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COMPLETE KÄHLER MANIFOLDS WITH ZERO RICCI CURVATURE. I

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The problem of constructing complete manifolds with zero Ricci curvature is important for both physicists and geometers. When the manifold is compact and Kähler, this problem was solved satisfactorily by the second author in 1976. While it is not difficult to construct explicit examples of noncompact manifolds with zero Ricci curvature, a complete understanding of complete noncompact manifolds with zero Ricci curvature is still needed. Therefore, immediately after the work in 1976, the second author proposed a scheme to classify these manifolds. This is the first part of the papers being written by the authors on a systematic research on the existence of these metrics. They have natural applications to algebraic geometry which shall be reported on later.

A typical theorem we prove is the following. Let D be a neat, almost ample smooth divisor in a projective manifold $\overline{M}$. Let Ω be any $(1, 1)$-form representing the first Chern class of $K_{\overline{M}}^{-1} \otimes D^{-1}$. Then there is a complete Kähler metric with Ω as its Ricci form. (We define a divisor to be neat if no compact algebraic curve in $\overline{M}\backslash D$ is homologous to a linear sum of curves supported in D.) In particular, if D is the anticanonical divisor, there is a complete Ricci flat Kähler metric on $\overline{M}\backslash D$. If $K_{\overline{M}}^{-1} \otimes D^{-1}$ is ample, $\overline{M}\backslash D$ carries a complete Kähler metric with positive Ricci curvature. Naturally, such theorems immediately imply theorems on the topology of $M\backslash D$. For example, D is connected and the fundamental group of $M\backslash D$ is almost nilpotent.

The assumption on the smoothness of D can be removed. When D has normal crossing and higher multiplicity, the situation is more complicated and we shall deal with it later.

This paper can be looked upon as a study of the solutions of a global complex Monge-Ampère equation on a Kähler manifold. The uniqueness of such an equation is very interesting and largely unknown. It is presumably related to the automorphism of the manifold.

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1. An existence theorem for some complex Monge-Ampère equations

Let (M, g) be a complete Kähler manifold and ω_g be the Kähler form associated to the metric g. In local coordinates $(z_1, \dots, z_n)$, the metric g is represented by the tensor $(g_{i\bar{j}})_{1\le i,j\le n}$, where $n = \dim_{\mathbf{C}} M$, and

$$\omega_g = \frac{\sqrt{-1}}{2\pi}\sum_{i,j=1}^{n} g_{i\bar{j}}\, dz_i \wedge d\bar{z}_j.$$

Consider the following complex Monge-Ampère equation on M,

$$(1.1)\qquad \begin{cases} \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^n = e^f \omega_g^n, \\ \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi > 0, \qquad \varphi \in C^\infty(M, R^1), \end{cases}$$

where $\omega_g^n = \omega_g \wedge \cdots \wedge \omega_g$ and f is a given smooth function satisfying the integrability condition

$$(1.2)\qquad \int_M (e^f - 1)\omega_g^n = 0.$$

For any solution φ of equation (1.1), the $(1, 1)$-form $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ defines a new Kähler metric. By the well-known expression of the Ricci curvature on a Kähler manifold, one can easily check (cf. [Y2]) that the Ricci curvature form of this new metric is given by $\mathrm{Ric}(g) - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f$. Thus, in order to construct the Kähler metric with prescribed Ricci curvature, it suffices to solve equation (1.1) with properly chosen function f. For instance, if $\mathrm{Ric}(g) - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f > 0$, then the solution of (1.1) gives a Kähler metric with positive Ricci curvature. In 1976, the second author solved the famous Calabi's conjecture by proving the solvability of (1.1) when M is compact. Here we study the solvability of (1.1) when M is noncompact. We will prove an existence theorem (Theorem 1.1) for (1.1) under certain assumptions on the decay of f at infinity. This existence theorem will be applied later to construct complex Ricci flat metrics and complete Kähler metrics with positive Ricci curvature on many complete Kähler manifolds.

In order to state our theorem, we need the following definitions.

Definition 1.1. Let (X, ds^2) be a complete Riemannian manifold, and let K, α, β be nonnegative numbers. We say that the manifold (X, ds^2) is of (K, α, β)-polynomial growth if its sectional curvature is bounded by K, $\mathrm{Vol}_{ds^2}(B_R(x_0)) \le CR^\alpha$ for all $R > 0$, and $\mathrm{Vol}_{ds^2}(B_1(x)) \ge C^{-1}(1 + \mathrm{dist}(x_0, x))^{-\beta}$ for some fixed point x_0 in M and some constant C independent of x. Here $B_R(x_0)$ denotes the geodesic ball with center at x_0 and radius R, and Vol_{ds^2} denotes the volume associated to the metric ds^2.

Definition 1.2. We say that the complete Kähler manifold (M, g) is of quasi-finite geometry of order $l + \delta$ if there are positive numbers $r > 0$, $r_1 > r_2 > 0$

such that for any x in M, there is a holomorphic map φ_x from a domain U_x in $\mathbf{C}^n$ containing the origin 0 onto the geodesic ball $B_r(x)$ satisfying:

(1) $\varphi_x(0) = x$, $D_{r_2} \subseteq U_x \subseteq D_{r_1}$, where $D_s = \{z \in \mathbf{C}^n \mid |z| \leq s\}$.

(2) The pull-back metric $\varphi_x^* g$ is a Kähler metric in U_x such that in the natural coordinate system on $\mathbf{C}^n$, the metric tensor of $\varphi_x^* g$ and its derivatives up to order l are bounded and δ-Hölder-continuously bounded.

In the following, denote by ∇_g and Δ_g the gradient and the laplacian of the metric g, respectively.

Theorem 1.1. *Let (M, g) be a complete Kähler manifold of quasi-finite geometry of order $2 + \frac{1}{2}$ and with $(K, 2, \beta)$-polynomial growth. Let f be a smooth function satisfying the integrability condition* (1.2) *and, for some constant C,*

$$(1.3) \qquad \sup_M\{|\nabla_g f|, |\Delta_g f|\} \leq C, \qquad |f(x)| \leq \frac{C}{(1+\rho(x))^N}, \quad x \in M,$$

where $N \geq 4 + 2\beta$ and $\rho(x) = \operatorname{dist}_g(x_0, x)$ is the distance function on M from a fixed point x_0.

Then equation (1.1) *has a bounded, smooth solution φ such that $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ defines a complete Kähler metric equivalent to g. In fact, the supreme norms of φ and its derivatives can be bounded by constants depending only on f, C, N, K, β, and the order of the derivative.*

We will prove this theorem in §§2 and 3.

Remarks. (1) The manifold (M, g) under consideration is parabolic, i.e., the volume growth is not greater than quadratic growth. This restriction on the volume growth could be removed by some estimates on the Green's kernel of some elliptic operator of second order.

(2) One should be able to prove that the resulting metric in Theorem 1.1 is asymptotically as close to g as possible if the function f and its derivatives decay sufficiently fast. We shall return to this question in the future.

We end this section by a proposition on the quasi-finite geometry of a Kähler manifold with some assumptions on the curvature.

Proposition 1.2. *Let (M, g) be a complete Kähler manifold with its sectional curvature and the covariant derivative of its scalar curvature bounded. Then (M, g) is of quasi-finite geometry of order $2 + \frac{1}{2}$.*

Proof. Denote by $R(g)$ and $S(g)$ the sectional curvature and scalar curvature, respectively. By scaling, we may assume that

$$(1.4) \qquad \sup_M\{\|R(g)\|_g(x), \|DS(g)\|_g(x)\} \leq 1.$$

Then for any point x in M, there is no conjugate point of x in the geodesic ball $B_{\pi/2}(x)$. Then the exponential map $\exp_x : B_{\pi/2}(0) \subset T_xM \to B_{\pi/2}(x) \subset M$ is locally diffeomorphic. By pulling back the complex structure of M and the Kähler metric g, we obtain a new Kähler manifold $(B_{\pi/2}(0), \exp_x^* g)$ with boundary $\partial B_{\pi/2}(0)$. Clearly, the injectivity radius $\operatorname{Inj}\operatorname{Rad}(0)$ is $\pi/2$ for the manifold $(B_{\pi/2}(0), \exp_x^* g)$. Let $\rho(y) = \operatorname{dist}_{\exp_x^* g}(y, 0)$ be the distance function from 0 on $(B_{\pi/2}(0), \exp_x^* g)$. Then ρ is smooth. By the Hessian comparison theorem (cf. [SY, GW]), the functions ρ^2 and $\log(\rho^2)$ are plurisubharmonic on $B_r = \{y \in T_x M | \rho(y) < r\}$ and

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\rho^2 \geq C > 0 \quad \text{on } B_r(0), \tag{1.5}$$

where r, C are two small positive numbers independent of x, and the operators ∂, $\overline{\partial}$ are induced from those on M by $\exp_x$.

Next, consider the $\overline{\partial}$-equation $\overline{\partial}u = v$ on $(B_r, \exp_x^* g)$, where v is a smooth $(0, 1)$-form with $\overline{\partial}v = 0$ and $u \in C^\infty(B_r, R^1)$. By using Hörmander's L^2-estimate with weight function $4n\log(\rho^2) + \varphi(\rho^2)$ for a suitable convex function φ and taking r smaller if necessary, we can construct local holomorphic coordinates $(Z_1, \dots, Z_n)$ on $(B_r, \exp_x^* g)$ (cf. [SY, GW]). Thus we conclude that B_r is a domain in C^n satisfying (1) in Definition 1.2. In the local system $(z_1, \dots, z_n)$, the metric $\exp_x^* g$ is represented by a hermitian matrix $(g_{\overline{i}\overline{j}})_{1\leq i,j\leq n}$. By the boundedness of the curvature tensor of $\exp_x^* g$, one can prove that

(i) $g_{i\bar{j}}(0) = \delta_{ij}$, $C^{-1}\,\mathrm{id} \leq (g_{i\bar{j}}) \leq C\,\mathrm{id}$,
(ii) $\sup_{B_r}\{|\partial g_{i\bar{j}}/\partial z_k| | 1 \leq i, j, k \leq n\} \leq C$,

where C is a uniform constant and id denotes the identity matrix. We refer readers to [Jo] for details of the proof of these.

Now we have the following elliptic equations:

$$-\Delta_g(\log\det(g_{i\bar{j}})) = S(\exp_x^* g), \tag{1.6}$$

$$-\Delta_g(g_{i\bar{j}}) + \sum_{s,t,k,l=1}^{n} g^{\bar{k}l} g^{\bar{s}t} \frac{\partial g_{i\bar{s}}}{\partial \bar{z}_l}\frac{\partial g_{t\bar{j}}}{\partial z_k} = \operatorname{Ric}(g)_{i\bar{j}} = -(\log\det(g_{i\bar{j}}))_{i\bar{j}}. \tag{1.7}$$

By the assumption, the scalar curvature of $\exp_x^* g$ is a uniformly bounded $\frac{1}{2}$-Hölder continuous function. Then from (i), (ii) above and the applications of the standard Schauder estimate [GT] to (1.6) and (1.7), it follows that (M, g) is of quasi-finite geometry of order $2 + \frac{1}{2}$.

2. Weighted Sobolev inequalities

In this section, we will prove some weighted Sobolev inequalities on a manifold with polynomial growth. These inequalities are needed in the proof of Theorem 1.1. They may not be optimal. We are satisfied by the fact that they are sufficient for our later use.

Proposition 2.1. *Let (X, ds^2) be an n-dimensional complete Riemannian manifold with (K, α, β)-polynomial growth, where $K > 0$ and $\alpha, \beta \geq 0$. Let $\rho(x) = \operatorname{dist}(x_0, x)$ be the distance function from the fixed point x_0 given in the definition of the manifold with polynomial growth. Let $l = \alpha + 2 + 2\beta$. Then for any Lipschitz function f on M with vanishing $(1+\rho)^{-l}$-average, i.e.,*

$$\int_M (1+\rho(x))^{-l} f(x)\, dV = 0, \tag{2.1}$$

we have

$$\left(\int_X (1+\rho(x))^{-l} |f(x)|^{2(n+1)/(n-1)}\, dV\right)^{(n-1)/2(n+1)} \leq C_0 \left(\int_X |\nabla f|^2\, dV\right)^{1/2}, \tag{2.2}$$

where dV is the volume form of the metric ds^2 and C_0 is a constant depending only on K, n, α, β.

Remark. To make the integral in (2.1) meaningful, we assume that the function f in Proposition 2.1 is absolutely L^1-integrable with respect to the weight $(1+\rho)^{-l}$.

