in\mathbf{R}\},\quad \mathbf{I}^{(k,n)}=\{z=b^{(k,n)}+ic^{(k)}y\,;y\in\mathbf{R}\}.$$

For $k_1\neq k_2$, the set $P^{(k_1,n_1)}\cap P^{(k_2,n_2)}$ consists of the unique point $\omega^{(k_1,n_1)(k_2,n_2)}$ satisfying the equation

$$\omega^{(k_1,n_1)(k_2,n_2)}=b^{(k_1,n_1)}+c^{(k_1)}\zeta_1=b^{(k_2,n_2)}+c^{(k_2)}\zeta_2,\quad \zeta_1,\zeta_2\in\mathbf{C}.$$

This equation (with respect to ζ_1 and ζ_2) has only one solution, which is real. We denote it by

$$x^{(k_1,n_1)}_{(k_2,n_2)} = \zeta_1, \quad x^{(k_2,n_2)}_{(k_1,n_1)} = \zeta_2.$$

We also let

$$\Omega^{(k_1,n_1)}_k = \{\omega^{(k_1,n_1)(k,n)}; n \in \mathbf{Z}\}, \qquad X^{(k_1,n_1)}_k = \{x^{(k_1,n_1)}_{(k,n)}; n \in \mathbf{Z}\},$$
$$\Omega^{(k_1,n_1)} = \bigcup_{k\neq k_1} \Omega^{(k_1,n_1)}_k \subset \mathbf{R}^{(k_1,n_1)}, \quad X^{(k_1,n_1)} = \bigcup_{k\neq k_1} X^{(k_1,n_1)}_k,$$

so that the set

$$\Omega = \bigcup_{\substack{k_1=1,\ldots,N\\ n_1\in\mathbf{Z}}} \Omega^{(k_1,n_1)} \subset \mathbf{R}^2.$$

The condition $\omega^{(k_1,n_1)(k_2,n_2)} = \omega^{(k_1,n_1)(k_3,n_3)}$ with $(k_2,n_2) \neq (k_3,n_3)$ implies certain linear relation between $n_1+\alpha_{k_1}$, $n_2+\alpha_{k_2}$ and $n_3+\alpha_{k_3}$ and thus defines a hyperplane in the parameter space $\mathbf{R}^N_{(\alpha)}$. The union of such hyperplanes for all (k_1,n_1), (k_2,n_2) and (k_3,n_3) forms the exceptional set $E_M \subset \mathbf{R}^N$.

A direct calculation shows that each sequence $X^{(k_1,n_1)}_k$ is an arithmetic progression of the step length $s^{(k_1)}_k$ defined by (15). Therefore, the points of $X^{(k_1,n_1)}_l \cup X^{(k_1,n_1)}_m$, $l\neq m$, are uniformly separated if and only if the steps $s^{(k_1)}_l$ and $s^{(k_1)}_m$ are commensurable, i.e. their quotient is rational.

The proof is complete.

In the next sections we will show that if Ω is uniformly separated, then it is a complete interpolating sequence for PW_M.

Let the numbers β^{k_1,n_1}_k be chosen so that the function

$$L^{(k_1,n_1)}_k(\zeta) = \sin \pi(\langle b^{(k)}, c^{(k_1)}\rangle \zeta - \beta^{k_1,n_1}_k)$$

vanishes at $X^{(k_1,n_1)}_k$. Then

$$L^{(k_1,n_1)}(\zeta) = \prod_{k\neq k_1} L^{(k_1,n_1)}_k(\zeta), \quad \zeta \in \mathbf{C},$$

is the generating function of $X^{(k_1,n_1)}$, i.e. an entire function whose zero set is $X^{(k_1,n_1)}_k$.

Let

$$\sigma^{(k_1)} = \pi \sum_{k=1}^{N} |\langle b^{(k)}, c^{(k_1)}\rangle|. \tag{16}$$

The function $L^{(k_1,n_1)}$ satisfies

$$|L^{(k_1,n_1)}(\zeta)| \asymp e^{\sigma^{(k_1)}|\operatorname{Im}\zeta|}, \quad \operatorname{dist}(\zeta, X^{(k_1,n_1)}) > \delta > 0.$$

Proposition 3.2. *$X^{(k_1,n_1)}$ is a uniqueness set in $PW_{\sigma^{(k_1)}}$. If $X^{(k_1,n_1)}$ is uniformly separated, then it is a complete interpolating sequence for $PW_{\sigma^{(k_1)}}$.*

Proof. Fix $\delta>0$. For $\zeta=\xi+i\eta\in\mathbf{C}$ we have

$$
\begin{aligned}
|L^{(k_1,n_1)}(\zeta)| &\asymp \exp\biggl(\pi|\eta|\sum_{k=1}^{N}|\langle b^{(k)},c^{(k_1)}\rangle|\biggr)\\
&= \exp\biggl(\pi\sum_{k=1}^{N}|\langle b^{(k)},\operatorname{Im}(b^{(k_1,n_1)}+c^{(k_1)}\zeta)\rangle|\biggr) = \exp(H_M(\operatorname{Im}(c^{(k_1)}\zeta))),
\end{aligned}
\tag{17}
$$

if $\operatorname{dist}(\zeta,X^{(k_1,n_1)})>\delta$.

Let $f\in PW_{\sigma^{(k_1)}}$ and $f|_{X^{(k_1,n_1)}}=\mathbf{0}$, then $\phi(\zeta)=f(\zeta)/L^{(k_1,n_1)}(\zeta)$ is an entire function. Now a standard reasoning (see e.g. [10, Lecture 18]) shows that (17) implies $\phi=\mathbf{0}$. That $X^{(k_1,n_1)}$ is a complete interpolating sequence for $PW_{\sigma^{(k_1)}}$ in the case when it is uniformly separated, follows now from Theorem B.

4. A uniqueness theorem

Theorem 1. *Let Ω be as in Proposition 3.1, $\alpha\notin E_M$, and $f\in PW_M$, $f|_\Omega=\mathbf{0}$. Then $f=\mathbf{0}$.*

Proof. For each $k_1=1,\dots,N$ and $n_1\in\mathbf{Z}$ consider the trace of f on the hyperplane $P^{(k_1,n_1)}$

$$f^{(k_1,n_1)}(\zeta)=f(b^{(k_1,n_1)}+c^{(k_1)}\zeta),\quad \zeta\in\mathbf{C}.$$

It follows from Proposition 2.2 that $f^{(k_1,n_1)}\in PW_{\sigma^{(k_1)}}$, here $\sigma^{(k_1)}$ is defined in (16). The assumption of the theorem yields $f^{(k_1,n_1)}|_{X^{(k_1,n_1)}}=\mathbf{0}$ and, by Proposition 3.2, $f^{(k_1,n_1)}(\zeta)=0$, $\zeta\in\mathbf{C}$, i.e. $f|_{P^{(k_1,n_1)}}=\mathbf{0}$. It means that the zero set of f contains the zero set $Z(S)$ of S. Furthermore, since the multiplicity of zeros of S (which is defined on the set of regular points of $Z(S)$, i.e. on $Z(S)\setminus\Omega$) equals 1, f divides S (see e.g. [23]), so

$$\Phi(z)=\frac{f(z)}{S(z)},\quad z\in\mathbf{C}^2,$$

is an entire function. On the other hand the Riemann–Lebesgue lemma yields $|f(x+(i,i))|\to0$, as $x\to\infty$, $x\in\mathbf{R}^2$, while $|S(x+(i,i))|\asymp1$. Therefore $\Phi(x+(i,i))\to0$, as $x\to\infty$. One can also see that $|\Phi(z)|$ is bounded. Indeed, by (14), for any $\delta>0$ there exists $C_\delta>0$ such that $\log|\Phi(z)|\le C_\delta$ for

$$z=\{z\in\mathbf{C}^2\,;\operatorname{dist}(\langle z,b^{(k)}\rangle-\alpha_k,\mathbf{Z})\ge\delta,\ k=1,\dots,N\},$$

and plurisubharmonicity arguments for $\log|\Phi(z)|$ extend this to any $z\in\mathbf{C}^2$.

