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COMPLETE LIFTS FROM A MANIFOLD TO ITS COTANGENT BUNDLE

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We consider the complete lift from a vector field of a manifold (base space) to its cotangent bundle. In §1, we shall prove that the complete lift is characterized to be an infinitesimal homogeneous contact transformation preserving each fibre in the cotangent bundle. In §2, we shall see that the cotangent bundle admits several structures (symplectic structure or homogeneous contact structure etc.). In §§3 and 4, we shall discuss complete lifts in the case that the base space is Riemannian.

1. Complete lift of a vector field.[1]

Let M be an n-dimensional differentiable manifold of class C^∞. Consider the set ${}^CT_P(M)$ of all non-zero covectors at a point $P\in M$. Then

$$ {}^CT(M)=\bigcup_{P\in M} {}^CT_P(M) $$

is, by definition, the cotangent bundle over the manifold M. A point P of ${}^CT(M)$ is an ordered pair (P, ω_P) of a point $P\in M$ and a covector $\omega_P\in {}^CT(M)$. We denote by π the projection ${}^CT(M)\to M$ given by $\tilde{P}=(P, \omega_P)\to P$. The set $\pi^{-1}(P)$, that is, ${}^CT_P(M)$ is called the fibre over P, and M is called the base space.

Suppose that the manifold M is covered by a system of coordinate neighbourhoods $\{U, x^h\}$ where (x^h) is a system of local coordinates in the neighbourhood U. Then, in the open set $\pi^{-1}(U)$ of ${}^CT(M)$ we can introduce local coordinates (x^h, p_i) for $\tilde{P}$, which we call coordinates in $\pi^{-1}(U)$, induced from (x^h) or simply induced coordinates in $\pi^{-1}(U)$.

We denote by $\mathfrak{X}(M)$ the set of all vector fields of class C^∞ in M. Suppose that $X\in\mathfrak{X}(M)$. The complete lift X^C of X is, by definition [9], given by

$$ X^C=(dX^V)\varepsilon^{-1} \tag{1.1} $$

where X^V is the vertical lift[2] of X and ε^{-1} is a tensor field of type (2, 0) whose components ε^{BA} in $\pi^{-1}(U)$ are

$$ \varepsilon^{BA}=\begin{pmatrix} 0 & -\delta_i^h \\ \delta_h^i & 0 \end{pmatrix}. $$

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1) As to the notations we follow Yano and Patterson [9].

2) As to its definition, see Yano and Patterson [9].

In $\pi^{-1}(U)$, the components of X^C are

$$X^C\colon \begin{pmatrix} X^h \\ -p_i\partial_h X^i \end{pmatrix}.$$

On the other hand, a vector field $X^A=(X^h, P_i)$ over ${}^CT(M)$ is said to be an infinitesimal homogeneous contact transformation [5] if it satisfies

$$\mathcal{L}(X)p=0 \tag{1. 2}$$

where p means the basic 1-form in ${}^CT(M)$. Locally, the equation (1. 2) is equivalent to

$$p_i\partial_h X^i=-P_h, \quad p_i\partial^h X^i=0, \quad \left(\partial^h=\frac{\partial}{\partial p_h}\right). \tag{1. 3}$$

Since the components of X^C satisfy (1. 3), it follows that the complete lift of a vector field is an infinitesimal homogeneous contact transformation.