The rest of this section is devoted to the proof of this proposition.

We start our proof with some notation. For each point x in X, every other point in X can be joined to x by a minimal geodesic. The exponential map $\exp_x$ identifies a domain $D(x)$ in the tangent space T_xX with an open set in X which is within the cut-locus of x in X. Denote by S_x the unit sphere in T_xX. Then we can write the domain in a polar coordinate system as

$$D(x) = \{(r, \theta) | \theta \in S_x,\ 0 \leq r \leq r(\theta)\}, \tag{2.3}$$

where $r(\theta)$ is a function defined on S_x.

For every y in $D(x)$, we can write the volume element of X at y as $\sqrt{g}(x, y) r_x(y)^{n-1}\, dr\, d\theta$, where g is the determinant of the metric ds^2 and r_x is the distance function on X from x. For every measurable subset E of $D(x)$, we define the cone of x over E to be

$$C_x(E) = \{(r, \theta) | \text{ for some } \bar{r},\ (\bar{r}, \theta) \in E\}, \tag{2.4}$$

where (r, θ) is the polar coordinate system at x. Note that these notations are taken from [Y1].

Lemma 2.1 [Y1, Lemma 4]. *Let f be a Lipschitz function defined on $D(x)$. Then for $E = \{y \in B_R(x) | f(y) = 0\}$, where $B_R(x)$ is the geodesic ball in X, we have*

$$\begin{aligned} &\mu(S_y \cap C_y(E))|f(y)| \\ &\quad \leq \int_{C_y(E)\cap B_{R+r_x(y)}(y)} \left|\frac{\partial f}{\partial r_y}\right| (z)(\sqrt{g}(y, z) r_y(z)^{n-1})^{-1}\, dV(z), \end{aligned} \tag{2.5}$$

where μ denotes the euclidean measure on S_y and $y \in B_R(x)$.

Lemma 2.2. *Let $R_0 = \pi/2\sqrt{K}$. Then for any $R \le \frac{r}{3}R_0$ and y in $B_R(x)\cap D(x)$,*

$$C(K, n)^{-1} \le |\sqrt{g}(x, y)| \le C(K, n), \tag{2.6}$$

where $C(K, n) > 0$ is a constant depending only on K and n.

Proof. The distance r_x is smooth in the open subset $B_R(x) \cap D(x)$. Since $X\backslash D(x)$ is the cut locus of (X, ds^2) with respect to the point x, any geodesic γ connecting a point y in $B_R(x)\cap D(x)$ to x lies entirely in $B_R(x)\cap D(x)$. By the assumption on the curvature of ds^2 and the Hessian comparison theorem (cf. [SY, GW]),

$$\Delta_{S^n} r_{S^n}(r_x(y)) \le \Delta r_x(y) \le \frac{n-1}{r_x(y)} + \frac{nK}{3} r_x(y),$$

where $y \in B_R(x)\cap D(x)$, Δ stands for the laplacian of (X, ds^2), and r_{S^n} and Δ_{S^n} are the distance function from the north pole and the laplacian of S^n with the standard metric with constant curvature K.

Hence, an easy computation shows

$$\left|\Delta r_x(y) - \frac{n-1}{r_x(y)}\right| \le C'(K, n) r_x(y), \qquad y \in B_R(x) \cap D(x), \tag{2.7}$$

where $C'(K, n)$ is a constant depending only on K and n. But

$$\Delta r_x(y) = \frac{n-1}{r_x(y)} + \frac{\partial}{\partial r_x} \log \sqrt{g}(x, y), \tag{2.8}$$

and so the lemma follows from (2.7) and $\sqrt{g}(x, x) = 1$.

Lemma 2.3. *For any $R > 0$ and any point x in X, $E \subset B_R(x)$ and $y \in B_R(x)$, we have*

$$\begin{aligned} &[\mu(S_y \cap C_y(E \cap D(y)))]^{-1} \\ &\qquad \le \mathrm{Vol}_{ds^2}(E)^{-1} \sup_{\theta\in S_y} \int_0^{r(\theta)} \sqrt{g}(y, (r, \theta)) r^{n-1}\, dr, \end{aligned} \tag{2.9}$$

where μ is the euclidean measure on S_y and the function $r(\theta)$ is defined by

$$\begin{aligned} r(\theta) = \sup\{r | &\text{the geodesic } \gamma_\theta(t) \text{ is minimal in } [0, r], \\ &r < R + r_x(y) \text{ and } \gamma_\theta(0) = y,\ \gamma_\theta'(0) = \theta \in S_u\}. \end{aligned} \tag{2.10}$$

Proof. It is well known that the cut locus of y in X has volume measure zero. Therefore,

$$\mathrm{Vol}_{ds^2}(E) = \mathrm{Vol}_{ds^2}(E \cap D(y)) \le \mathrm{Vol}_{ds^2}(C_y(E \cap D(y)) \cap B_R(x)). \tag{2.11}$$

Note that any point in $C_y(E \cap D(y)) \cap B_R(x)$ can be joined to y by a unique minimal geodesic γ_θ with $D(y) \cap B_{R+r_x(y)}(y)$. It follows from (2.11) that

$$\begin{aligned} \mathrm{Vol}_{ds^2}(E) &\le \int_{S_y\cap C_y(E\cap D(y))} d\theta \int_0^{r(\theta)} \sqrt{g}(y, (r, \theta)) r^{n-1}\, dr \\ &\le \mu(S_y \cap C_y(E \cap D(y))) \sup_{\theta\in S_y} \int_0^{r(\theta)} \sqrt{g}(y, (r, \theta)) r^{n-1}\, dr. \end{aligned}$$

Then the lemma follows.

Now for the given Lipschitz function f on X and a fixed $R > 0$, we define a new function $k_R(f)$ as

(2.12)
$$k_R(f)(x) = \sup\{k \mid \mathrm{Vol}_{ds^2}(\{y \in B_R(x) \mid f(y) \geq k\}) \geq \tfrac{1}{2}\,\mathrm{Vol}_{ds^2}(B_R(x))\} \quad \text{for } x \in X.$$

Then $k_R(f)$ is a measurable function on X. Actually, one can easily show that $k_R(f)$ is upper semicontinuous.

Lemma 2.4. *Let (X, ds^2) be a complete Riemannian manifold with its sectional curvature bounded by K and $\mathrm{Vol}_{ds^2}(B_1(x)) \geq C_0/(1 + \mathrm{dist}(x_0, x))^\beta$ for some constant C_0 and fixed point x_0 in X. Put $\rho(x) = \mathrm{dist}(x_0, x)$. Then for any Lipschitz function f on X and $R \leq \min\{R_0, 1\}$, where R_0 is defined in Lemma 2.3,*

$$\left(\int_X (1 + \rho(x))^{-2\beta\delta} |f(x) - k_R(f)(x)|^{2\delta}\, dV\right)^{1/\delta} \leq CR^{2\varepsilon} \int_X |\nabla f|^2\, dV_g, \tag{2.14}$$

where ∇ denotes the gradient of (X, ds^2), $\varepsilon = (n - \delta(n-2))/2\delta$, $1 \leq \delta < n/(n-2)$, and C is a constant depending only on K and n.

Proof. Define $f_+ = \max\{f - k_R(f), 0\}$, $f_- = \max\{-f + k_R(f), 0\}$, and $E_\pm(x) = \{y \in B_R(x) \mid f_\pm(y) = 0\}$ for any x in X. Then $\mathrm{Vol}_{ds^2}(E_\pm) \geq \frac{1}{2}\mathrm{Vol}_{ds^2}(B_R(x))$ by the definitions of $k_R(f)$ and $f_\pm$.

Applying Lemma 2.1 to f_+, f_-, respectively, we have

$$\mu(S_x \cap C_x(E_+))|f_+(x)| \leq \int_{C_x(E_+)\cap B_R(x)} \left|\frac{\partial f_+}{\partial r_x}\right|(y)(\sqrt{g}(x, y) r_x(y)^{n-1})^{-1}\, dV(y), \tag{2.13}$$

$$\mu(S_x \cap C_x(E_-))|f_-(x)| \leq \int_{C_x(E_-)\cap B_R(x)} \left|\frac{\partial f_-}{\partial r_x}\right|(y)(\sqrt{g}(x, y) r_x(y)^{n-1})^{-1}\, dV(y). \tag{2.15}$$

In the following, we will always use C to denote a constant depending only on K and n.

By Lemmas 2.2 and 2.3, and the Schwartz inequality, we conclude from (2.14) and (2.15) that for $\varepsilon = (n - \delta(n-2))/2\delta$,

$$|f(x) - k_R(f)(x)|^2 \leq \frac{CR^{m+\varepsilon}}{\mathrm{Vol}_{ds^2}(B_R(x))^2} \int_{B_R(x)} |\nabla f|^2(y) r_x(y)^{9n+2-\varepsilon}\, dV(y). \tag{2.16}$$

Define a function $\chi(x, y)$ on $X \times X$ by $\chi(x, y) = 1$ for $r_x(y) < R$, and $\chi(x, y) = 0$ for $r_x(y) \geq R$. Then by (2.16),

$$\begin{aligned}(2.17)\quad &\left(\int_X (1+\rho(x))^{-2\beta\delta} |f(x) - k_R(f)(x)|^{2\delta}\, dV(x)\right)^{1/\delta} \\ &\leq CR^{2n+\varepsilon}\left(\int_X (1+\rho(x))^{-2\beta\delta} (\mathrm{Vol}_{ds^2}(B_R(x)))^{-2\delta}\, dV(x)\right. \\ &\qquad \left.\cdot \left[\int_X \chi(x, y) |\nabla f|^2(y) r_x(y)^{-n+2-\varepsilon}\, dV(y)\right]^\delta\right)^{1/\delta}.\end{aligned}$$

By the Volume Comparison Theorem [Bi], we obtain

$$(2.18)\qquad \mathrm{Vol}_{ds^2}(B_R(x)) \geq CR^n \,\mathrm{Vol}_{ds^2}(B_1(x)) \geq \frac{CR^n}{(1+\rho(x))^\beta}.$$

Hence, by (2.17), (2.18), Young's inequality, and Lemma 2.2,

(2.19)

$$\begin{aligned}&\left(\int_X (1+\rho(x))^{-2\beta\delta} |f(x) - k_R(f)(x)|^{2\delta}\, dV\right)^{1/2\delta} \\ &\leq CR^{\varepsilon/2}\left(\int_X dV(x)\left(\int_X \chi(x, y)|\nabla f|^2(y) r_x(y)^{2-n-\varepsilon}\, dV(y)\right)^\delta\right)^{1/2\delta} \\ &\leq CR^{\varepsilon/2}\left(\int_X |\nabla f|^2(y)\, dV(y) \cdot \sup_{y\in X}\left(\int_{B_R(y)} r_y(x)^{-(n-2+\varepsilon)\delta}\, dV(x)\right)^{1/\delta}\right)^{1/2} \\ &\leq CR^\varepsilon \left(\int_X |\nabla f|^2(y)\, dV(y)\right)^{1/2}.\end{aligned}$$

The lemma is proved.

Lemma 2.5. *Let (X, ds^2) and f be as in Lemma 2.4, and let $R \leq \min\{1, R_0\}$, where R_0 is given in Lemma 2.2. Then for $1 \leq \delta < n/(n-2)$,*

$$\begin{aligned}(2.20)\quad &\left(\int_{B_R(x)} |f(y) - k_R(f)(x)|^{2\delta}\, dV(y)\right)^{1/2\delta} \\ &\leq CR^\varepsilon (1+\rho(x))^\beta \left(\int_{B_{3R}(x)} |\nabla f|^2(y)\, dV\right)^{1/2},\end{aligned}$$

where $\varepsilon = (n-(n-2)\delta)/2\delta$ and C is a constant depending only on K and n.

Proof. Define $f_\pm$ and $E_\pm$ as in the proof of Lemma 2.4. Then for each $y \in B_R(x)$, by Lemma 2.1,

$$\begin{aligned}(2.21)\quad &\mu(S_y \cap C_y(E_+))|f_+(y)| \\ &\leq \int_{C_y(E_+)\cap B_{3R}(x)} |\nabla f_+|(z)(\sqrt{g}(y, z) r_y(z)^{n-1})^{-1}\, dV(z),\end{aligned}$$

$$\begin{aligned}(2.22)\quad &\mu(S_y \cap C_y(E_-))|f_-(y)| \\ &\leq \int_{C_y(E_-)\cap B_{3R}(x)} |\nabla f_-|(z)(\sqrt{g}(y, z) r_y(z)^{n-1})^{-1}\, dV(z).\end{aligned}$$

As before, we can derive from (2.21), (2.22), Lemma 2.2, and Lemma 2.3 the estimate

$$(2.23)\quad |f(y)-k_R(f)(x)|^2 \le \frac{CR^{2n+\varepsilon}}{(1+\rho(x))^{2\beta}} \int_{B_{3R}(x)} |\nabla f|^2(z) r_y(z)^{-n+2-\varepsilon}\, dV(z)$$

for any y in $B_R(x)$. Then this lemma follows from the same argument as that in the proof of Lemma 2.4.