Now the Liouville theorem yields $\Phi=\mathbf{0}$ and hence $f=\mathbf{0}$, which completes the proof.

5. Solution to the interpolation problem

In this section we solve the interpolation problem (2) under the assumption that $\alpha\notin E_M$ and Ω is uniformly separated.

For each $\omega=\omega^{(k_1,n_1)(k_2,n_2)}\in\Omega$ define

$$\phi_\omega(z)=\prod_{k\neq k_1,k_2}\sin(\pi(\langle z,b^{(k)}\rangle-\alpha_k))r_{k_1,n_1}(z)r_{k_2,n_2}(z), \tag{18}$$

where

$$r_{k,n}(z)=\frac{\sin(\pi\langle z,b^{(k)}\rangle-\alpha_k)}{(\langle z,b^{(k)}\rangle-n-\alpha_k)}.$$

The function ϕ_ω vanishes on $\Omega\setminus\{\omega\}$, and also

$$c_\omega:=|\phi_\omega(\omega)|\asymp 1,\quad \omega\in\Omega,$$

as follows from the fact that Ω is uniformly separated.

Theorem 2. *Let $\alpha\notin E_M$ and the set Ω be uniformly separated. Then, given a sequence $a=\{a_\omega\}\in l^2(\Omega)$, the solution $f=f_a\in PW_M$ to the interpolation problem*

$$f(\omega)=a_\omega,\quad \omega\in\Omega,$$

exists and has the form

$$f_a(z)=\sum_{\omega\in\Omega}\frac{a_\omega}{c_\omega}\phi_\omega(z),\quad z\in\mathbf{C}^2. \tag{19}$$

The series converges in PW_M norm and also uniformly on compact sets in $\mathbf{C}^2$.

Proof. It suffices to assume that $a_\omega\neq 0$ for a finite number of ω's only and prove that the function f_a, represented by (19) (for finite a no convergence problems appear) satisfies

$$\|f_a\|\leq\text{Const}\,\|a\|_{l^2(\Omega)}.$$

For arbitrary $a\in l^2(\Omega)$ the statement will then follow as a limit case.

Further we may fix some $k_1,k_2\in\{1,\dots,N\}$, $k_1\neq k_2$, and assume that all nonzero a_ω correspond to points ω of the form $\omega=\omega^{(k_1,n_1)(k_2,n_2)}$, $n_1,n_2\in\mathbf{Z}$. This is because each a can be represented as a union of at most $\frac{1}{2}N(N+1)$ such sequences with pairwise disjoint supports.

Now (after fixing k_1 and k_2) set

$$\omega_{n_1,n_2}=\omega^{(k_1,n_1),(k_2,n_2)}, \quad a_{n_1,n_2}=a_{\omega_{n_1,n_2}}, \quad c_{n_1,n_2}=c_{\omega_{n_1,n_2}}.$$

Taking (18) into account, we have

$$f_a(z)=\left\{\prod_{k\neq k_1,k_2}\sin(\pi(\langle z,b^{(k)}\rangle-\alpha_k))\right\}g_a(z),$$

where

$$g_a(z)=\sum_{n_1} r_{k_1,n_1}(z)\left\{\sum_{n_2}\frac{a_{n_1,n_2}}{c_{n_1,n_2}}r_{k_2,n_2}(z)\right\}=\sum_{n_1}V_{n_1}(\langle z,b^{(k_2)}\rangle)r_{k_1,n_1}(z),$$

and

$$V_{n_1}(\zeta)=\sum_{n_2}\frac{a_{n_1,n_2}}{c_{n_1,n_2}}\frac{\sin(\pi(\zeta-\alpha_{k_2}))}{\zeta-n_2-\alpha_{k_2}}\in PW_\pi.$$

Set $a_{(n_1)}=\{a_{n_1,n_2}\}_{n_2\in\mathbf{Z}}$. We have $a_{(n_1)}\in l^2$, and $\sum_{n_1}\|a_{(n_1)}\|^2=\|a\|^2$. By Theorem B,

$$\|V_{n_1}\|_{L^2(\mathbf{R})}\leq \text{Const}\,\|a_{(n_1)}\|_{l^2(\mathbf{Z})},$$

and

$$\sum_{n_1}\|V_{n_1}\|^2_{L^2(\mathbf{R})}\leq \text{Const}\,\|a\|^2_{l^2(\Omega)}.$$

Now assume for simplicity that $b^{(k_2)}$ is directed along the x_2-axis (one can always achieve this by an appropriate affine transformation) and apply Theorem B once again. We obtain

$$\begin{aligned}
&\int_{\mathbf{R}^2}|g_a(x)|^2\,dm_x\\
&\quad=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\sum_{n_1}V_{n_1}(x_2b_2^{(k_2)})\frac{\sin(\pi(x_1b_1^{(k_1)}+x_2b_2^{(k_1)}-\alpha_{k_1}))}{x_1b_1^{(k_1)}+x_2b_2^{(k_1)}-n_1+\alpha_{k_1}}\right|^2dx_1\,dx_2\\
&\quad\leq \text{Const}\int_{-\infty}^{\infty}\sum_{n_1}|V_{n_1}(x_2b_2^{(k_2)})|^2\,dx_2\leq \text{Const}\sum_{n_1}\|V_{n_1}\|^2_{L^2(\mathbf{R})}\leq\|a\|^2_{l^2(\Omega)}.
\end{aligned}$$

This completes the proof of the theorem.

In the general case the sequence Ω, as constructed above, need not be uniformly separated. This forces one to implement the block interpolation procedure as presented in Section 6. Another approach is to modify the generating function S. One can easily see that reasoning is still valid if each of the k-th factors in (13) be replaced by $s_k(\langle z, b^{(k)}\rangle)$, where $s_k(\zeta)$ is a sine-type function of type π with real zeros. We do not know whether it is possible to find the factors $s_k(\zeta)$ in a way to make the corresponding set Ω uniformly separated.

In view of possible image analysis applications, it may be important to approximate the original polygon by one with a separation property. To this end the following approximation procedure can be proposed.

Proposition 5.1. *Any convex symmetric polygon can be approximated, from both inside and outside, by polygons with the separation property.*

Proof. By Proposition 3.1, a polygon M produces a uniformly separated interpolating sequence Ω if and only if for each k_1 the steps $s_k^{(k_1)}$ in (15) are commensurable.