It is known [9] that

$$[X^C, Y^C]=[X, Y]^C \qquad X, Y\in \mathfrak{X}(M).$$

Therefore, the set L_0^C of all complete lifts of vector fields in M is a subalgebra in Lie algebra L^C of all infinitesimal homogeneous contact transformations and obviously L_0^C is isomorphic to the Lie algebra $\mathfrak{X}(M)$. As $\mathfrak{X}(M)$ is infinite dimensional, so is L_0^C. Thus we have

THEOREM 1. 1. [5]. *The Lie algebra L^C of all infinitesimal homogeneous contact transformation of M is infinite dimensional.*

A diffeomorphism

$$f\colon {}^CT(M)\to{}^CT(M)$$

is said to be a homogeneous contact transformation of M if and only if f leaves invariant the basic 1-form p, i.e. $f^*p=p$, where f^* is the dual map induced by f on differential forms over ${}^CT(M)$. Suppose f_0 be a diffeomorphism of M onto itself. Then f_0 naturally induces a diffeomorphism f of ${}^CT(M)$ onto itself. It is easy to see that f is a homogeneous contact transformation. This map f is called an extension of the diffeomorphism f_0 of M.

LEMMA [5] *A homogeneous contact transformation f of ${}^CT(M)$ onto itself is an extension of a diffeomorphism of M onto itself if and only if f is a fibre preserving map.*

Corresponding to the above Lemma, we see easily the following

THEOREM 1. 2. *In order that an infinitesimal homogeneous contact transformation of M be a complete lift of a vector field over M, it is necessary and sufficient that* *it is a fibre preserving one.*

Next let $X^A=(X^h, P_i)$ be components of an infinitesimal homogeneous contact transformation X of M. The function $U=p_iX^i$ is said to be the characteristic function of X. Then we have the following

THEOREM 1. 3. *In order that an infinitesimal homogeneous contact transformation of M be a complete lift of a vector field over M, it is necessary and sufficient that the characteristic function of the infinitesimal homogeneous contact transformation be homogeneous and linear with respect to p_i.*

2. Symplectic structure.

The exterior differential dp of the basic 1-form $p=p_idx^i$ is the 2-form F given by

$$F\equiv dp=dp_i\wedge dx^i.$$

Hence if we write

$$F=\frac{1}{2}\varepsilon_{CB}dx^C\wedge dx^B,$$

we have

$$\varepsilon_{CB}=\begin{pmatrix} 0 & \delta_i^h \\ -\delta_h^i & 0 \end{pmatrix}.$$

The matrix (ε_{CB}) being non-singular, the 2-form F, the exterior differential of the basic 1-form p, furnishes a symplectic structure in ${}^CT(M)$. Because of $F^n=F\wedge F\wedge \cdots\wedge F\neq 0$, owing to Yano and Mutô [8], ${}^CT(M)$ admits a homogeneous contact structure and consequently ${}^CT(M)$ is non-compact.

Consider an action of an infinitesimal transformation X over F:

$$\mathcal{L}(X)F=di(X)F+i(X)dF$$

that is, F being closed,

$$\mathcal{L}(X)F=di(X)F.$$

X defines a *symplectic infinitesimal automorphism* if it leaves F invariant, that is,

$$\mathcal{L}(X)F=di(X)F=0. \tag{2. 1}$$

We denote by L^S the Lie algebra of all symplectic infinitesimal automorphisms.

Next we make a 1-form ξ correspond to $X\in\mathfrak{X}({}^CT(M))$ by

$$\xi=-i(X)F. \tag{2. 2}$$

Thereby we define an isomorphism μ of $\mathcal{X}({}^{C}T(M))$ onto $\mathcal{X}^*({}^{C}T(M))$:

$$\xi=\mu(X), \qquad X\in\mathcal{X}({}^{C}T(M)), \qquad \xi\in\mathcal{X}^*({}^{C}T(M)).$$

We denote by L_0^S the set of all vectors of ${}^{C}T(M)$ such that its image by μ is derived. If X, $Y\in L^S$, we have

$$\mu([X, Y])=d\{i(X\wedge Y)F\},$$

that is, $[X, Y]\in L_0^S$ [3]. Consequently we see that L_0^S is an ideal of L^S. From (2. 2) 1-form ξ is locally expressed by

$$\xi_A=-X^B\varepsilon_{BA}=-(P_i, X^h)$$

where we put $X^A=(X^h, P_i)$. Especially if $X\in L_0^S$, there exist a function f over ${}^{C}T(M)$ such that $\xi_A=\partial_A f$. Consequently, the components of X can be written as

$$X^A=(X^h, P_i)=(\partial^h f, -\partial_i f). \tag{2. 3}$$

Conversely, $X\in L^S$ having the components of type (2. 3) belongs to L_0^S.