Next we need to estimate the growth of the function $k_R(f)$.

Lemma 2.6. *Let* (X, ds^2) *and* f *be as in Lemma* 2.4. *Then for* $1 \le \delta < n/(n-2)$,

$$(2.4)\quad \begin{aligned} &|k_R(f)(x) - k_R(f)(x_0)| \\ &\quad \le CR^{-n(\delta+1)/2\delta}(1+\rho(x))^{\beta(1+1/\delta)+1}\left(\int_X |\nabla f|^2(z)\, dV\right)^{1/2}. \end{aligned}$$

Proof. Given x in X, let γ be the minimal geodesic joining x_0 to x. Choose a sequence of points $\{y_j\}_{0\le j\le N}$ on the geodesic γ such that $y_0 = x_0$, $y_N = x$, and $y_j = \gamma(t_j)$, where $t_j = \frac{j}{N}\rho(x)$. Take N to be $[\rho(x)/R] + 1$, where $[\cdot]$ denotes the integer part. We will always use C to denote a constant depending only on K and n.

By Lemma 2.5, for $0 \le j \le N-1$,

$$(2.25)\quad \begin{aligned} &\left(\int_{B_R(y_j)\cap B_R(y_{j+1})} |k_R(f)(y_j) - k_R(f)(y_{j+1})|^\delta\, dV(y)\right)^{1/\delta} \\ &\quad \le CR^2(1+\rho(y_j))^\beta \cdot \left(\int_{B_{3R}(y_j)\cup B_{3R}(y_{j+1})} |\nabla f|^2(z)\, dV(z)\right)^{1/2}. \end{aligned}$$

Let $z_j = \gamma((t_j + t_{j+1})/2)$; then $B_{R/2}(z_j) \subset B_R(y_j) \cap B_R(y_{j+1})$. By the Volume Comparison Theorem [Bi],

$$\begin{aligned} \mathrm{Vol}_{ds^2}(B_R(y_j)\cap B_R(y_{j+1})) &\ge \mathrm{Vol}_{ds^2}(B_{R/2}(z_j)) \ge CR^n\, \mathrm{Vol}_{ds^2}(B_1(z_j)) \\ &\ge CR^n/(1+\rho(z_j))^\beta. \end{aligned}$$

It follows from (2.5) that

$$(2.26)\quad \begin{aligned} &|k_R(f)(y_j) - k_R(f)(y_{j+1})| \le CR^{\varepsilon-n/\delta}(1+\rho(y_j))^{\beta(1+1/\delta)} \\ &\qquad \cdot \left(\int_{B_{3R}(y_j)\cup B_{3R}(y_{j+1})} |\nabla f|^2(z)\, dV(z)\right)^{1/2}. \end{aligned}$$

Note that by our choices of y_j, the number of the balls $B_{3R}(y_j)$ having nonempty intersection is less than 5. Thus by summing (2.26) over j, we have completed the proof of the lemma.

Now we can finish the proof of Proposition 2.1. Let $l = \alpha + 2 + 2\beta$, and put $\delta = (n+1)/(n-1)$ and $R = \min\{R_0, 1\} = \min\{\pi/2\sqrt{K}, 1\}$. By the equation (2.1),

$$\left|\int_X (1+\rho(x))^{-l} k_R(f)(z)\,dV\right| \le \int_X (1+\rho(x))^{-l} |f(x) - k_R(f)|(z)\,dV. \tag{2.27}$$

Then it follows from Lemmas 2.4 and 2.6 that

$$\begin{aligned} |k_R(f)(x_0)| &\left(\int_X (1+\rho(x))^{-l}\,dV\right) \\ &\le C\left(\int_X |\nabla f|^2\,dV\right)^{1/2}\left(1 + \int_X (1+\rho(x))^{-l}\,dV\right), \end{aligned} \tag{2.28}$$

where $C = C(K, n)$ depends only on K and n.

Since $\mathrm{Vol}_{ds^2}(B_r(x_0)) \le Cr^\alpha$ by assumption, by a standard argument using Fubini's theorem, we can show the last integral in (2.28) is bounded. Thus

$$\begin{aligned} |k_R(f)(x)| &\cdot \left(\int_X (1+\rho(x))^{-l}\,dV\right) \\ &\le C(1+\rho(x))^{\beta 2n/(n+1)+1}\left(\int_X |\nabla f|^2\,dV\right)^{1/2}. \end{aligned} \tag{2.29}$$

Applying Lemma 2.4 again, we have

$$\left(\int_X (a+\rho(x))^{-l} |f(x)|^{2(n+1)/2(n-1)}\,dV\right)^{(n-1)/2(n+1)} \le C_0\left(\int_X |\nabla f|^2\,dV\right)^{1/2},$$

where $C_0 = C_0(K, n, \alpha, \beta)$ is a constant depending only on K, n, α, and β. Thus the proposition is proved.

Remark. The same proof shows that (2.2) still holds for the weight function $(1+\rho(x))^{-l}$ with $l = \alpha + 1 + \varepsilon + 2\beta$ and $\varepsilon > 0$, except that the constant C_0 may depend on ε.

3. The proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We shall use the perturbation method. We start with an approximation lemma. We will adopt the notation in Theorem 1.1.

Lemma 3.1. *Let f be the function in Theorem 1.1. Then there is a sequence of smooth functions f_m $(m \ge 1)$ such that each f_m has compact support, the sequence $\{f_m\}$ converges to f uniformly on M as $m \to \infty$, and*

$$\int_M (e^{f_m} - 1)\omega_g^n = 0, \tag{3.1}$$

$$\max_M\{|\nabla_g f|(x), |\Delta_g f|(x)\} \le C, \qquad |f_m|(x) \le \frac{C}{(1+\rho(x))^N}, \tag{3.2}$$

where $N \ge 4 + 2\beta$ as in Theorem 1.1 and C is some uniform constant.

Proof. First we produce an exhaustive smooth function. By smoothing the distance function ρ, one can construct a positive function ψ on $M \backslash B_1(x_0)$ satisfying

(i) $\rho(x) \leq C\psi(x)$ for $\rho(x)$ sufficiently large (say, for instance, $\rho(x) \geq C$),
(ii) $\sup_M\{|\nabla_g \psi|, |\Delta_g \psi|\} \leq C$,

where C is some constant depending only on K and β (cf. [Wu]).

Choose a positive function η on R^1 such that $\eta(t) \geq 1$, $\eta(t) = 1$ for $t \leq 2C$, and $\lim_{t\to\infty} \eta(t) = +\infty$. We can take our exhaustive function to be $\eta(\psi)$. For simplicity, we denote it by ψ.

Let $r' > 0$ be a small number such that $\rho^2(x)$ is smooth in $B_{r'}(x_0)$. Let $\zeta : [0, \infty) \to [0, 1]$ be a cut-off function with $\zeta(t) \equiv 1$ for $t \leq 1$ and $\zeta(t) \equiv 0$ for $t \geq 2$, $|\zeta'|$, $|\zeta''| \leq 1$. Define

$$f_m(x) = \zeta\left(\frac{\psi(x)}{m}\right)\left(f(x) + \varepsilon_m \zeta\left(\left(\frac{\rho(x)}{r'}\right)^2\right)\right),$$

where ε_m is a constant determined by the integrability condition (3.1). Then f_n has compact support and satisfies (3.2). By (1.2) and the decay assumption of f, one can easily check that $\lim_{m\to\infty} \varepsilon_m = 0$. Therefore, f_m converges to f uniformly. The lemma is proved.

Replacing f in equation (1.1) by f_m, we obtain a sequence of perturbed complex Monge-Ampère equations,

$$(3.3)_m \qquad \begin{cases} \left(\omega_g + \dfrac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^n = e^{f_m}\omega_g^n \quad \text{on } M, \\ \left(\omega_g + \dfrac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right) > 0, \qquad \varphi \in C^\infty(M, R^1). \end{cases}$$

Our strategy to solving (1.1) is to show that $(3.3)_m$ admits solutions and a subsequence of those solutions converges to that of (1.1). First we prove that $(3.3)_m$ is solvable.

Lemma 3.2. *Let (M, g) be a complete Kähler manifold of quasi-finite geometry of order $2 + \frac{1}{2}$. Then the following equation for $\varepsilon > 0$ has a unique solution*:

$$(3.4)_{m,\varepsilon} \qquad \begin{cases} \left(\omega_g + \dfrac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^n = e^{f_m + \varepsilon\varphi}\omega_g^n \quad on\ M, \\ \left(\omega_0 + \dfrac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right) > 0, \qquad \varphi \in C^\infty(M, R^1). \end{cases}$$

Moreover, the $(1, 1)$-form $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$ defines a Kähler metric equivalent to g.

Proof. This is due to S. Y. Cheng and S. T. Yau [CY]. One can also find a detailed proof of a slightly general version of it in [TY].

Denote by $\varphi_{m,\varepsilon}$ the unique solution of $(3.4)_{m,\varepsilon}$. We want to find a subsequence $\{\varepsilon_j\}_{j\geq 1}$ such that $\lim_{j\to\infty} \varepsilon_j = 0$ and the $\varphi_{m,\varepsilon_j}$ converge to a solution of $(3.3)_m$. By standard elliptic theory, it suffices to prove the uniform $C^{2,1/2}$-estimate of the solutions $\varphi_{m,\varepsilon}$. We will use the integral method to estimate the C^0-norms of $\varphi_{m,\varepsilon}$. The next lemma guarantees that we can do integration by parts on M for the equation $(3.4)_{m,\varepsilon}$. Recall that for a function ψ on M, $\psi_+(x) = \max\{0, \psi(x)\}$ and $\psi_-(x) = \max\{0, -\psi(x)\}$.

Lemma 3.3. *For any constants $\varepsilon > 0$, $p \geq 1$, and $q \geq 0$, we have*

$$\tag{3.5} \int_M [((1+\rho(x))^q|\varphi_{m,\varepsilon}|)^p + |\nabla_g((1+\rho(x))^q\varphi_{m,\varepsilon})|^2|(1+\rho(x))^q\varphi_{m,\varepsilon}|^{2p-2}]\omega_g^n < \infty.$$

Proof. Let η be a cut-off function $\eta(t) = 1$ for $t \leq 1$, $\eta(t) = 0$ for $t \geq 2$, and $|\eta'(t)|$, $|\eta''(t)| \leq 2$. For simplicity, write $\psi_{\tilde{q}} = (1+\rho(x))^{\tilde{q}}(\varphi_{m,\varepsilon})_+$.

Define $\eta_R(x) = \eta(\rho(x)/R)$ for $R > 0$. Multiplying $(1+\rho(x))^{\tilde{q}}\psi_{\tilde{q}}^{2p-1}\eta_R^2$ by both sides of $(3.4)_{m,\varepsilon}$, we obtain

$$\tag{3.6} \begin{aligned} &\frac{\sqrt{-1}}{2\pi}\int_M (1+\rho(x))^{\tilde{q}}\eta_R^2(x)\psi_{\tilde{q}}^{2p-1}\partial\bar{\partial}\varphi_{m,\varepsilon}\wedge(\omega_{m,\varepsilon}^{n-1}+\cdots+\omega_g^{n-1}) \\ &\quad = \int_M (1+\rho(x))^{\tilde{q}}\eta_R^2(x)\psi_{\tilde{q}}^{2p-1}(e^{f_m+\varepsilon\varphi_{m,\varepsilon}}-1)\omega_g^n, \end{aligned}$$

where $\omega_{m,\varepsilon} = \omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_{m,\varepsilon}$.