Denote by $\mathbf{S}$ the set of all reals x such that $\sin x\in\mathbf{Q}$ and $\cos x\in\mathbf{Q}$. It forms a dense subgroup of the (additive) group $\mathbf{R}$, since

$$\mathbf{S}\supset\left\{\frac{\pi k}{2}+\arcsin\frac{2l^2+2lm}{2l^2+2lm+m^2}\,;k,l,m\in\mathbf{Z}\right\}.$$

Given a polygon M and $\varepsilon>0$, put $\tilde a^{(0)}=0$ and choose $\tilde a^{(1)}$ such that $|\tilde a^{(1)}-a^{(1)}|<\frac12\varepsilon$ and $|\tilde a^{(1)}|\in\mathbf{Q}$. Then, assuming the vertices $\tilde a^{(j+1)}$ be already constructed for $j<k<N-1$, denote by τ_j the angle between $\tilde a^{(j+1)}-\tilde a^{(j)}$ and $\tilde a^{(j)}-\tilde a^{(j-1)}$, $j\geq1$, and choose $\tilde a^{(k+1)}$ such that $|\tilde a^{(k+1)}-a^{(k+1)}|<\frac12\varepsilon$, $|\tilde a^{(k+1)}-\tilde a^{(k)}|\in\mathbf{Q}$ and $\tau_k\in\mathbf{S}$. It gives us the values of all the new steps

$$\tilde s_k^{(k_1)}\in\pi\mathbf{Q} \tag{20}$$

except for possibly $k=N$ and $k_1=N$. Now we can shift the points $\tilde a^{(j)}$, $1\leq j\leq N$, by $h\in\left(0,\frac12\varepsilon\right)$ along the vector $\tilde a^{(1)}$ with no changes in $\tilde a^{(j)}-\tilde a^{(j-1)}$ and τ_j for $j<N$, the value h being chosen to provide the angle $\tau_N\in\mathbf{S}$ (by the definition, $\tilde a^{(N+1)}=-\tilde a^{(1)}$). It gives us automatically $|\tilde a^{(N+1)}-\tilde a^{(N)}|\in\mathbf{Q}$, and so for the polygon $\widetilde M$ with the vertices $\tilde a^{(j)}$ we have (20) with all k and k_1.

As can be easily seen from the construction, the points $\tilde a^{(j)}$ can be chosen such that $\widetilde M\subset M$ as well as $\widetilde M\supset M$.

The proof is complete.

By applying a standard duality reasoning (see e.g. [7]), one can read Theorem 2 as one about Fourier series expansions in M.

Theorem 3. *Under the assumptions of Theorem 2 each function $\phi \in L^2(M)$ admits an expansion*

$$\phi(x) = \sum_{\omega\in\Omega} c_\omega(\phi) e^{i\langle x,\omega\rangle}.$$

The series converges in $L^2(M)$-norm, and

$$\|\phi\|^2_{L^2(M)} \asymp \sum_{\omega\in\Omega} |c_\omega(\phi)|^2.$$

6. Block interpolation procedure

Main lemmas

In this section we consider the case when the sequence Ω is not uniformly separated, but still contains no multiple points, i.e. $\alpha\notin E_M$.

Then, for each $\Delta>1$ there exists $\varepsilon_0=\varepsilon_0(\delta)>0$ and $A=A(\Delta)>0$ with the following property.

For each $\varepsilon<\varepsilon_0$ one can represent Ω as

$$\Omega = \bigcup_{l=1}^{\infty} \Omega_l \tag{21}$$

such that

$$\operatorname{dist}(\Omega_l, \Omega_m) \geq \max\{\varepsilon, \Delta \operatorname{diam} \Omega_l, \Delta \operatorname{diam} \Omega_m\}, \quad l\neq m, \tag{22}$$

and

$$\operatorname{diam} \Omega_l \leq A\varepsilon. \tag{23}$$

Such a partition can be obtained in several steps. Fix an $\varepsilon>0$ and take ε-neighbourhoods of all points from Ω. If they are disjoint, we are done. If not, split their union into connected components A^1_k and let $\Omega^1_k=A^1_k\cap\Omega$. Take $\max(\varepsilon, \Delta \operatorname{diam} \Omega^1_k)$-neighbourhoods of each Ω^1_k. If they are disjoint, we are done. If not, take the connected components of their union and repeat the procedure. It is easy to see that, if the initial ε was small enough, then after a finite number of steps the procedure will be completed with the desired partition.

We shall call such a representation an (ε, Δ)-*partition* of Ω, keeping in mind its main characteristics: ε – the order of the sizes of the blocks, and Δ – the relative distance between the blocks.

We may also assume ε to be chosen small enough such that, given $k_1, k_2 \in \{1, \dots, N\}$, each Ω_l contains at most one point of the form $\omega^{(k_1,n_1)(k_2,n_2)}$, and the condition $\omega^{(k_1,n_1)(k_2,n_2)}, \omega^{(k_2,n_2)(k_3,n_3)} \in \Omega_l$ implies $\omega^{(k_1,n_1)(k_3,n_3)} \in \Omega_l$, $l=1, 2, \dots$.

Given such an (ε, Δ)-partition (22), we consider the subspaces

$$X_m = \{f \in PW_M\,;\, f|_{\Omega \setminus \Omega_m} = \mathbf{0}\}.$$

It follows from the uniqueness theorem that

$$\dim X_m = \#\Omega_m$$

and $\mathrm{Span}(\bigcup_m X_m)$ is dense in PW_M.

Theorem 4. *There exists $\Delta_0 > 0$ such that, for $\Delta > \Delta_0$ and $\varepsilon < \varepsilon_0(\Delta)$, the spaces $\{X_m\}$ form a Riesz basis from subspaces in PW_M.*

The proof consists of two parts. According to Theorem C it suffices to prove the following two lemmas.

Lemma 6.1. *Let a sequence of functions $\{f_m\} \in l^2(\{X_m\})$ be given. Then the series $\sum_m f_m$ converges in PW_M and*

$$\Big\| \sum_m f_m \Big\|^2_{PW_M} \le \mathrm{Const} \sum_m \|f_m\|^2_{PW_M}, \tag{24}$$

where the constant is independent of $\{f_m\}$.

Lemma 6.2. *For each m there exists a projector*

$$\mathcal{P}^{(m)}: PW_M \longrightarrow X_m$$

such that

$$\mathcal{P}^{(m)}|_{X_m} = \mathrm{id}, \quad \mathcal{P}^{(m)}|_{X_l} = \mathbf{0}, \quad l \ne m,$$

and, for each $f \in PW_M$,

$$\{\mathcal{P}^{(m)} f\} \in l^2(\{X_m\}). \tag{25}$$

Before proving the lemmas we need to fix a convenient parameterization of the set $\{\Omega_m\}$. First note that in the case when Ω_m consists of one point, the construction of $\mathcal{P}^{(m)}$ as well as the verification of (24) related to such Ω_m's are quite straightforward and similar to the one-dimensional case. We omit the details. We say that Ω_m is a *bunch* if it contains more than one point. Each such bunch is a union of pairwise intersections of the lines $\mathbf{R}^{(k_1,n_1)}$, $\mathbf{R}^{(k_2,n_2)}, \dots, \mathbf{R}^{(k_s,n_s)}$, where the indices $k_1, k_2, \dots, k_s$ are pairwise distinct. The collection $\{k_1, k_2, \dots, k_s\}$ defines the *type* of the bunch. Given the type, one can define the bunch uniquely just by fixing one of its points, $\omega^{(k_1,n_1)(k_s,n_s)}$, say. So each bunch is defined uniquely by its type $\{k_1, k_2, \dots, k_s\}$ and a pair of indices. Since the total number of possible different types is finite, we may assume that all bunches have the same type $\{1, 2, \dots, s\}$.