On the other hand, every infinitesimal homogeneous contact transformation X is locally expressed as

$$X^A=(X^h, P_i)=(\partial^h U, -\partial_i U), \tag{2. 4}$$

where U is the characteristic function of X, [5]. From (2. 3) and (2. 4), we can see that algebra L^C is a subalgebra of L_0^S. Summerizing the above facts, we get the following.

THEOREM 2. 1. $\qquad L^S\supset L_0^S\supset L^C\supset L_0^C\approx\mathcal{X}(M).$

COROLLARY [3]. *The Lie algebra of all symplectic infinitesimal automorphisms of* ${}^{C}T(M)$ *is infinite dimensional.*

The skew-symmetric tensor ε is of maximum rank $2n$, and, using a theorem due to Lichnerowicz [2] and Hatakeyama [1], we can introduce a positive definite metric G_{CB} in ${}^{C}T(M)$ such that

$$\varphi_B{}^A=\varepsilon_{BE}G^{EA}$$

defines an almost complex structure:

$$\varphi_B{}^E\varphi_E{}^A=-\delta_B^A$$

and consequently this metric is Hermitian with respect to the almost complex structure. The almost Hermitian manifold thus defined is almost Kählerian since the form dp is closed.

THEOREM 2. 2. *In a cotangent bundle associated with an almost Kählerian structure* φ, *in order that an infinitesimal homogeneous contact transformation* X *is infinitesimal isometry, it is necessary and sufficient that* X *is almost analytic*, *Proof.* Necessity: Since an infinitesimal homogeneous contact transformation is always a symplectic automorphism by virtue of Theorem 3. 1, $\mathcal{L}(X)F=0$ follows from $\mathcal{L}(X)p=0$. Then we have

$$(\mathcal{L}(X)\varphi)_B{}^A=(\mathcal{L}(X)\varepsilon)_{BE}G^{EA}+\varepsilon_{BE}(\mathcal{L}(X)G)^{EA}=0.$$

Sufficiency: We have

$$0=(\mathcal{L}(X)\varphi)_B{}^A=(\mathcal{L}(X)\varepsilon)_{BE}G^{EA}+\varepsilon_{BE}(\mathcal{L}(X)G)^{EA}$$

from which we get

$$\mathcal{L}(X)G=0.$$

3. Cotangent bundle of a Riemannian manifold.

Let M be a Riemannian manifold with the fundamental metric tensor g and g_{jk} the components of g with respect to a coordinate neighbourhood $U(x^i)$ in M. We define a line element in a coordinate neighbourhood $\pi^{-1}(U)(x^i, p_i)$ of ${}^CT(M)$ by

$$g_{ji}dx^j dx^i+g^{ji}Dp_jDp_i \tag{3. 1}$$

where Dp_i means the covariant differential of p_i. The components of the fundamental metric tensor of ${}^CT(M)$ can be obtained by putting (3. 1) in the form

$$\tilde{g}_{CB}dx^C dx^B, \tag{3. 2}$$

from which we have

$$\tilde{g}_{ji}=g_{ji}+\gamma_{jc}\gamma_{ib}g^{cb}, \qquad \tilde{g}_{\bar{j}i}=-g^{ja}\gamma_{ai} \qquad \tilde{g}_{\bar{j}\bar{i}}=g^{ji} \qquad (\gamma_{ji}=\Gamma_{ji}{}^a p_a).$$

The contravariant components of the fundamental metric tensor are given by

$$\tilde{g}^{ih}=g^{ih}, \qquad \tilde{g}^{i\bar{h}}=\gamma_{ia}g^{ah}, \qquad \tilde{g}^{\bar{i}\bar{h}}=g_{ih}+\gamma_{ic}\gamma_{hb}g^{cb}.$$

If we put $\tilde{F}_B{}^A=\varepsilon_{BE}\tilde{g}^{EA}$, then it gives us an almost complex structure on ${}^CT(M)$. Moreover, it is known that $(\tilde{g}, \tilde{F})$ is an almost Kählerian structure on ${}^CT(M)$, [4].