Integrating (3.6) by parts and using the fact that $\omega_{m,\varepsilon}$ is equivalent to ω_g, one can derive the following from (3.6):

$$\tag{3.7} \begin{aligned} &\int_M (1+\rho(x))^{\tilde{q}}\eta_R^2(x)\psi_{\tilde{q}}^{2p-1}(e^{\varepsilon\varphi_{m,\varepsilon}}-1)e^{f_m}\omega_g^n + \frac{1}{p}\int_M |\nabla_g\psi_{\tilde{q}}^p|^2\eta_R^2(x)\omega_g^n \\ &\quad \leq C\left\{\int_M (1+\rho(x))^{\tilde{q}}\eta_R^2(x)\psi_{\tilde{q}}^{2p-1}|e^{f_m}-1|\omega_g^n\right. \\ &\qquad \left. + \int_M (1+\rho(x))^{-2}\psi_{\tilde{q}}^{2p}\eta_R^2(x)\omega_g^n + \int_M \frac{|\eta'|^2(\rho/R)}{R^2}\psi_{\tilde{q}}^{2p}\omega_g^n\right\}, \end{aligned}$$

where C is a constant.

Since f_m has compact support, the first integral on the right of (3.7) is finite. Thus by the definition of η_R, we obtain

$$\begin{aligned}(3.8)\qquad &\int_M \eta_R^2(x)\psi_{\tilde{q}}^{2p}\omega_q^n + \int_M \eta_R^2(x)|\nabla_g \psi_{\tilde{q}}^p|^2\omega_g^n \\ &\le C'\left\{1 + \int_M \eta_{2R}^2(x)\psi_{\tilde{q}'}^{2p}\omega_g^n\right\},\end{aligned}$$

where C' is a constant and $\tilde{q}' = \tilde{q} - 1/p$.

The solution $\varphi_{m,\varepsilon}$ is bounded, so by the assumptions of Theorem 1.1, for $\tilde{q} = -N/2p$, the integral $\int_M \psi_{\tilde{q}}^{2p}\omega_g^n < +\infty$. Thus by using (3.8) inductively and letting $R \to \infty$, we can easily see

$$\int_M \{((1+\rho(x))^q\varphi_{m,\varepsilon})_+^{2p} + |\nabla_g((1+\rho(x))^q\varphi_{m,\varepsilon})_+^p|^2\}\omega_g^n < \infty.$$

One can estimate the integral of $((1+\rho(x))^q\varphi_{m,\varepsilon})_-$ similarly. The lemma then follows.

Corollary. *Let (M, g) and $\varphi_{m,\varepsilon}$ be as above. Then*

$$(3.9)\qquad \int_M (e^{f_m+\varepsilon\varphi_{m,\varepsilon}} - 1)\omega_g^n = 0.$$

Proof. We adopt the notations used in the proof of Lemma 3.3. By $(3.4)_{m,\varepsilon}$, we may have

$$\begin{aligned}(3.10)\qquad &\frac{\sqrt{-1}}{2\pi}\int_M \bar{\partial}\varphi_{m,\varepsilon}\wedge\partial\eta_R(x)\wedge(\omega_{m,\varepsilon}^{n-1}+\cdots+\omega_g^{n-1}) \\ &= \int_M (e^{f_m+\varepsilon\varphi_{m,\varepsilon}} - 1)\omega_g^n.\end{aligned}$$

The integral on the left-hand side of (3.10) is dominated by $\frac{1}{R}\int_M |\nabla_g\varphi_{m,\varepsilon}|\omega_g^n$. Thus, in order to show (3.9), it suffices to prove that $\int_M |\nabla_g\varphi_{m,\varepsilon}|\omega_g^n < +\infty$. Let $q = \alpha + 1$, where α is the rate of volume growth of (M, g). Then $\int_M (1+\rho(x))^{-2q}\omega_g^n < +\infty$. On the other hand, by Lemma 3.3,

$$\int_M (1+\rho(x))^{2q}|\nabla_g\varphi_{m,\varepsilon}|^2\omega_g^n < +\infty.$$

Thus by the Hölder inequality,

$$\begin{aligned}\int_M |\nabla\varphi_{m,\varepsilon}|\omega_g^n &\le \left(\int_M (1+\rho(x))^{2q}|\nabla\varphi_{m,\varepsilon}|^2\omega_g^n\right)^{1/2} \\ &\quad\cdot\left(\int_M (1+\rho(x))^{-2q}\omega_g^n\right)^{1/2} \\ &< +\infty.\end{aligned}$$

Lemma 3.4. *Let (M, g) be as in Theorem 1.1. Then there is a constant C independent of m and ε, such that*

$$(3.11)\qquad \int_M (1+\rho(x))^{-N}|\varphi_{m,\varepsilon} - \mathrm{Ave}_\rho(\varphi_{m,\varepsilon})|^{2(2n+1)/(2n-1)}\omega_g^n \le C,$$

where $\mathrm{Ave}_\rho(\varphi_{m,\varepsilon})$ *is the average of* $\varphi_{m,\varepsilon}$ *with respect to the weight* $(1+\rho(x))^{-N}$, *i.e.*,

$$\mathrm{Ave}_\rho(\varphi_{m,\varepsilon}) = \frac{\int_M (1+\rho(x))^{-N}\varphi_{m,\varepsilon}\omega_g^n}{\int_M (1+\rho(x))^{-N}\omega_g^n}. \tag{3.12}$$

Proof. For simplicity, we put $\psi = \varphi_{m,\varepsilon} - \mathrm{Ave}_\rho(\varphi_{m,\varepsilon})$. Multiplying $\varphi_{m,\varepsilon}$ by both sides of $(3.4)_{m,\varepsilon}$ and integrating by parts, which is justified by Lemma 3.3, we obtain

$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}\int_M \partial\psi\wedge\bar\partial\psi\wedge(\omega_{m,\varepsilon}^{n-1}+\omega_{m,\varepsilon}^{n-2}\wedge\omega_g+\cdots+\omega_g^{n-1})\\ = \int_M \varphi_{m,\varepsilon}(1-e^{f_m})\omega_g^n + \int_M \varphi_{m,\varepsilon}(1-e^{\varepsilon\varphi_{m,\varepsilon}})e^{f_m}\omega_g^n.\end{aligned} \tag{3.13}$$

By integrability condition (3.1) of f_m and the fact $\varphi_{m,\varepsilon}(e^{\varepsilon\varphi_{m,\varepsilon}}-1)\geq 0$, it follows from (3.13) that

$$\int_M |\nabla_g\psi|^2\omega_g^n \leq \int_M |\psi||e^{f_m}-1|\omega_g^n. \tag{3.14}$$

Applying Proposition 2.1 to (3.14) and using (3.2),

$$\int_M (1+\rho(x))^{-N}|\psi|^{2(2n+1)/(2n-1)}\omega_g^n \leq C\int_M |\psi|(1+\rho(x))^{-N}\omega_g^n,$$

where C is a constant independent of m and ε.

Now the lemma follows from the Hölder inequality and the assumption on the volume growth of (M, g).

Lemma 3.5. *There is a constant* C *independent of* m *and* ε, *such that*

(1) $-\inf_M\varphi_{m,\varepsilon}\leq C$, $\sup_M(\varphi_{m,\varepsilon}-\mathrm{Ave}_\rho(\varphi_{m,\varepsilon}))\leq C$ *whenever* $\mathrm{Ave}_\rho(\varphi_{m,\varepsilon})\geq 0$.

(2) $\sup_M\varphi_{m,\varepsilon}\leq C$, $-\inf_M(\varphi_{m,\varepsilon}-\mathrm{Ave}_\rho(\varphi_{m,\varepsilon}))\leq C$ *whenever* $\mathrm{Ave}_\rho(\varphi_{m,\varepsilon})\leq 0$.

Proof. Since the proofs are the same for both cases, we just prove (1). We will use Moser's iteration.

Put $\psi = (\varphi_{m,\varepsilon}-\mathrm{Ave}(\varphi_{m,\varepsilon}))_+$; then $\varphi_{m,\varepsilon}(x)\geq 0$ whenever $\psi(x)\geq 0$. It follows that $\psi(e^{\varepsilon\varphi_{m,\varepsilon}}-1)\geq 0$ on M. By $(3.4)_{m,\varepsilon}$, we may have

$$\int_M |\nabla_g\psi^{(q+1)/2}|^2\omega_g^n \leq \frac{q+1}{4}\int_M |\psi|^q|e^{f_m}-1|\omega_g^n.$$

Using (3.2) and Proposition 2.1,

$$\begin{aligned}&\left(\int_M (1+\rho(x))^{-N}|\psi^{(q+1)/2}-\mathrm{Ave}_\rho(\psi^{(q+1)/2})|^{2(2n+1)/(2n-1)}\omega_g^n\right)^{(2n-1)/(2n+1)}\\ &\qquad\leq C^q\int_M (1+\rho(x))^{-N}|\psi|^q\omega_g^n.\end{aligned} \tag{3.15}$$

Note that C always denotes a constant independent of m and ε in this proof.

Since $\int_M (1+\rho(x))^{-N}\omega_g^n$ is bounded, it follows from (3.15) that

$$\left(\int_M (1+\rho(x))^{-N}(1+|\psi|)^{(q+1)(2n+1)/(2n-1)}\omega_g^n\right)^{(2n-1)/(2n+1)} \le C^q\int_M (1+\rho(x))^{-N}(1+|\psi|)^{q+1}\omega_q^n. \tag{3.16}$$

Put $q_0 = 2(2n+1)/(2n-1)$ and $q_{j+1} = q_j(2n+1)/(2n-1)$ for $j\ge 0$. We use $\|\cdot\|_q$ to denote the L^q-norm with respect to the weight $(1+\rho(x))^{-N}$. Then (3.16) implies

$$\begin{aligned}\|(1+|\psi|)\|_{q_j+1} &\le \prod_{i=0}^{j}\left(2C\left(\frac{2n+1}{2n-1}\right)^j\right)^{((2n-1)/(2n+1))^j/2}\|(1+|\psi|)\|_{q_0}\\ &\le C\|(1+|\psi|)\|_{q_0}.\end{aligned}$$

Note that the last constant C may be different from the previous one, but it is still independent of m and ε. Now by Lemma 3.4 and letting j go to infinity, we obtain

$$\sup_M(\varphi_{m,\varepsilon} - \mathrm{Ave}_\rho(\varphi_{m,\varepsilon})) = \lim_{j\to\infty}\|\psi\|_{q_{j+1}} \le C.$$

Since we assume that $\mathrm{Ave}_\rho(\varphi_{m,\varepsilon})\ge 0$ in case (1), it follows from Lemma 3.4 that $\|(\varphi_{m,\varepsilon})_-\|_{q_0}\le C$. Then by the same argument as above, we can also prove that $-\inf_M\varphi_{m,\varepsilon}\le C$.

Lemma 3.6. *Let (M, g) be as in Theorem 1.1. Then there are two constants C_3 and C_4 independent of m and ε, such that*

$$0\le n+\Delta_g\varphi_{m,\varepsilon}\le C_3 e^{C_4(\varphi_{m,\varepsilon}-\inf_M\varphi_{m,\varepsilon})}. \tag{3.17}$$

Proof. We refer readers to [Y2] for the proof of this. Note that we still have the maximum principle for our manifold (M, g), since the curvature is bounded (cf. [TY]).

Now we are ready to estimate the C^0-norms of the solutions $\varphi_{m,\varepsilon}$.

Lemma 3.7. *For each m, there is a constant $C(m)$ such that $\sup_M|\varphi_{m,\varepsilon}|\le C(m)$.*

Proof. By (3.1) and (3.9),

$$\int_M e^{f_m}(e^{\varepsilon\varphi_{m,\varepsilon}}-1)\omega_g^n = 0. \tag{3.18}$$

Since $\varphi_{m,\varepsilon}$ is not identically zero, it follows from (3.18) that both $\sup_M\varphi_{m,\varepsilon}$ and $-\inf_M\varphi_{m,\varepsilon}$ are strictly positive. Note that the maximum principle holds on (M, g). Applying the maximum principle to $(3.4)_{m,\varepsilon}$, we conclude that both the maximum and the minimum of $\varphi_{m,\varepsilon}$ are attained in the compact support of f_m. Let $x_{\max}$ and $x_{\min}$ be in $\mathrm{Supp}(f_m)$, satisfying

$$\varphi_{m,\varepsilon}(x_{\max}) = \sup_M\varphi_{m,\varepsilon} > 0,\qquad \varphi_{m,\varepsilon}(x_{\min}) = \inf_M\varphi_{m,\varepsilon} < 0.$$

Now we have two cases:

(1) $\mathrm{Ave}_\rho(\varphi_{m,\varepsilon}) \geq 0$.
(2) $\mathrm{Ave}_\rho(\varphi_{m,\varepsilon}) < 0$.