Making a change of variables if necessary, we may reduce the problem to the case

$$0=\arg b^{(1)}<\arg b^{(2)}<\dots<\arg b^{(s)}=\tfrac{1}{2}\pi. \tag{26}$$

We also need some additional notation. Let

$$\mathcal{M}=\{(n_1,n_s)\in\mathbf{Z}^2\,;\omega^{(1,n_1)(s,n_s)}\in\Omega_m \text{ for some } m=m(n_1,n_s)\}.$$

Now, given $(n_1,n_s)\in\mathbf{Z}^2$, set

$$\mathcal{M}^{(1)}_{n_1}=\{n_s\,;(n_1,n_s)\in\mathcal{M}\},\quad \mathcal{M}^{(s)}_{n_s}=\{n_1\,;(n_1,n_s)\in\mathcal{M}\},$$

and

$$\mathcal{M}^{(1)}=\{n_1\,;\mathcal{M}^{(1)}_{n_1}\neq\emptyset\},\qquad \mathcal{M}^{(s)}=\{n_s\,;\mathcal{M}^{(s)}_{n_s}\neq\emptyset\}.$$

For $(n_1,n_s)\in\mathcal{M}$, we may introduce the set $\{n_2,\dots,n_{s-1}\}$ of *complementary indices* such that

$$\Omega_{m(n_1,n_s)}=\bigcup_{1\le p<q\le s}\{\omega^{(p,n_p)(q,n_q)}\}.$$

For $1<p<s$ we denote by $n_p(n_1,n_s)$ the corresponding complementary index. Thus the functions in X_m have the form

$$f(z)=S(z)\sum_{1\le p<q\le s}\frac{a^{(n_1,n_s)}_{p,q}}{(\langle z,b^{(p)}\rangle-n_p-\alpha_p)(\langle z,b^{(q)}\rangle-n_q-\alpha_q)},\quad z\in\mathbf{C}^2.$$

Series convergence (proof of Lemma 6.1)

Lemma 6.1 will follow from the statement below.

Given a finite sequence $\{f^{(n_1,n_s)}\}_{(n_1,n_s)\in\mathcal{M}}$, *the following relation holds*

$$\Big\| \sum_{(n_1,n_s)\in\mathcal{M}} f^{(n_1,n_s)} \Big\|^2_{L^2(\mathbf{R}^2)} \le \mathrm{Const} \sum_{(n_1,n_s)\in\mathcal{M}} \|f^{(n_1,n_s)}\|^2_{L^2(\mathbf{R}^2)}. \tag{27}$$

Set $\varkappa=(1,1)$. According to Proposition 2.1 it suffices to obtain (27) in the norm of $L^2(\mathbf{R}^2+i\varkappa)$. Since

$$|S(\xi+i\varkappa)|\asymp 1, \quad \xi\in\mathbf{R}^2,$$

we may switch to the rational functions

$$\phi^{(n_1,n_s)}(\zeta)=\sum_{1\le p<q\le s} \frac{a_{pq}^{(n_1,n_s)}}{(\langle\zeta,b^{(p)}\rangle-\gamma_p^{(n_p)})(\langle\zeta,b^{(q)}\rangle-\gamma_q^{(n_q)})},$$

here $\gamma_p^{(n_p)}=n_p+\alpha_p-i\beta_p$ with

$$\beta_p=\langle\varkappa,b^{(p)}\rangle>0, \quad p=1,\dots,s,$$

where the inequality follows from (26). The norm of $\sum_{(n_1,n_s)\in\mathcal{M}}\phi^{(n_1,n_s)}$ should be estimated in $L^2(\mathbf{R}^2)$. One can see directly that $\phi^{(n_1,n_s)}$ belongs to $\mathcal{H}^+$. Therefore

$$\Big\|\sum_{(n_1,n_s)\in\mathcal{M}}\phi^{(n_1,n_s)}\Big\|_{L^2(\mathbf{R}^2)}=\sup\bigg\{\bigg|\int_{\mathbf{R}^2}h(\xi)\sum_{(n_1,n_s)\in\mathcal{M}}\phi^{(n_1,n_s)}(\xi)\,dm_\xi\bigg|\,;h\in\mathcal{B}(\mathcal{H}^-)\bigg\},$$

where $\mathcal{B}(\mathcal{H}^-)$ denotes the unit ball in $\mathcal{H}^-$. We have

$$\begin{aligned}\int_{\mathbf{R}^2}h(\xi)\sum_{(n_1,n_s)\in\mathcal{M}}\phi^{(n_1,n_s)}(\xi)\,dm_\xi &= \sum_{(n_1,n_s)\in\mathcal{M}}\int_{\mathbf{R}^2}h(\xi)\phi^{(n_1,n_s)}(\xi)\,dm_\xi\\ &= \sum_{(n_1,n_s)\in\mathcal{M}} I^{(n_1,n_s)},\end{aligned}$$

and

$$I^{(n_1,n_s)}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}h(\xi_1,\xi_2)\sum_{1\le p<q\le s}\frac{a_{pq}^{(n_1,n_s)}}{(\langle\xi,b^{(p)}\rangle-\gamma_p^{(n_p)})(\langle\xi,b^{(q)}\rangle-\gamma_q^{(n_q)})}\,d\xi_2\,d\xi_1. \tag{28}$$

For $p=2,\dots,s$ we let $\varrho_p=\langle\xi,b^{(p)}\rangle=\xi_1b_1^{(p)}+\xi_2b_2^{(p)}$ and consider the function of two variables

$$\zeta_{(p)}(\xi_1,\varrho)=\frac{1}{b_2^{(p)}}\varrho-\frac{b_1^{(p)}}{b_2^{(p)}}\xi_1. \tag{29}$$

This function satisfies the relation $\langle(\xi_1,\zeta_{(p)}(\xi_1,\varrho)),b^{(p)}\rangle=\varrho$. The inner integrand in (28) is a rational function which has its poles when $\varrho_p=\gamma_p^{(n_p)}$, $p=2,\dots,s$, that is at the points $\zeta_{2,p}=\zeta_{(p)}(\xi_1,\gamma_p^{(n_p)})$. The imaginary parts of these points $\operatorname{Im}\zeta_{2,p}=-1-b_p^{(1)}/b_p^{(2)}$ are all different, since $b_1^{(p)}b_2^{(q)}-b_1^{(q)}b_2^{(p)}\neq 0$ when $p\neq q$. Therefore

$$I^{(n_1,n_s)}=\sum_{1<p\le s} I_p^{(n_1,n_s)}, \tag{30}$$

where

$$I_p^{(n_1,n_s)}=\int_{-\infty}^{\infty} h(\xi_1,\zeta_{(p)}(\xi_1,\gamma_p^{(n_p)}))\operatorname*{Res}_{\xi_2=\zeta_{(p)}(\xi_1,\gamma_p^{(n_p)})}\phi^{(n_1,n_s)}(\xi_1,\xi_2)\,d\xi_1.$$

For each $p=2,\dots,s$ we estimate

$$\begin{aligned}\sum_{(n_1,n_s)\in\mathcal{M}} I_p^{(n_1,n_s)} &=\sum_{n_p}\sum_{n_p(n_1,n_s)=n_p} I_p^{(n_1,n_s)}\\ &=\sum_{n_p}\int_{-\infty}^{\infty} h(\xi_1,\zeta_{(p)}(\xi_1,\gamma_p^{(n_p)})\operatorname*{Res}_{\xi_2=\zeta_{(p)}(\xi_1,\gamma_p^{(n_p)})}\phi^{(n_1,n_s)}(\xi_1,\xi_2)\,d\xi_1.\end{aligned} \tag{31}$$

It follows from (29), that, for $p=s$, $h(\xi_1,\zeta_{(p)}(\xi_1,\gamma_p^{(n_p)}))$ is holomorphic for all ξ_1, $\operatorname{Im}\xi_1<0$. For $p=2,\dots,s-1$ this function is holomorphic in the strip $0>\operatorname{Im}\xi_1>-1-b_2^{(p)}/b_1^{(p)}$.