The proof for the second case is similar to that for the first one. Thus we may assume that $\mathrm{Ave}_\rho(\varphi_{m,\varepsilon}) \geq 0$. By Lemma 3.5, there is a constant C independent of m and ε such that

$$\varphi_{m,\varepsilon}(x_{\min}) \geq -C, \qquad \sup_M(\varphi_{m,\varepsilon} - \mathrm{Ave}_\rho(\varphi_{m,\varepsilon})) \leq C. \tag{3.19}$$

Put $\psi = (\varphi_{m,\varepsilon} - \inf_M \varphi_{m,\varepsilon} - 1)_-$. Then $0 \leq \psi \leq 1$ and $\psi(x_{\min}) = 1$. Choose $r > 0$ such that $B_r(x_{\min})$ is a convex geodesic ball of M. Without loss of generality, we may assume $r = 1$. Let $G(x, y)$ be the Green's function of the Dirichlet problem on $B_1(x_{\min})$ and let η be a cut-off function on $B_1(x_{\min})$ such that $\eta(x) \equiv 1$ for $x \in B_{1/2}(x_{\min})$ and $\eta(x) \equiv 0$ for x outside $B_{3/4}(x_{\min})$. By Lemma 3.5 and (3.19), there are two constants C_3' and C_4, such that

$$\Delta_g \varphi_{m,\varepsilon} + n \leq C_3' e^{C_4 \varphi_{m,\varepsilon}}. \tag{3.20}$$

Multiplying $\eta^2(x)G(x_{\min}, x)\psi(x)$ by both sides of (3.20), we obtain

$$\begin{aligned} -\int_M \Delta_g \psi(x) \cdot \eta^2(x)\psi(x)G(x_{\min}, x)\omega_g^n \\ \leq C_4' \int_M \psi(x)\eta^2(x)G(x_{\min}, x)\omega_g^n, \end{aligned}$$

where $C_4' = C_3' e^{C_4}$. Integrating by parts and using the inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, we deduce

$$\begin{aligned} &\frac{1}{2}\int_{B_1(x_{\min})} \nabla_g(\eta^2(x)\psi^2(x))\nabla_g G(x_{\min}, x)\omega_g^n \\ &\quad + \frac{1}{2}\int_{B_1(x_{\min})} |\nabla_g \psi|^2 \eta^2(x) G(x_{\min}, x)\omega_g^n \\ &\leq C_4' \int_{B_1(x_{\min})} \psi(x) G(x_{\min}, x)\omega_g^n \\ &\quad + \frac{1}{2}\int_{B_1(x_{\min})} |\psi|^2(x)|\nabla_g \eta(x)|^2 G(x_{\min}, x)\omega_g^n \\ &\quad + \frac{1}{2}\int_{B_1(x_{\min})} \psi^2(x)\nabla_g \eta^2(x)\nabla_g G(x_{\min}, x)\omega_g^n. \end{aligned} \tag{3.21}$$

The functions $G(x_{\min}, x)$ and $\nabla_g G(x_{\min}, x)$ are bounded independently of ε on $\mathrm{Supp}(\nabla_g \eta) \subseteq B_1(x_{\min}) \backslash B_{1/2}(x_{\min})$. Therefore by Green's formula and (3.21),

$$1 = \psi^2(x_{\min}) \leq C_4'' \left(\int_{B_1(x_{\min})} \psi G(x_{\min}, x)\omega_g^n + \int_{B_1(x_{\min})} |\psi|^2 \omega_g^n \right), \tag{3.22}$$

where C_4'' is a constant independent of ε. By the Hölder inequality,

$$1 \le C_4''\left\{\left(\int_{B_1(x_{\min})} |\psi|^{2n-1}\omega_g^n\right)^{1/(2n-1)} \cdot \left(\int_{B_1(x_{\min})} G(x_{\min}, x)^{(2n-1)/(2n-2)}\omega_g^n\right)^{(2n-2)/(2n-1)} + \int_{B_1(x_{\min})} |\psi|^2\omega_g^n\right\}.$$

Thus for some constant C_5 independent of ε,

$$1 \le C_5 \operatorname{meas}\{\operatorname{Supp}(\psi) \cap B_1(x_{\min})\}.$$

It follows that

(3.23)
$$\begin{aligned}\int_M (1+\rho(x))^{-N}\varphi_{m,\varepsilon}\omega_g^n &\le \int_{M\setminus B_1(x_{\min})\cap\operatorname{Supp}(\psi)} (1+\rho(x))^{-N}\varphi_{m,\varepsilon}\omega_g^n + C_6 \\ &\le kC_6 + \sup_M \varphi_{m,\varepsilon}\left(\int_M (1+\rho(x))^{-N}\omega_g^n - (2+\rho(x_{\min}))^{-N}C_5^{-1}\right),\end{aligned}$$

where C_6 is a constant depending only on (M, g). Then by (3.19) and (3.23), $\sup_M \varphi_{m,\varepsilon} \le C(m)$ for some constant $C(m)$. The proof for the second case is similar except that we use $\Delta_g\varphi_{m,\varepsilon} + n > 0$ instead of (3.20). The lemma is proved.

The following high order estimate is essentially proved in [Y2].

Lemma 3.8. *There is an a priori estimate of the derivatives* $\nabla_g^3\varphi_{m,\varepsilon}(x)$ *in terms of the geometry of* (M, g), *and*

$$\sup_M\{|\varphi_{m,\varepsilon}|, |\Delta_g\varphi_{m,\varepsilon}|\} \quad \text{and} \quad \sup_{B_1(x)}\{|f_m|, |\nabla_g f_m|, |\nabla_g^2 f_m|, |\nabla_m^3 f_m|\}.$$

Corollary. *For each* m, *the complex Monge-Ampère equation* $(3.3)_m$ *admits a solution* φ_m *satisfying*

(i) $\sup_M |\varphi_m| \le C(m)$ *for some constant* $C(m)$.

(ii) $\int_M |\nabla\varphi_m|^2\omega_g^n < +\infty$.

Proof. It follows from Lemmas 3.7, 3.6, 3.8, and the elliptic theory [GT] that there is a sequence $\{\varphi_{m,\varepsilon_j}\}$ with $\lim_{j\to\infty}\varepsilon_j = 0$ and the $\varphi_{m,\varepsilon_j}$ converge as $j\to\infty$. Let $\varphi_m = \lim_{j\to\infty}\varphi_{m,\varepsilon_j}$. Then φ_m satisfies equation $(3.3)_m$ and (i) in the above. For (ii), we multiply $\varphi_{m,\varepsilon}$ by both sides of $(3.4)_{m,\varepsilon}$ and integrate the resulting equation:

(3.24)
$$\begin{aligned}\frac{\sqrt{-1}}{2\pi}\int_M \varphi_{m,\varepsilon}\partial\overline{\partial}\varphi_{m,\varepsilon} \wedge (\omega_{m,\varepsilon}^{n-1} + \cdots + \omega_g^{n-1}) \\ = \int_M \varphi_{m,\varepsilon}(1 - e^{f_m+\varepsilon\varphi_{m,\varepsilon}})\omega_g^n.\end{aligned}$$

By Lemma 3.3, we can integrate (3.24) by parts and obtain

$$\int_M |\nabla_g \varphi_{m,\varepsilon}|^2 \omega_g^{n-1} \le 4 \int_M |e^{f_m} - 1||\varphi_{m,\varepsilon}|\omega_g^n < \infty.$$

Then Corollary (ii) follows from (i) and Fatou's lemma.

Next we will prove that a subsequence $\{m_j\}_{j>1}$ with $\lim_{j\to\infty} m_j = \infty$, the functions $\varphi_{m_j} - \mathrm{Ave}_\rho(\varphi_{m_j})$ converge to a solution φ of (1.1). As before, it suffices to prove the $C^{2,1/2}$-estimate of $\varphi_m - \mathrm{Ave}_\rho(\varphi_m)$. By Lemmas 3.6 and 3.8, it is equivalent to showing

$$\sup_M |\varphi_m - \mathrm{Ave}_\rho(\varphi_m)| \le C, \tag{3.25}$$

where C is a constant independent of m.

Obviously, the function $\varphi_m - \mathrm{Ave}_\rho(\varphi_m)$ is still a solution of $(3.3)_m$. So we may assume that $\mathrm{Ave}_\rho(\varphi_m) = 0$.

Lemma 3.9. *For each* m, *we have*

$$\lim_{R\to\infty} \frac{1}{R} \int_{B_{2R}(X_0)\backslash B_R(X_0)} |\nabla\varphi_m|\omega_g^n = 0. \tag{3.26}$$

Proof. By the Hölder inequality,

$$\begin{aligned} &\int_{B_{2R}(X_0)\backslash B_R(X_0)} |\nabla\varphi_m|\omega_g^n \\ &\quad \le (\mathrm{Vol}_g(B_{2R}(X_0)))^{1/2} \left(\int_{B_{2R}(X_0)\backslash B_R(X_0)} |\nabla\varphi_m|^2\omega_g^n\right)^{1/2}. \end{aligned}$$

Then the lemma follows from Corollary (ii) of Lemma 3.8 and the assumption that $\mathrm{Vol}_g(B_{2R}(X_0)) \le CR^2$ for some constant C.

Remark. This is the only place we need the quadratic growth of the volume of (M, g).

Note that Lemma 3.4 holds for φ_m. Now Lemma 3.9 guarantees that we can apply the same argument in the proof of Lemma 3.5 to the solution φ_m and the equation $(3.3)_m$. Thus (3.25) follows from Lemma 3.4 and the same proof as that of Lemma 3.5. Then we obtain a solution φ of (1.1). It is easy to see from the above proof that $\sup_M\{|\varphi|, |\nabla^2\varphi|\}$ can be bounded by a constant depending only on K, β, and the datum in (1.3). Thus Theorem 1.1 is proved.

4. Prescribed Ricci curvature problem on quasi-projective manifolds

Let $M = \overline{M}\backslash D$ be a quasi-projective manifold, where $\overline{M}$ is a projective manifold and $D \subset \overline{M}$ is a smooth divisor with normal crossings. We pose the following problem.

Problem. Given a $(1, 1)$-form $\Omega \in C_1(K_{\overline{M}}^{-1} \otimes [D]^{-1})$, is there a complete Kähler metric with its Ricci curvature equal to $\Omega|_M$?

In case M is compact, i.e., $D = 0$, the answer to the above problem is "yes" by the second author [Y2]. In this section, we will apply Theorem 1.1 to give a partial solution to the problem in case M is noncompact. Precisely, we want to prove the following

Theorem 4.1. *Let $M = \overline{M} \backslash D$ with $\overline{M}$ a projective manifold and D a smooth divisor in $\overline{M}$. Suppose that D is ample. Then for any $(1, 1)$-form Ω in $C_1(K_{\overline{M}}^{-1} \otimes [D]^{-1})$, there is a complete Kähler metric g on M with its Ricci curvature* $\mathrm{Ric}(g)$ *equal to $\Omega|_M$. Moreover, the Kähler form ω_g of g is defined by*

$$\omega_g = \frac{\sqrt{-1}}{2\pi}\{\partial\overline{\partial}(-\log\|S\|^2)^{(n+1)/n} + \partial\overline{\partial}\varphi\}, \tag{4.1}$$

where $n = \dim_{\mathbf{C}} M$, S is the defining section of the divisor D, $\|\cdot\|$ is a norm of $[D]$ with positive definite curvature form, and φ is a bounded smooth function on M such that the derivatives of φ are uniformly bounded with respect to the metric induced by the form $\partial\overline{\partial}(-\log\|S\|^2)^{(n+1)/n}$

Note that the $(1, 1)$-form $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(-\log\|S\|^2)^{(n+1)/n}$ in (4.1) is indeed positive definite. The assumptions in Theorem 4.1 can be weakened, especially for complex surfaces, i.e., $\dim_{\mathbf{C}} M = 2$. This will be discussed in the next section. For simplicity, we prefer to adopt this cleaner version (i.e., Theorem 4.1).

Theorem 4.1 has the following two important corollaries.

Theorem 4.2. *Suppose that $M = \overline{M} \backslash D$, where $\overline{M}$ is a projective manifold and D is a smooth anticanonical divisor. Also suppose that $K_{\overline{M}}^{-1}$ is ample. Then M admits a complete Ricci-flat Kähler metric of form* (4.1).

Proof. Now $[D] = K_{\overline{M}}^{-1}$, so we can take $\Omega = 0$ in $C_1(K_{\overline{M}}^{-1} \otimes [D]^{-1})$. Theorem 4.1 implies this theorem.