Let now

$$r_p^{(n_1,n_s)}(\xi_1)=\operatorname*{Res}_{\xi_2=\zeta_{(p)}(\xi_1,\gamma_p^{(n_p)})}\phi^{(n_1,n_s)}(\xi_1,\xi_2).$$

These are rational functions of degree at most $s-1$. They decay at infinity and their poles are located at points xi_1 which satisfy $\langle(\xi_1,\xi_2),b^{(p)}\rangle=\gamma_p^{(n_p)}$ and $\langle(\xi_1,\xi_2),b^{(p)}\rangle=\gamma_q^{(n_q)}$. The corresponding points $(xi_1,\xi_2)+\varkappa$ belong to the same bunch $\Omega^{(n_1,n_s)}$. Therefore all poles of $r_p^{(n_1,n_s)}$ are located in a disk of diameter at most $A\varepsilon$.

On the other hand n_1 is different for different summands in (31). Since each $r_p^{(n_1,n_s)}$ has pole at the point ξ_1 satisfying $b_1^{(1)}\xi_1=\gamma_1^{n_1}$, assuming ε taken small enough, we obtain that all poles of $r_p^{(n_1,n_s)}$ are located in small disks around the points $\gamma_1^{(n_1)}/b_1^{(1)}$. Applying now Lemma 2.1 we obtain for sufficiently small δ,

$$\int_{-\infty}^{\infty}\Bigl|\sum_{n_p(n_1,n_s)=n_p} r_p^{(n_1,n_s)}(\xi_1-i\delta)\Bigr|^2 d\xi_1\le \sum_{n_p(n_1,n_s)=n_p}\int_{-\infty}^{\infty}|r_p^{(n_1,n_s)}(\xi_1-i\delta)|^2\,d\xi_1. \tag{32}$$

In order to estimate each $I_p^{(n_1,n_s)}$ we need two more inequalities. The inequality

$$\|r_p^{(n_1,n_s)}(\,\cdot\,-i\delta)\|_{L^2(\mathbf{R})}\le \operatorname{Const}\|\phi^{(n_1,n_s)}\|_{\mathbf{R}^2}$$

follows from the definition of $r_p^{(n_1,n_s)}$, and the triangle inequality.

The inequality

$$\sum_{n_p=-\infty}^{\infty}\int_{-\infty}^{\infty}|h(\xi_1-i\delta,\zeta_{(p)}(\xi_1-i\delta,\gamma_p^{(n_p)}))|^2\,d\xi_1\le \mathrm{Const}\,\|h\|^2_{\mathcal{H}^-}$$

follows from Proposition 2.6.

Now it remains to change the path of integration in (30) to $\mathbf{R}-i\delta$ and then apply the Cauchy inequality twice,

$$\begin{aligned}
I_p^{(n_1,n_s)} &= \sum_{n_p=-\infty}^{\infty}\int_{-\infty}^{\infty}h(\xi_1-i\delta,\zeta_{(p)}(\xi_1-i\delta,\gamma_p^{(n_p)}))\sum_{n_p(n_1,n_s)=n_p}r_p^{(n_1,n_s)}(\xi_1-i\delta)\\
&\le \sum_{n_p=-\infty}^{\infty}\left(\int_{-\infty}^{\infty}|h(\xi_1-i\delta,\zeta_{(p)}(\xi_1-i\delta,\gamma_p^{(n_p)}))|^2\,d\xi_1\right)^{1/2}\\
&\quad\times\left(\int_{-\infty}^{\infty}\left|\sum_{n_p(n_1,n_s)=n_p}r_p^{(n_1,n_s)}(\xi_1-i\delta)\right|^2 d\xi_1\right)^{1/2}\\
&\le \left(\sum_{n_p=-\infty}^{\infty}\int_{-\infty}^{\infty}|h(\xi_1-i\delta,\zeta_{(p)}(\xi_1-i\delta,\gamma_p^{(n_p)}))|^2 d\xi_1\right)^{1/2}\\
&\quad\times\left(\sum_{n_p=-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\sum_{n_p(n_1,n_s)=n_p}r_p^{(n_1,n_s)}(\xi_1-i\delta)\right|^2 d\xi_1\right)^{1/2}\\
&\le \mathrm{Const}\,\|h\|_{\mathcal{H}^-}\left(\sum_{n_p=-\infty}^{\infty}\sum_{n_p(n_1,n_s)=n_p}\|\phi^{(n_1,n_s)}\|^2_{L^2(\mathbf{R}^2)}\right)^{1/2}\\
&\le \mathrm{Const}\,\|h\|_{\mathcal{H}^-}\left(\sum_{n_p=-\infty}^{\infty}\sum_{n_p(n_1,n_s)=n_p}\|f^{(n_1,n_s)}\|^2_{PW_M}\right)^{1/2}.
\end{aligned}$$

This completes the proof of Lemma 6.1.

Construction and estimate of projectors (proof of Lemma 6.2)

The main step in the proof of Lemma 6.2 is a construction of the projector $\mathcal{P}^{(m)}$.

Let a bunch Ω_m be fixed. As in Lemma 6.1 we assume for definiteness that its type is $\{1,\ 2,\dots,s\}$ and the corresponding indices are $\{n_1,\ n_2,\dots,n_s\}$. The corresponding space X_m is

$$X_m=S(z)Y_m,$$

where $S(z)$ is the generating function (see (13)), and Y_m is the space of rational functions of the form

$$\psi(z)=\sum_{1\le p<q\le s}\frac{a_{pq}}{(\langle z,b^{(p)}\rangle-n_p-\alpha_p)(\langle z,b^{(q)}\rangle-n_q-\alpha_q)}. \tag{33}$$

We start by clarifying the block condition, i.e. we express the fact that all points of Ω_m are located in a disk of diameter at most $\operatorname{diam}\Omega$. Consider the linear forms

$$\lambda_l(z)=\lambda_l^{(n_1,n_s)}(z)=\langle z,b^{(l)}\rangle-n_l-\alpha_l,\quad l=1,\ 2,\dots,s. \tag{34}$$

Each such form vanishes at the points $\omega^{(l,n_l)(p,n_p)}$, $p\neq l$. Since $\operatorname{diam}(\Omega_m)<A\varepsilon$, we have

$$|\lambda_l(z)|\le c\varepsilon \tag{35}$$

when $\operatorname{dist}(z,\Omega_m)<10\varepsilon$, say. The constant c can be chosen independent of m.

We need an additional construction. Fix $j\in\{2,\dots,s\}$. Then each vector $b^{(l)}$ admits a (unique) representation

$$b^{(l)}=c_{1,l}(j)b^{(1)}+c_{j,l}b^{(j)}. \tag{36}$$

Similarly we have

$$\lambda_l(\zeta)=c_{1,l}(j)\lambda_1(\zeta)+c_{j,l}\lambda_j(\zeta)+n_{jl}, \tag{37}$$

where

$$n_{jl}=c_{1,l}(j)(n_1+\alpha_1)+c_{j,l}(n_j+\alpha_j)-(n_l+\alpha_l). \tag{38}$$

Let

$$K=\sup_{j,l}\{|c_{1,l}(j)|,\ |c_{j,l}|\}.$$

It follows from (35) and (37) that

$$|n_{j,l}|\le 2Kc\varepsilon,$$

the estimate still being independent of the block number.

We also note that (keeping j fixed) we can take λ_1 and λ_j as independent variables. Then $\zeta=\zeta^{(j)}(\lambda_1,\lambda_j)$ is the solution to the equations

$$\lambda_1(\zeta)=\lambda_1,\quad \lambda_j(\zeta)=\lambda_j.$$

Let $0<r<R<\infty$ be some numbers, that will be specified later, and $\varepsilon_m=\max\{\varepsilon,\operatorname{diam}\Omega_m\}$. Let

$$T_j=T_j^{(m)}=\{(\lambda_1,\lambda_j)\in\mathbf{C}^2\,;\,|\lambda_1|=R\varepsilon_m,\ |\lambda_j|=r\varepsilon_m\}.$$

Given a function $\psi(\zeta)$, $\zeta\in T_j$, we put

$$Q_j^{(m)}\psi(z)=\frac{1}{(2i\pi)^2}\int_{T_j}\psi(\zeta^{(j)}(\lambda_1,\lambda_j))\frac{d\lambda_j\,d\lambda_1}{(\lambda_j(z)-\lambda_j)(\lambda_1(z)-\lambda_1)}.$$

Lemma 6.3. *There exist* $0<r<R<\infty$ *and* $a>0$ *such that*

$$\psi(z)=\sum_{1<j\leq s} \mathcal{Q}_j^{(m)}\psi(z) \tag{39}$$

for all $\psi\in Y_m$ *and* $z\in\mathbf{C}^2$ *satisfying*

$$|S(z)|e^{-H_M(z)}>a\varepsilon. \tag{40}$$

The numbers r, R *and* a *can be chosen to be the same for all bunches* Ω_m.

Proof. Let $\psi\in Y_m$ have the form (33). We have

$$\mathcal{Q}_j^{(m)}\psi(z)=\frac{1}{(2i\pi)^2}\int_{|\lambda_1|=R\varepsilon_m}\sum_{1\leq p<q\leq s} a_{pq}J_{pq}\,\frac{d\lambda_1}{\lambda_1(z)-\lambda_1},$$

where

$$J_{pq}=\int_{|\lambda_j|=r\varepsilon_m}\frac{d\lambda_j}{\lambda_p(\zeta^{(j)}(\lambda_1,\lambda_j))\lambda_q(\zeta^{(j)}(\lambda_1,\lambda_j))(\lambda_j(z)-\lambda_j)}.$$

In the integral J_{pq} we have $|\lambda_1|=R\varepsilon_m$. Relation (37) yields

$$|\lambda_p|\geq|c_{1,p}(j)|\,|\lambda_1|-K|\lambda_j|-2Kc\varepsilon$$

for $p\neq j$, 1. Taking R and r so that

$$R\min_{l,j}|c_{1,l}(j)|>2K(r+2c)$$

we see that for $p\neq j$, $\lambda_p(\zeta^{(j)}(\lambda_1,\lambda_j))$, if considered as a function with respect to λ_j, does not vanish in $|\lambda_j|\leq r\varepsilon_m$. Take $a>2R$. Relation (40) yields $\lambda_j(z)-\lambda_j\neq 0$ for $|\lambda_j|\leq r\varepsilon_m$. Therefore $J_{pq}=0$ if $j\neq p$, q, and the only summands we are left with are those for which either p or q equals j.

Up to now the coefficients a_{pq} have been defined for $p<q$ only. Set $a_{qp}=a_{pq}$. Then

$$\mathcal{Q}_j^{(m)}\psi(z)=\sum_{\substack{1\leq l\leq s\\ l\neq j}}\frac{a_{jl}}{2i\pi}\int_{|\lambda_1|=R\varepsilon_m} I_{jl}\,\frac{d\lambda_1}{\lambda_1(z)-\lambda_1}, \tag{41}$$

where

$$I_{jl}=\frac{1}{2i\pi}\int_{|\lambda_j|=r\varepsilon_m}\frac{1}{\lambda_j\lambda_l(\zeta^{(j)}(\lambda_1,\lambda_j))}\,\frac{d\lambda_j}{\lambda_j(z)-\lambda_j}.$$

Let $l>1$. We use the identity

$$\frac{1}{\lambda_j\lambda_l}=\frac{1}{\lambda_l-c_{jl}\lambda_j}\left(\frac{1}{\lambda_j}-\frac{c_{jl}}{\lambda_l}\right).$$

If $\lambda_l=\lambda_l(\zeta^{(j)}(\lambda_1,\lambda_j))$, relation (37) yields

$$\frac{1}{\lambda_j\lambda_l}=\frac{1}{c_{1l}(j)\lambda_1+n_{jl}}\left(\frac{1}{\lambda_j}-\frac{c_{jl}}{\lambda_l}\right),\quad \lambda_l=\lambda_l(\zeta^{(j)}(\lambda_1,\lambda_j)). \tag{42}$$

Back to (41) we obtain

$$\begin{aligned}I_{jl}&=\frac{1}{2i\pi}\int_{|\lambda_j|=r\varepsilon_m}\frac{d\lambda_j}{\lambda_j(c_{1l}(j)\lambda_1+n_{jl})(\lambda_j(z)-\lambda_j)}\\&\quad-\frac{1}{2i\pi}\int_{|\lambda_j|=r\varepsilon_m}\frac{c_{jl}\,d\lambda_j}{\lambda_l(\zeta^{(j)}(\lambda_1,\lambda_j))(c_{1l}(j)\lambda_1+n_{jl})(\lambda_j(z)-\lambda_j)}\\&=\frac{1}{\lambda_j(z)(c_{1l}(j)\lambda_1+n_{jl})},\end{aligned}$$

here the first summand in the middle expression is evaluated by the residue theorem while the second just vanishes since $\lambda_l(\zeta^{(j)}(\lambda_1,\lambda_j))\neq0$ when $|\lambda_j|\le r\varepsilon_m$ and $|\lambda_1|=R\varepsilon_m$. Besides, a straightforward calculation gives

$$I_{1j}=\frac{1}{\lambda_1\lambda_j(z)}.$$

Now

$$\begin{aligned}Q_j^{(m)}\psi(z)&=\frac{a_{1j}}{2i\pi}\int_{|\lambda_1|=R\varepsilon_m}\frac{d\lambda_1}{\lambda_1\lambda_j(z)(\lambda_1(z)-\lambda_1)}\\&\quad+\sum_{\substack{1<l\le s\\ l\neq j}}\frac{a_{lj}}{2i\pi}\int_{|\lambda_1|=R\varepsilon_m}\frac{d\lambda_1}{\lambda_j(z)(c_{1l}(j)\lambda_1+n_{jl})(\lambda_1(z)-\lambda_1)}.\end{aligned}$$

Consider the two factors $c_{1l}(j)\lambda_1+n_{jl}$ and $\lambda_1(z)-\lambda_1$ as functions with respect to λ_1. One of them vanishes outside while the second vanishes inside the circle $|\lambda_1|=R\varepsilon_m$. Therefore

$$Q_j^{(m)}\psi(z)=\frac{a_{1j}}{\lambda_1(z)\lambda_j(z)}+\sum_{\substack{1<l\le s\\ l\neq j}}\frac{a_{lj}}{\lambda_j(z)(c_{1l}(j)\lambda_1(z)+n_{jl})}.$$

Summing this up we obtain

$$\sum_{1<j\le s} \mathcal{Q}_j^{(m)}\psi(z) = \sum_{1<j\le s} \frac{a_{1j}}{\lambda_1(z)\lambda_j(z)}$$

$$(43)\qquad + \sum_{1<j<l\le s} a_{jl}\left(\frac{1}{\lambda_j(z)(c_{1l}(j)\lambda_1(z)+n_{jl})} + \frac{1}{\lambda_l(z)(c_{1j}(l)\lambda_1(z)+n_{lj})}\right).$$

Now we observe that

$$(44)\qquad c_{1j}(l) = -\frac{c_{1l}(j)}{c_{jl}}, \quad c_{lj} = \frac{1}{c_{jl}}, \quad n_{lj} = -\frac{n_{jl}}{c_{jl}}.$$

Therefore, for $\lambda_1=\lambda_1(z)$, $\lambda_j=\lambda_j(z)$, $\lambda_l=\lambda_l(z)$, we get

$$(45)\qquad \begin{aligned} \frac{1}{(c_{1l}(j)\lambda_1+n_{jl})\lambda_j} + \frac{1}{(c_{1j}(l)\lambda_1+n_{lj})\lambda_l} &= \frac{1}{(c_{1l}(j)\lambda_1+n_{jl})\lambda_j} - \frac{c_{jl}}{(c_{1l}(j)\lambda_1+n_{jl})\lambda_l} \\ &= \frac{1}{\lambda_j\lambda_l}. \end{aligned}$$

The latter equality follows from (42). Substitution of this into (43) yields the desired relation (39).