Example 1. Let $\overline{M} = CP^n$, and D be a smooth hypersurface of degree $n + 1$. Then $M = CP^n \backslash D$ admits a complete Ricci-flat Kähler metric.

Theorem 4.3. *Let $M = \overline{M} \backslash D$ be as in Theorem* 4.1, *and $(K_{\overline{M}} \otimes [D])^{-1}$ be ample. Then M admits a complete Kähler metric with positive Ricci curvature.*

Proof. We simply take Ω in $C_1((K_{\overline{M}} \otimes [D])^{-1})$ to be a positive form and apply Theorem 4.1.

Example 2. Let $\overline{M} = CP^n$, and D be a smooth hypersurface of degree $\leq n$. Then $M = CP^n \backslash D$ admits a complete Kähler metric with positive Ricci curvature.

In the rest of this section, we prove Theorem 4.1. We will first construct a complete Kähler metric, the Ricci curvature of which is asymptotic to Ω near

D. Then we will verify that this metric has those properties stated in Theorem 1.1. Finally, Theorem 1.1 is used to complete the proof of Theorem 4.1.

By the ampleness of D, we can choose a hermitian metric $\|\cdot\|$ of the line bundle $[D]$ such that its curvature form ω is positive definite and $\|S\| < 1$ on $\overline{M}$. We fix the $(1, 1)$-form Ω in $C_1(K_{\overline{M}}^{-1} \otimes [D]^{-1})$. By the adjunction formula, the anticanonical line bundle K_D^{-1} is just $(K_{\overline{M}}^{-1} \otimes [D]^{-1})|_D$. Thus $\Omega|_D$ is in the cohomology class $C_1(D)$, where $C_1(D)$ means the first Chern class of the submanifold D. By the solution of the Calabi conjecture [Y2], there is a smooth function φ on D such that $\omega|_D + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi$ defines a Kähler metric on D with $\Omega|_D$ as its Ricci form. Note that such a φ is not unique, but unique up to a constant, i.e., any such φ is equal to $\varphi_0 + C'$, where φ_0 is fixed and C' is a constant. We will determine this constant C' later. Extend φ to the whole manifold $\overline{M}$, still denoted by φ, and let $\|\cdot\|_\varphi$ be the norm $e^{-\varphi}\|\cdot\|$ of $[D]$. Put $\omega_\varphi = \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi$. Then ω_φ is the curvature form of $\|\cdot\|_\varphi$. By adding a function of form $C\|S\|^2$ to φ, we may assume that ω_φ is equivalent to ω in a neighborhood of D. Thus there is a $\delta_\varphi > 0$ such that in the open neighborhood $\{x \in \overline{M} | \|S(x)\| < \delta_\varphi\}$,

$$C_\varphi^{-1}\omega \le \omega_\varphi \le C_\varphi \omega, \tag{4.2}$$

where C_φ is some constant and may depend on φ. Note that since $S = 0$ on D, we may always choose the extension of $\varphi|_D$ to $\overline{M}$ such that $\|S\|_\varphi < 1$ on $\overline{M}$. Now for any positive number $N > 0$, we define

(4.3)
$$\omega_N = \frac{\sqrt{-1}}{2\pi}\left\{\frac{n^{1+1/n}}{n+1}\partial\overline{\partial}(-\log\|S\|_\varphi^2)^{(n+1)/n} - C_N\partial\overline{\partial}(-\log(\lambda_N\|S\|^2))^{-N}\right\},$$

where λ_N and C_N are two constants determined later. Put

$$C_N = \frac{1}{N}(-2n\log\delta_\varphi)^{1/n}\left(-2\log\delta_\varphi - \max_{\overline{M}}|\varphi|\log\lambda_N\right)^{N+1}\mu_\varphi,$$

and $\lambda_N \le 1$, where μ_φ is a constant such that $\omega_\varphi \ge -\mu_\varphi\omega$ on $\overline{M}$. By a straightforward computation, we have the following expression for ω_N:

$$\begin{aligned}\omega_N = {} & (-n\log\|S\|_\varphi^2)^{1/n}\omega_\varphi + \frac{\sqrt{-1}}{2\pi}\frac{\partial\log\|S\|_\varphi^2 \wedge \overline{\partial}\log\|S\|_\varphi^2}{(-n\log\|S\|_\varphi^2)^{(n-1)/n}} \\ & + NC_N\frac{\omega}{(-\log(\lambda_N\|S\|^2))^{N+1}} \\ & - N(N+1)C_N\frac{\sqrt{-1}}{2\pi}\frac{\partial\log\|S\|^2 \wedge \overline{\partial}\log\|S\|^2}{(-\log(\lambda_N\|S\|^2))^{N+2}}.\end{aligned} \tag{4.4}$$

Then from this expression, one can check that ω_N is equivalent to the form $\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(-\log\|S\|^2)^{(n+1)/n}$ when λ_N is sufficiently small. Therefore, we have proved

Lemma 4.1. *For some C_N and λ_N, the $(1, 1)$-form ω_N defines a complete Kähler metric g_N on M such that its associated Kähler form is ω_N.*

Remark. It is easy to prove that if the induced metric of the form

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(-\log\|S\|^2)^{(n+1)/n}$$

is complete, so is g_N.

The following lemma follows from straightforward computations by using (4.4).

Lemma 4.2. *Fix a point x_0 in M. Let $r(x)$ be the distance function of (M, g_N) from x_0. Then the volume growth of (M, g_N) is of order $O(r^{2n/(n+1)})$ and $\mathrm{Vol}_{g_N}(B_1(x)) = O(r(x)^{-(n-1)/(n+1)})$ for x in M, where $B_1(x)$ is the geodesic ball of (M, g_N) with center at x and radius one.*

Remark. For the fixed points x_0 in M, the distance function $r(x)$ is of order $O((-\log\|S\|_\varphi^2)^{(n+1)/2n}(x))$ for x close to D.

Lemma 4.3. *Let $R(g_N)$ be the bisectional curvature tensor of the metric g_N. Then*

$$\|R(g_N)\|_{g_N}(x) = O(r(x)^{-2/(n+1)}), \tag{4.5}$$

where $\|\cdot\|_{g_N}$ is the induced norm by the metric g_N.

Proof. Note that it is not even obvious that $R(g_N)$ is bounded. So we will sketch a proof of (4.5) here. It is based on some complicated computations.

Put

$$\tilde{\omega} = \left(\frac{\sqrt{-1}}{2\pi}\right)^{n^{1+1/n}/(n+1)}\partial\overline{\partial}(-\log\|S\|_\varphi^2)^{(n+1)/n}.$$

Then both $\tilde{\omega}$ and ω_φ are positive definite in a neighborhood U of D, and $\tilde{\omega}$ is equivalent to ω_N in U. Since the bisectional curvature is dominated by holomorphic sectional curvature, estimate (4.5) is equivalent to

$$|R(g_N)(\xi, \xi, \xi, \xi)|(x) = \tilde{g}(\xi, \xi)_x^2 \cdot O(r(x)^{-2/(n+1)}) \tag{4.6}$$

for any point x in $U \cap M$ and ξ in $T_x M$, where $\tilde{g}$ is the induced hermitian metric by $\tilde{\omega}$.

For any given point x in $U \cap M$, we may choose a local frame of $[D]$ at x and local coordinates $(z_1, \ldots, z_n)$ of M, such that in such a local system,

(i) the holomorphic section S of $[D]$ is locally represented by the holomorphic function $z_n + \mu$, where μ is a constant such that $|\mu| = \|S(x)\|$.

(ii) The hermitian metric $\|\cdot\|_\varphi$ is locally represented by a positive function a with $a(x) = 1$, $da(x) = 0$, and $d(\partial a/\partial z_\gamma)(x) = 0$.

(iii) If $g_{\alpha\overline{\beta}} = g_\varphi(\partial/\partial z_\alpha, \partial/\partial z_\beta)$, where g_φ is the induced metric in U by ω_φ, then $g_{\alpha\overline{\beta}}(x) = \delta_{\alpha\overline{\beta}}$, $(\partial g_{\alpha\overline{\beta}}/\partial z_\gamma)(x) = 0$ for $\beta < n$, and $(\partial g_{\alpha\overline{\beta}}/\partial \overline{z}_\gamma)(x) = 0$ for $\alpha < n$.

For simplicity, put $F = -\log \|S\|_{\varphi}^2$. At the point x, by (i), (ii), (iii) above, we have

$$F_\alpha = \frac{\partial F}{\partial z_\alpha} = \delta_{\alpha n} F_n, \qquad F_{\alpha\gamma} = \frac{\partial^2 F}{\partial z_\alpha \partial z_\gamma} = \delta_{\alpha n}\delta_{\gamma n} F_n^2, \tag{4.7}$$

$$\tilde{g}(\xi, \xi)(x) = \left(\sum_{\alpha=1}^{n-1} |\xi^\alpha|^2\right) \cdot (nF)^{1/n} + |F_n|^2 (nF)^{-(n-1)/n} |\xi^n|^2, \tag{4.8}$$

where $\xi = \sum_{\alpha=1}^n \xi^\alpha \partial/\partial z_\alpha \in T_x M$.

Note that $r(x) = O(F^{(n+1)/2n})$. Thus (4.6) is equivalent to

$$\left| \sum_{\alpha,\beta,\gamma,\delta=1}^{n} R(g_N)_{\alpha\overline{\beta}\gamma\overline{\delta}} \xi^\alpha \overline{\xi}^\beta \xi^\gamma \overline{\xi}^\delta \right|(x) \le CF(x)^{-1/n} \cdot \tilde{g}(\xi, \xi)(x), \tag{4.9}$$

where C is a universal constant independent of x and

$$R(g_N)_{\alpha\overline{\beta}\gamma\overline{\delta}}(x) = -\frac{\partial^2 g_{N\alpha\overline{\beta}}}{\partial z_\gamma \partial \overline{z}_\delta}(x) + (g_N^{-1})^{\overline{u}v} \frac{\partial g_{N\alpha\overline{u}}}{\partial z_\gamma} \frac{\partial g_{N\nu\overline{\beta}}}{\partial \overline{z}_\delta}(x), \text{ 4.10}$$

where $g_{N\alpha\overline{\beta}} = g_N(\partial/\partial z_\alpha, \partial/\partial z_\beta)$.

Without loss of generality, we may assume $\lambda_N = 1$. Let $(g'_{\alpha\overline{\beta}})$ be the metric tensor associated to ω in $(z_1, \ldots, z_n)$. We will use $O(F(x)^{-1/n})$ to denote a quantity bounded by $CF^{-1/n}$ with a constant independent of x.

By computation, we have

(4.11)

$$\begin{aligned}
\frac{\partial g_{N\alpha\overline{\beta}}}{\partial z_\gamma}(x) &= (nF)^{1/n} \frac{\partial g_{\alpha\overline{\beta}}}{\partial z_\gamma}(x) + (nF)^{-(n-1)/n} F_n(x) \delta_{\gamma n} \delta_{\alpha\beta} \\
&\quad + \frac{|F_n|^2 F_n \delta_{\alpha n}\delta_{\beta n}\delta_{\gamma n} + F_n \delta_{\alpha n}\delta_{\gamma\beta}}{(nF)^{(n-1)/n}} \\
&\quad - \frac{n-1}{n^2} F^{-(2n-1)/n} F_n(x) \delta_{\alpha n}\delta_{\beta n}\delta_{\gamma n} + NC_N (f - \varphi)^{N-1}(x) \frac{\partial g'_{\alpha\overline{\beta}}}{\partial z_\gamma}(x) \\
&\quad - N(N+1) C_N (F - \varphi)^{-N-2}(x) g'_{\alpha\overline{\beta}}(x) \cdot (F_n \delta_{\gamma n} - \varphi_n) \\
&\quad + N(N+1)(N+2) C_N \frac{(F_n \delta_{\alpha n} - \varphi_\alpha)(F_{\overline{n}} \delta_{\beta n} - \varphi_{\overline{\beta}})(F_n \delta_{\gamma n} - \varphi_\gamma)}{(F - \varphi)^{N+3}}(x),
\end{aligned}$$

$$\frac{\partial^2 g_{N\alpha\overline{\beta}}}{\partial z_\gamma \partial \overline{z}_\delta}(x) \xi^\alpha \overline{\xi}^\beta \xi^\gamma \overline{\xi}^\delta = O(F(x)^{-1/n}) \tilde{g}(\xi, \xi)^2(x) + \frac{|F_n|^4 |\xi^n|^4}{(nF)^{(n-1)/n}}(x). \tag{4.12}$$