In what follows we need to locate the points $\zeta\in\mathbf{C}^2$ which correspond to the points $(\lambda_1,\lambda_j)\in T_j^{(m)}$. Given a bunch Ω_m of the type $(k_1,\dots,k_s)$, set

$$\mathcal{T}_j^{(m)} = \{\zeta = \zeta^{(k_j)}(\lambda_{k_1},\lambda_{k_j}) ; |\lambda_{k_1}| = R\varepsilon_m,\ |\lambda_{k_j}| = r\varepsilon_m\},$$

$$\mathcal{T}^{(m)} = \bigcup_{j=2}^{s(m)} \mathcal{T}_j^{(m)}, \quad \text{and} \quad \mathcal{T} = \bigcup_{m=1}^{\infty} \mathcal{T}^{(m)}.$$

Since the choice of r and R is independent of Δ, we can now find Δ_0 such that, for each (ε,Δ)-partition with $\Delta>\Delta_0$, we have

$$(46)\qquad \operatorname{dist}(\mathcal{T}, Z(S)) > 0,$$

here $Z(S)$ is the zero set of the function S. Besides

$$(47)\qquad L := \sup\{|\operatorname{Im} z| ; z\in\mathcal{T}\} < \infty.$$

Now define

$$(48)\qquad \mathcal{Q}^{(m)} = \sum_{j=2}^{s(m)} \mathcal{Q}_j^{(m)}$$

and

$$(49)\qquad \mathcal{P}^{(m)} : f \longmapsto S(z)\mathcal{Q}^{(m)}\left(\frac{f}{S}\right)(z);$$

the right-hand side of (49) is well-defined for z satisfying (40). Actually, it has an extension to the whole $\mathbf{C}^2$.

Lemma 6.4. *For any $f \in PW_M$ the function $\mathcal{P}^{(m)} f$ can be extended to a function from PW_M.*

Proof. We start by giving an explicit expression for $\mathcal{P}^{(m)}$. Let $f \in PW_M$ and z satisfy (40). We keep the notation of Lemma 6.3,

$$\mathcal{Q}_j^{(m)}\left(\frac{f}{S}\right)(z) = \frac{1}{2i\pi}\int_{|\lambda_1|=R\varepsilon_m} J_j(z)\,\frac{d\lambda_1}{\lambda_1(z)-\lambda_1},$$

where

$$J_j(z) = \frac{1}{2i\pi}\int_{|\lambda_j|=r\varepsilon_m} \frac{f(\zeta^{(j)}(\lambda_1,\lambda_j))}{S(\zeta^{(j)}(\lambda_1,\lambda_j))}\,\frac{d\lambda_j}{\lambda_j(z)-\lambda_j}$$

Let

$$S_j(\lambda_1,\lambda_j) = \frac{S(\zeta^{(j)}(\lambda_1,\lambda_j))}{\lambda_j}.$$

As was noticed in the proof of Lemma 6.3, the functions $\lambda_p(\zeta^{(j)}(\lambda_1,\lambda_j))$, $p \neq j$, do not vanish on $\{(\lambda_1,\lambda_j); |\lambda_1| = R\varepsilon_m,\ |\lambda_j| \leq r\varepsilon_m\}$, and so the same is true for the function S_j. Therefore,

$$J_j(z) = \frac{1}{2i\pi}\int_{|\lambda_j|=r\varepsilon_m} \frac{f(\zeta^{(j)}(\lambda_1,\lambda_j))}{\lambda_j S_j(\lambda_1,\lambda_j)}\,\frac{d\lambda_j}{\lambda_j(z)-\lambda_j} = \frac{f(\zeta^{(j)}(\lambda_1,0))}{\lambda_j S_j(\lambda_1,0)\lambda_j(z)},$$

here $\zeta^{(j)}(\lambda_1,0)$ is the unique solution to the system

$$\lambda_1(\zeta) = \lambda_1, \quad \lambda_j(\zeta) = 0, \tag{50}$$

and so,

$$\mathcal{Q}_j^{(m)}\left(\frac{f}{S}\right)(z) = \frac{1}{2i\pi}\int_{|\lambda_1|=R\varepsilon_m} \frac{f(\zeta^{(j)}(\lambda_1,0))}{\lambda_j S_j(\lambda_1,0)\lambda_j(z)}\,\frac{d\lambda_1}{\lambda_1(z)-\lambda_1}. \tag{51}$$

The integrand in (51) has its singularities at the zeros of the function $S_j(\lambda_1,0)$. It follows from (50) that these are precisely $\lambda_1(\zeta_l^j)$, $\zeta_l^j = \omega^{(l,n_l)(j,n_j)}$, $l \neq j$. Besides, relations (34) and (36)–(38) imply

$$c_{1,j}(l) = -\frac{n_{jl}}{c_{1,l}(j)}.$$

Thus

$$\mathcal{Q}_j^{(m)}\left(\frac{f}{S}\right)(z) = \frac{f(\omega^{(1,n_1)(j,n_j)})}{\lambda_1(z)\lambda_j(z)s_j} + \sum_{\substack{l>1\\ l\neq j}} f(\omega^{(l,n_l)(j,n_j)})A_{jl}(z) \tag{52}$$

with

$$A_{jl}(z)=\frac{1}{\lambda_j(z)(\lambda_1(z)-\lambda_1(\zeta_l^j))\lambda_1(\zeta_l^j)s_{j,l}}.$$

Here, by direct calculation, $s_j=S_j(0,0)\neq 0$ and

$$s_{j,l}=\pi^2 c_{1,l}(j)\prod_{k\neq j,l}\sin\pi\lambda_k(\omega^{(l,n_l)(j,n_j)}).$$

In particular, by (44), $s_{l,j}=-s_{j,l}/c_{j,l}$. Therefore,

$$A_{l,j}(z)=\frac{1}{\lambda_l(z)(\lambda_1(z)+n_{lj}/c_{1,j}(l))\lambda_1(\zeta_l^j)s_{l,j}}=-\frac{c_{j,l}}{\lambda_l(z)(\lambda_1(z)+n_{jl}/c_{1,l}(j))\lambda_1(\zeta_l^j)s_{j,l}}$$

and, in view of (45),

$$A_{j,l}(z)+A_{l,j}(z)=\frac{c_{1l}(j)}{\lambda_j(z)\lambda_l(z)s_{jl}}. \tag{53}$$

Now (52) and (53) give

$$\begin{aligned}\mathcal{Q}^{(m)}\left(\frac{f}{S}\right)(z)&=\sum_{1<j<l\leq s}f(\omega^{(l,n_l)(j,n_j)})(A_{j,l}(z)+A_{l,j}(z))=\sum_{1<j\leq s}\frac{f(\omega^{(1,n_1)(j,n_j)})}{\lambda_1(z)\lambda_j(z)s_j}\\&=\sum_{1<j<l\leq s}\frac{f(\omega^{(l,n_l)(j,n_j)})c_{1l}(j)}{\lambda_j(z)\lambda_l(z)s_{jl}}+\sum_{1<j\leq s}\frac{f(\omega^{(1,n_1)(j,n_j)})}{\lambda_1(z)\lambda_j(z)s_j}.\end{aligned}$$

Therefore, if $|S(z)|>a\varepsilon$,

$$\mathcal{P}^{(m)}f(z)=S(z)\left(\sum_{1<j\leq s}\frac{f(\omega^{(1,n_1)(j,n_j)})}{\lambda_1(z)\lambda_j(z)s_j}+\sum_{1<j<l\leq s}\frac{f(\omega^{(l,n_l)(j,n_j)})c_{1l}(j)}{\lambda_j(z)\lambda_l(z)s_{jl}}\right). \tag{54}$$

Representation (54) implies that $\mathcal{P}^{(m)}f$ can be extended to an entire function in $\mathbf{C}^2$. A direct estimate shows that $\mathcal{P}^{(m)}f\in PW_M$. The proof of Lemma 6.4 is complete.

The operator $\mathcal{P}^{(m)}$ thus constructed is the desired projector. Indeed, it follows from Lemma 6.3 that $\mathcal{P}^{(m)}|_{X_m}=\mathrm{id}$. On the other hand, (54) yields that $\mathcal{P}^{(m)}f=\mathbf{0}$ for all $f\in PW_M$ satisfying $f|_{\Omega_m}=\mathbf{0}$. In particular $\mathcal{P}^{(m)}|_{X_k}=\mathbf{0}$, $m\neq k$.

Now we prove (25). Let $\varkappa=(1,1)$. It suffices to prove that

$$\sum_m\|\mathcal{P}^{(m)}f\|^2_{L^2(\mathbf{R}^2+2iL\varkappa)}\leq \mathrm{Const}\,\|f\|^2_{L^2(\mathbf{R}^2)}.$$

Consider the set

$$E_a = \{z \in \mathbf{R}^2 + 2iL\varkappa\,;\, |S(z)| > a\varepsilon\}.$$

This set consists of the whole plane $\mathbf{R}^2+2iL\varkappa$ from which small strips around intersections of this plane with $Z(S)$ are eliminated. For $z \in E_a$, one can use representation (48), (47), and (41).

On the other hand, E_a is relatively dense in $\mathbf{R}^2+2iL\varkappa$, that is there exist $A>0$ and $\delta>0$ such that

$$\operatorname{mes}(Q_A \cap E_a) > \delta$$

for each square $Q_A \subset \mathbf{R}^2+2iL\varkappa$ with side length A. It now follows from [12] that, for $f \in PW_M$,

$$\|f\|_{L^2(\mathbf{R}^2+2iL\varkappa)} \le \text{Const}\, \|f\|_{L^2(E_a)}.$$

So (25) will follow from

$$\sum_m \|\mathcal{P}^{(m)}\|^2_{L^2(E_a)} \le \text{Const}\, \|f\|^2_{L^2(\mathbf{R}^2)}. \tag{55}$$

Fix a number m. We have

$$\mathcal{P}^{(m)} f(z) = S(z) \sum_{j=2}^{s(m)} \mathcal{Q}_j^{(m)}\left(\frac{f}{S}\right)(z),$$

and since $|S(z)| \le \text{Const}$, $z \in \mathbf{R}^2+2iL\varkappa$, one can replace $\mathcal{P}^{(m)}$ by $\mathcal{Q}^{(m)}$ in (55). Furthermore we have

$$\mathcal{Q}^{(m)} f(z) = \sum_{j=2}^{s(m)} \mathcal{Q}_j^{(m)}\left(\frac{f}{S}\right)(z), \quad z \in E_a,$$

and

$$\mathcal{Q}_j^{(m)}\left(\frac{f}{S}\right)(z) = \frac{1}{(2i\pi)^2} \int_{T_j^{(m)}} \frac{f(\zeta^{(j)}(\lambda_1,\lambda_j))}{S(\zeta^{(j)}(\lambda_1,\lambda_j))} \, \frac{d\lambda_j\, d\lambda_1}{(\lambda_j(z)-\lambda_j)(\lambda_1(z)-\lambda_1)}.$$

Now the triangle inequality yields

$$\left\| \mathcal{Q}_j^{(m)}\left(\frac{f}{S}\right)(z) \right\|_{L^2(E_a)} \le \frac{1}{2i\pi} \int_{T_j^{(m)}} \left| \frac{f(\zeta^{(j)}(\lambda_1,\lambda_j))}{S(\zeta^{(j)}(\lambda_1,\lambda_j))} \right| |d\lambda_1|\, |d\lambda_j|$$
$$\times \sup_{(\lambda_1,\lambda_j)\in T_j^{(m)}} \left\| \frac{1}{(\lambda_j(z)-\lambda_j)(\lambda_1(z)-\lambda_1)} \right\|_{L^2(E_a)}.$$

The second factor on the right-hand side is uniformly bounded. In order to estimate the first factor we note that (46) and (47) yield

$$|S(z)| \asymp 1, \quad z \in \mathcal{T}.$$

Therefore

$$\begin{aligned}\int_{T_j^{(m)}} \left| \frac{f(\zeta^{(j)}(\lambda_1, \lambda_j))}{S(\zeta^{(j)}(\lambda_1, \lambda_j))} \right| |d\lambda_1|\,|d\lambda_j| &\asymp \int_{T_j^{(m)}} |f(\zeta^{(j)}(\lambda_1, \lambda_j)|\,|d\lambda_1|\,|d\lambda_j| \\ &\leq \mathrm{Const} \left(\int_{T_j^{(m)}} |f(\zeta^{(j)}(\lambda_1, \lambda_j))|^2\,|d\lambda_1|\,|d\lambda_j| \right)^{1/2} \\ &\leq \mathrm{Const} \left(\int_{T_j^{(m)}} |f(\zeta)|^2\, dm_\zeta \right)^{1/2}.\end{aligned}$$

Finally we have

$$\sum_m \|\mathcal{P}^{(m)} f\|^2_{L^2(\mathbf{R}^2 + 2iL\varkappa)} \leq \mathrm{Const} \sum_m \int_{T_j^{(m)}} |f(\zeta)|^2\, dm_\zeta$$

and it remains to apply the inequality

$$\sum_m \int_{T_j^{(m)}} |f(\zeta)|^2\, dm_\zeta \leq \mathrm{Const}\, \|f\|^2_{PW_M}, \quad f \in PW_M.$$

The proof of this inequality is similar to that in Proposition 2.6. It follows from subharmonicity arguments combined with estimate (3).

Now given a function $f \in PW_M$, set $f_m = \mathcal{P}^{(m)} f$. By Lemma 6.2 we have $\sum_m \|f_m\|^2 < \infty$, and by Lemma 6.1 the series

$$g(z) = \sum_m f_m(z)$$

converges both in $L^2(\mathbf{R}^2)$-norm and compactwise. It follows that $(g-f)|_\Omega = \mathbf{0}$, so by Theorem 1, $g = f$. This completes the proof of Theorem 4.

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