By using (4.11) and (i), (ii), (iii) above, we can prove

$$\sum_{u,v<n} (g_N^{-1})^{\overline{u}v} \frac{\partial g_{N\alpha\overline{u}}}{\partial z_\gamma} \frac{\partial g_{N\nu\overline{\beta}}}{\partial \overline{z}_\delta}(x) = O(F(x)^{-1/n}) \tilde{g}(\xi, \xi)^2(x). \tag{4.13}$$

It follows from (4.8) and the definitions of $\tilde{g}$ and g_N that

$$\begin{aligned} \det(g_N)(x) &= \det(\tilde{g})(x)(1+O(F(x)^{-1-1/n-N})) \\ &= (|F_n|^2+nF)(x)(1+O(F(x)^{-1-1/n-N})) \\ &= (|F_n|^2+nF)(x)(1+O(F(x)^{-1/n-1-N})). \end{aligned} \tag{4.14}$$

Therefore,

$$(g_N^{-1})^{\bar{n}\nu}(x) = O\left(\frac{1+|F_n|F^{-1}}{|F_n|^2+nF}F^{-1-N-1/n}(x)\right), \tag{4.15}$$

$$(g_N^{-1})^{\bar{n}n}(x) = \frac{(nF)^{(n-1)/n}}{|F_n|^2+nF}(x)(1+O(F(x)^{-1-N-1/n})). \tag{4.16}$$

Using (4.11) and (4.15), one can show

(4.17)

$$\begin{aligned} &\left(\sum_{\nu<n}(g_N^{-1})^{\bar{n}\nu}\frac{\partial g_{N\alpha\bar{n}}}{\partial z_\gamma}\frac{\partial g_{N\nu\bar{\beta}}}{\partial \bar{z}_\delta}(x) + \sum_{u<n}(g_N^{-1})^{\bar{u}n}\frac{\partial g_{N\alpha\bar{u}}}{\partial z_\gamma}\frac{\partial g_{Nn\bar{\beta}}}{\partial \bar{z}_\delta}(x)\right)\cdot \xi^\alpha\bar{\xi}^\beta\xi^\gamma\bar{\xi}^\delta \\ &\quad = O(F^{-1/n})\tilde{g}(\xi,\xi)^2(x), \end{aligned}$$

$$\begin{aligned} &(g_N^{-1})^{\bar{n}n}\frac{\partial g_{N\alpha\bar{n}}}{\partial z_\gamma}\frac{\partial g_{Nn\bar{\beta}}}{\partial \bar{z}_\delta}(x)\xi^\alpha\bar{\xi}^\beta\xi^\gamma\bar{\xi}^\delta \\ &\quad = O(F^{-1/n})\tilde{g}(\xi,\xi)^2(x) + (nF+|F_n|^2)^{-1}\frac{|F_n|^6|\xi^n|^4}{(nF)^{(n-1)/n}}. \end{aligned} \tag{4.18}$$

Combining (4.10), (4.12), (4.13), (4.17), and (4.18) we obtain the required estimate (4.9). The lemma is proved.

Lemma 4.4. *Let $S(g_N)$ be the scalar curvature of the metric g_N. Then*

$$\|\partial S(g_N)\|_{g_N} = O(1). \tag{4.19}$$

Proof. The proof follows from some computations similar to those in the proof of Lemma 4.3. We omit it.

Remark. Actually, we can prove that $\max\{|S(g_N)|(x), \|\partial S(g_N)\|_{g_N}(x)\}$ has fast decay at infinity, i.e., near D. But since we do not need this strong property of $S(g_N)$, we will not prove it here.

In order to apply Theorem 1.1 to our case here, we need to find a function f_N such that

$$\mathrm{Ric}(g_N) - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}f_N = \Omega|_M. \tag{4.20}$$

As in the proof of Lemma 4.3, let g' be the Kähler metric associated to ω. Then $\mathrm{Ric}(g') - \omega$ is in the cohomology class $C_1(K_{\overline{M}}^{-1}\otimes[D]^{-1})$. It implies that there is a function ψ on $\overline{M}$ such that

$$\Omega = \mathrm{Ric}(g') - \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\psi. \tag{4.21}$$

Note that this ψ is unique up to a constant. Our function f_N will be defined by

$$f_N = -\log \|S\|^2 - \log\left(\frac{\det(g_N)}{\det(g')}\right) - \psi, \tag{4.22}$$

where ψ is a function satisfying (4.21). We will first choose ψ properly such that f_N has fast decay along D.

Choose a finite covering $\{U_j\}_{1\le j\le J}$ of D such that

(1) $\bigcup_{1\le j\le J} U_j \subseteq \{x\in\overline{M} \mid \|S(x)\| < \delta_\varphi\} \subset \{x\in\overline{M} \mid \omega_\varphi(x) > 0\}$.

(2) For each U_j, there is a local coordinate system $(z_1^j, \dots, z_n^j)$ such that the section S is represented by the function Z_n^j in U_j and $(z_1^j, \dots, z_{n-1}^j)$ is a local system of $U_j \cap D$.

Lemma 4.5. *There is a ψ satisfying* (4.21) *such that the function f_N defined in* (4.22) *is asymptotically of order $O(r^{-(2nN+2n+2)/(n+1)})$. Furthermore, the gradient ∇f_N and the laplacian Δf_N with respect to g_N are uniformly bounded.*

Proof. The boundedness on ∇f_N and Δf_N can be proved by some direct computations. So it suffices to check that $f_N = O(r^{-(2nN-2n-1)/(n+1)})$ in each open subset U_j. For simplicity, we drop the subscripts j, N, etc., and put $f = f_N$, $z_\alpha = z_\alpha^j$, $g_N = (G_{\alpha\overline{\beta}})_{1\le\alpha,\beta\le n}$, and $g_\varphi = (g_{\alpha\overline{\beta}})_{1\le\alpha,\beta\le n}$ in local coordinates $(z_1, \dots, z_n)$.

By definition, we compute

$$\begin{aligned}\|S\|^2\omega_{g_N}^n = {} & \|S\|^2\left((-n\log\|S\|_\varphi^2)^{1/n}\omega_\varphi + NC_N\frac{\omega}{(-\log\|S\|^2)^{N+1}}\right)^{n-1} \\ & \wedge \partial\log\|S\|_\varphi^2 \wedge \overline{\partial}\log\|S\|_\varphi^2 \\ & \cdot\left(\frac{1}{(-n\log\|S\|_\varphi^2)^{(n-1)/n}} - \frac{N(N+1)C_N}{(-\log\|S\|^2)^{N+2}}\right) + O(\|S\|^{1/2}),\end{aligned} \tag{4.23}$$

where $O(\|S\|^{1/2})$ denotes a volume form bounded by $C\|S\|^{1/2}\omega^n$ near D for some constant.

Let $\|\cdot\|$ be represented by a positive function a_j on U_j such that $\|S\|^2|_{U_j} = a_j|z_n^j|^2$. Also let $(g'_{j\alpha\overline{\beta}})$ be the metric tensor of g' in $(z_1^j, \dots, z_n^j)$. Then we can define a volume form V_D on D by

$$V_D|_{U_i} = \frac{\det(g'_{i\alpha\overline{\beta}})_{1\le\alpha,\beta\le n}}{a_i}, \qquad 1\le i\le J. \tag{4.24}$$

By the definition of V_D, we have

$$\Omega|_D = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log V_D + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\psi|_D. \tag{4.25}$$

Thus by the choice of ω_φ,

$$\det(g_{\alpha\overline{\beta}})_{1\le\alpha,\beta\le n}|_D = \widetilde{C}e^{-\psi}V_D, \tag{4.26}$$

where $\widetilde{C}$ is a positive constant. We choose ψ such that $\widetilde{C}=1$ and ψ satisfies (4.21). Then in U_j,

$$\begin{aligned}\|S\|^2\det(g_N) &= a_j|z_n|^2\det(G_{\alpha\overline{\beta}})_{1\le\alpha,\beta\le n}\\ &= a_j\det(g_{\alpha\overline{\beta}})_{1\le\alpha,\beta\le n-1} + O((-\log\|S\|^2)^{-N-1-1/n})\\ &= e^{-\psi}\det(g'_{\alpha\overline{\beta}})_{1\le\alpha,\beta\le n} + O((-\log\|S\|^2)^{-N-1-1/n}).\end{aligned} \tag{4.27}$$

It follows that $f|_{U_j} = 1 + O((-\log\|S\|^2)^{-N-1-1/n})$. Using the fact that $r(x) = O((-\log\|S\|(x))^{(n+1)/2n})$, the lemma is proved.

Finally, we come to see whether or not f_N satisfies the integrability condition (1.2). As we have already seen, g_N depends not only on the extension of $\varphi|_D$ to $\overline{M}$, but also on the choice of $\varphi|_D$, which is unique only up to a constant. We fix the function φ in the previous discussions. Note that $\omega_{\varphi+\lambda} = \omega_\varphi$ for any constant λ. Then the function ψ in Lemma 4.5 remains unchanged when φ is replaced by $\varphi+\lambda$ in the definition of g_N.

Lemma 4.6. *There is a λ such that if we use $\varphi+\lambda$ to replace φ in the above definition of g_N, then the resulting f_N satisfies*

$$\int_M (e^{f_N}-1)\omega_N^n = 0, \tag{4.28}$$

where ω_N is the Kähler form of g_N.

Proof. By using integration by parts, it is easy to check

$$\int_M \left\{\omega_N^n - \left(\frac{\sqrt{-1}}{2\pi}\frac{n^{1+1/n}}{n+1}\partial\overline{\partial}(-\log\|S\|^2_{\varphi+\lambda})^{(n+1)/n}\right)\right\}^n = 0.$$

Note that now ω_N is defined as

(4.29)
$$\omega_N = \frac{\sqrt{-1}}{2\pi}\left\{\frac{n^{1+1/n}}{n+1}\partial\overline{\partial}(-\log\|S\|^2_{\varphi+\lambda})^{(n+1)/n} - C_N\partial\overline{\partial}(-\log\lambda_N\|S\|^2)^{-N}\right\}.$$

On the other hand, $e^f\omega_N^n = e^{-\psi}\omega^n/\|S\|^2$. It is independent of λ by the remark before this lemma. An easy computation shows

$$\left(\frac{n^{1+1n}}{n+1}\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(-\log\|S\|^2_{\varphi+\lambda})^{(n+1)/n}\right)^n \tag{4.30}$$
$$= n\frac{\sqrt{-1}}{2\pi}\partial\{(\log\|S\|^2_\varphi - \lambda)\overline{\partial}\log\|S\|^2_\varphi \wedge \omega_\varphi^{n-1}\}$$
$$= \left(\frac{n^{1+1/n}}{n+1}\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(-\log\|S\|^2_\varphi)^{(n+1)/n}\right)^n - n\lambda\omega_\varphi^n.$$

Therefore,

$$\int_M (e^{f_N}-1)\omega_N^n = \int_M \left\{\frac{1}{\|S\|^2}e^{-\psi}\omega^n - \left(\frac{\sqrt{-1}}{2\pi}\frac{n^{1+1/n}}{n+1}\partial\overline{\partial}(-\log\|S\|^2_\varphi)^{(n+1)/n}\right)^n\right\}$$
$$- n\lambda\int_{\overline{M}}\omega_\varphi^n.$$

The first integral is finite and independent of λ, while the second one is $n\lambda\int_{\overline{M}}\omega^n$. So we can choose λ such that (4.28) holds.

Now we can finish the proof of Theorem 4.1. Choose a sufficiently large N, say $100n$. Then by Lemmas 4.1, 4.2, and 4.3, (M, g_N) is a complete Kähler manifold of $(K, 2n/(n+1), (n-1)/(n+1))$-polynomial growth for some constant K. By Lemmas 4.4 and 4.5, we can apply Theorem 1.1 to equation (1.1) with $g = g_N$ and $f = f_N$. Then for the solution φ, the induced metric $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi$ is the complete metric we want in Theorem 4.1. Theorem 4.1 is proved.

5. Generalizations of Theorem 4.1 and their applications

In the main theorem of the last section, the divisor D corresponding to infinity is assumed to be ample. This is a rather restrictive assumption. In order to produce more complete Ricci-flat Kähler manifolds and complete Kähler manifolds with nonnegative Ricci curvature, we should weaken the assumption on the ampleness of the divisor D. We will take up this task in this section. Because the proofs of the theorems here are essentially the same as that of Theorem 4.1, we will not give the details.

Definition 5.1. Let $\overline{M}$ be a projective manifold, $D \subset \overline{M}$ a divisor. Then

(i) D is called *almost ample* if there is an integer m such that the global section in $H^0(\overline{M}, \mathscr{O}(mD))$ gives a birational morphism from $\overline{M}$ into some projective space CP^N and this morphism is biholomorphic on a neighborhood of D.

(ii) D is said to be *neat* if no curve C in $\overline{M}$ disjoint from D can be written as a linear sum of the curves supported in D in $H_2(\overline{M}, R)$.

Remark. If $\overline{M}$ is a complex surface and $D^2 > 0$, then D is automatically neat. It follows from the Hodge Index Theorem. In general, we do not know how to characterize the neatness of a divisor D in higher dimensions.

Theorem 5.1. *Let $M = \overline{M}\backslash D$, where $\overline{M}$ is a projective manifold and $D \subset \overline{M}$ is a neat, almost ample smooth divisor. Then given any* $(1, 1)$*-form* Ω *representing the cohomology class* $C_1(K_{\overline{M}}^{-1} \otimes [D]^{-1})$, *there is a complete Kähler metric with* Ω *as its Ricci form. Moreover, this metric is asymptotically equivalent to the one in* (4.1).

Proof. By the Hahn-Banach theorem and the neatness of D, one can find a cohomology class ω_0 in $H^2(\overline{M}, Z) \cap H^{1,1}(\overline{M}, C)$ such that

$$\int_\gamma \omega_0 > 0 \tag{5.1}$$

for any effective curve γ with $\gamma \cdot D = 0$, and $\omega_0|_D = 0$.

It follows that for a large integer $k > 0$, $\omega_0 + kC_1(D)$ is numerically effective and big, i.e., $\int_\gamma(\omega_0 + kC_1(D)) > 0$ for any effective curve γ in $\overline{M}$ and $\int_{\overline{M}}(\omega_0 + kC_1(D))^n > 0$, where $n = \dim_{\mathbf{C}} \overline{M}$. Then by Nakai's criterion [Ha], the class $\omega_0 + kC_1(D)$ gives an ample line bundle L on $\overline{M}$.

Take a positive $(1, 1)$-form ω_L representing the Chern class $c_1(L)$. Let ω be the semipositive form obtained by pulling back the standard Fubini-Study form on CP^N under the morphism Φ_m, where CP^N and Φ_m are given in Definition 1.1. By the choice of our L, the class $\omega_L|_D$ is cohomological to $\omega|_D$, so there is a function ψ on $\overline{M}$ such that

$$\omega_L|_D = \omega|_D + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\psi|_D \tag{5.2}$$

and ψ vanishes outside a neighborhood of D. Define

$$\omega_E = \omega_L - \omega - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\psi; \tag{5.3}$$

then $\omega_E|_D = 0$.

As in the proof of Theorem 4.1, we let φ be a function on $\overline{M}$ such that $\omega|_D + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi|_D$ defines a Kähler metric with $\Omega|_D$ as its Ricci form. Let $\|\cdot\|$ be the hermitian metric of $[D]$ with curvature form ω, and $\|\cdot\|_\varphi = e^{-\varphi}\|\cdot\|$. Then the approximated metric is

$$\begin{aligned}\omega_N = \frac{\sqrt{-1}}{2\pi}\Big\{&\frac{n^{1+1/n}}{n+1}\partial\overline{\partial}(-\log\|S\|_\varphi^2)^{(n+1)/n} \\ &+ C_N\partial\overline{\partial}(-\log\lambda_N\|S\|^2)^{-N}\Big\} + \mu_N\omega_E,\end{aligned} \tag{5.4}$$

where S is the defining section of D, and λ_N, μ_N, and C_N are constants. Note that we may assume that φ vanishes outside a neighborhood of D. Then one can check that for properly chosen φ, C_N, λ_N, and μ_N, the form ω_N induces a complete Kähler metric g_N on M with the same properties as stated in Lemmas 4.1–4.5. Now this theorem follows from Theorem 1.1 by the same arguments as in the proof of Theorem 4.1.

Next, we suppose that $\overline{M}$ is a fiber space over a smooth algebraic curve $\overline{S}$ with connected fibers. Let $\pi : \overline{M} \to \overline{S}$ be the projection. We further assume that $D = \pi^{-1}(D_{\overline{S}})$ with $D_{\overline{S}} \subset \overline{S}$ consisting of finitely many smooth reduced fibers.

Theorem 5.2. *Let $M = \overline{M} \backslash D$ with $\overline{M}$, D described as above. Then given any $(1, 1)$-form Ω in the cohomology class $C_1(K_{\overline{M}}^{-1} \otimes [D]^{-1})$, there is a complete Kähler metric with Ω as its Ricci form. Moreover, this metric has the volume growth of linear order.*

Proof. As before, we still denote by S the global section defining D. In this special case, the section S is actually the pull-back of a section S_D on $\overline{S}$ defining $D_{\overline{S}}$. Let L be an ample line bundle on $\overline{M}$ and ω be a positive $(1, 1)$-form representing $C_1(L)$. By [Y2], there is a smooth function φ on D such that $\omega|_D + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi$ defines a Kähler metric on D with $\Omega|_D$ as its Ricci form. Note that such a φ is unique up to constants. We extend this φ to $\overline{M}$, still denoted by φ. Let η be a cut-off function, i.e., $\eta : R^1 \to R^1$, $\eta(t) \equiv 1$ for $t \leq 1$, $\eta(t) \equiv 0$ for $t \geq 2$, and $|\eta| \leq 1$. Choose a small number $R > 0$, such that the function $\eta(\|S\|^2/R)$ vanishes outside a small neighborhood of D and, in the support of $\eta(\|S\|^2/R)$, the form $\omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\varphi$ is positive along the fiber directions. Now we define

$$\omega_\varphi = \omega + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\left(\eta\left(\frac{\|S\|^2}{R}\right)\varphi\right). \tag{5.5}$$

Then $\omega_\varphi|_{\pi^{-1}(t)}$ is always positive definite for any t in $\overline{S}$. Our approximated metric g_A is defined by assigning its Kähler form ω_A as

$$\omega_A = \lambda\omega_\varphi + \mu\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}(-\log\|S\|^2)^2 + \nu\pi^*\omega_0, \tag{5.6}$$

where λ, μ, and ν are properly chosen constants and ω_0 is a Kähler form on $\overline{S}$. Then this theorem follows from Theorem 1.1 by the same arguments as in the proof of Theorem 4.1.

Remarks. (1) Theorem 5.2 can be generalized in the following situation. Let $\pi : \overline{M} \to \overline{S}$ be a fiber space over another projective manifold $\overline{S}$ with connected fibers. We further assume that $D = \pi^{-1}(D_{\overline{S}})$ is biholomorphic to $D_{\overline{S}} \times F$ with $D_{\overline{S}} \subset \overline{S}$ smooth and F a smooth, reduce fiber in D of $\overline{M}$ over $\overline{S}$. We also assume that $D_{\overline{S}} \subset \overline{S}$ is almost ample and neat. Let $p_1 : D \cong D_{\overline{S}} \times F \to D_{\overline{S}}$ and $p_2 : D \cong D_{\overline{S}} \times F \to F$ be two natural projections. Then one can prove: given any $(1, 1)$-form Ω in $C_1(K_{\overline{M}}^{-1} \otimes [D]^{-1})$ such that $\Omega|_D = p_1^*\Omega_1 + p_2^*\Omega_2$, where $\Omega_1 \in C_1(K_{\overline{S}}^{-1} \otimes [D_{\overline{S}}]^{-1})$ and $\Omega_2 \in C_1(F)$, there is a complete Kähler metric on $M = \overline{M}|D$ with Ω as its Ricci form and this metric has the volume growth of order less than two.

(2) Let $\overline{M}$ be a fiber space over $\overline{S}$ and $D \subset \overline{M}$ as in (1). Then if M admits a complete Kähler metric with nonnegative Ricci curvature, there must

be some constraints on the base $\overline{S}$. In fact, by the generalized Schwartz lemma [Y3], one can prove the following statement: Let (M, g) be a complete Kähler manifold with nonnegative Ricci curvature. Then there is no holomorphic map from M into a Kähler manifold N such that its Ricci curvature is bounded from above by a negative constant and this map is of full rank at least at one point.

Corollary 5.1. *Let $M = \overline{M}\backslash D$ be either in Theorem 5.1, Theorem 5.2, or in remark (1) above. Assume that D is in the anticanonical class. Then M admits a complete Ricci flat Kähler metric.*

Corollary 5.2. *Let $M = \overline{M}\backslash D$ be either in Theorem 5.1 or in Theorem 5.2. Let $C_1(K_{\overline{M}}^{-1} \otimes [D]^{-1})$ contain a nonnegative form. Then M admits a complete Kähler metric with nonnegative Ricci curvature.*

We end this section with an application of Theorem 5.1, etc., to the upper bound for the growth of finitely generated subgroups in $\pi_1(M)$ for some quasi-projective manifold M. We define the growth function γ associated with a finitely generated group and a specified choice of generators $\{g_1, \dots, g_p\}$ for the group as follows. For each positive integer s let $\gamma(s)$ be the number of distinct group elements which can be expressed as words of length $\leq s$ in the specified generators and their inverses [Mi].

Definition 5.2 [Mi]. Let X be a smooth manifold. We say that the fundamental group $\pi(X)$ is of polynomial growth of order less than k if for any finitely generated subgroup of $\pi_1(X)$ and a specified choice of generators, the associated growth function γ satisfies

$$\gamma(s) \leq Cs^k \quad \text{for some constant } C.$$

Theorem 5.3. *Let $M = \overline{M}\backslash D$ be either in Theorem 5.1 or Theorem 5.2. Let $C_1(K_{\overline{M}}^{-1} \otimes [D]^{-1})$ contain a nonnegative form. Then the fundamental group $\pi_1(M)$ is of polynomial growth of order less than the real dimension of M.*

Proof. It follows from Corollary 5.2 and [Mi].

Note that in our case, the fundamental group $\pi_1(M)$ is indeed finitely generated since M can be compactified. In general, it is not yet known whether or not the fundamental group of a complete Riemannian manifold with nonnegative Ricci curvature is necessarily finitely generated.

6. A FINAL REMARK

In this last section, we summarize the previous results of §§4 and 5 on complete Ricci-flat metrics in a simple and clean form for complex surfaces. We start with an easy lemma.

Lemma 6.1. *Let $\overline{M}$ be a projective complex surface and D be a smooth anti-canonical divisor. Then $\overline{M}$ is biholomorphic to one of the following:*

(1) *The surface obtained by blowing up CP^2 at finitely many points along a smooth cubic curve in CP^2.*

(2) $CP^1 \times CP^1$.

(3) $CP^1 \times T^1_C$, *where T^1_C is the complex torus of dimension* 1.

Proof. First note that if $\overline{M}$ contains an exceptional curve E of type one, then we can blow down this E to obtain a surface satisfying the assumptions of the lemma. Thus we may assume that $\overline{M}$ is relatively minimal. On the other hand, since $K^{-1}_{\overline{M}} = [D]$, the Kodaira dimension of $\overline{M}$ is $-\infty$. It follows from classification theory [BPV] that $\overline{M}$ is a ruled surface. If $\overline{M}$ is rational, then $\overline{M}$ is either CP^2 or one of the Hirzebruch surfaces. So $D^2 = K^2_{\overline{M}} = 9$ or 8. It follows that $C_1(\overline{M})$ is numerically positive. So $\overline{M}$ is either CP^2 or $CP^1 \times CP^1$. If $\overline{M}$ is not rational, it is a ruled surface over a curve C of genus ≥ 1. By the adjunction formula, K_D is trivial and D intersects with each fiber at two points. It implies that the genus of C is one and D consists of two disjoint tori. One easily checks that $\overline{M}$ must then be $CP^1 \times T^1_C$. The lemma is proved.

Combining this lemma with Theorems 5.1 and 5.2, we have

Theorem 6.1. *Let $M = \overline{M} \backslash D$ be a quasi-projective surface with $\overline{M}$ smooth and D a smooth anticanonical divisor in $\overline{M}$. Then if $D^2 \geq 0$, M admits a complete Ricci-flat Kähler metric.*

Remark. Let $M = \overline{M} \backslash D$ as in Theorem 6.1. We do not know whether or not the condition $D^2 \geq 0$ is necessary for the existence of a complete Ricci-flat metric on M. For such a pair (M, D), by Lemma 6.1, $D^2 < 0$ if and only if $\overline{M}$ is a surface obtained by blowing up CP^2 at more than nine points along a cubic curve in CP^2 and D is the quadratic transformation of that cubic curve.

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