On the other hand let $\mathcal{T}(M)$ be the tangent bundle of M. Then $T(M)\equiv\mathcal{T}(M)-M$, a set of all non-zero tangent vectors, is an open submanifold of $\mathcal{T}(M)$. In the remaining part of this paper, by $T(M)$ we mean the tangent bundle of M. It is known that the tangent bundle $T(M)$ of a Riemannian manifold M is naturally reduced to an almost Kählerian manifold. The fundamental metric tensor G and the almost complex structure F are given by [6]

$$G_{CB}=\begin{pmatrix} g_{ji}+\Gamma_j{}^c\Gamma_i{}^b g_{cb} & \Gamma_{ij} \\ \Gamma_{ji} & g_{ji} \end{pmatrix}$$

and

$$F_B{}^A=\begin{pmatrix} \Gamma_j{}^h & \delta_j^h \\ -\Gamma_j{}^a\Gamma_a{}^h-\delta_j^h & -\Gamma_j{}^h \end{pmatrix}$$

where we put

$$\Gamma_j{}^h=\Gamma_{ja}{}^h y^a, \qquad \Gamma_{ji}=\Gamma_j{}^a g_{ai}.$$

(x^i, y^i) being coordinates in $T(M)$.

Now we consider a mapping

$$f:\ T(M)\to{}^C T(M), \qquad (x^i, y^i)\to(x^i, p_i)$$

which is locally expressed by

$$\begin{cases} x^i=x^i, \\ p_i=g_{ia}y^a. \end{cases}$$

It is easily verified that at any point P of $T(M)$

$$G(X, Y)=\tilde{g}(f_*X, f_*Y)$$

for all $X, Y\in T_P(M)$. This shows that $T(M)$ is isometric to ${}^C T(M)$. Next let F and $\tilde{F}$ be fundamental 2-form of $T(M)$ and ${}^C T(M)$ respectively. Then we get also

$$F(X, Y)=\tilde{F}(f_*X, f_*Y)$$

for all $X, Y\in T_P(M)$. Consequently, $T(M)$ is not only equivalent to ${}^C T(M)$ as Riemannian manifold but also as almost Kählerian manifold.

Next let X^C and $\tilde{X}^C$ be complete lift to $T(M)$ and ${}^C T(M)$ from a vector field X of M respectively. Then we have the following

THEOREM. 3. 1. *In order that the image of X^C by the isometry f of $T(M)$ onto ${}^C T(M)$ coincides with $\tilde{X}^C$, it is necessary and sufficient that a vector X of M is an infinitesimal isometry.*

Proof. Locally, X^C and $f_*(X^C)$ can be written as

$$X^C=(X^h,\ y^a\partial_a X^h)$$

and

$$f_*(X^C)=(X^h,\ -p_a\partial_h X^a+(\nabla_h X_a+\nabla_a X_h)y^a)$$

from which our assertion follows immediately.

N. B. It is verified that the image of the horizontal lift[3] to $T(M)$ by the isometry f of $T(M)$ onto ${}^C T(M)$ coincides with the horizontal lift to ${}^C T(M)$.

3) As to its definition, see [9].

4. Complete lifts.

In this section, complete lift means only that of a vector field of Riemannian base space. Recently Yano [7] studied properties of complete lift from Riemannian manifold to its tangent bundle. Complete lift to $T(M)$ leaves the basic 1-form invariant if and only if the vector field of M is a Killing vector, [6]. However as we saw in § 2, complete lift to ${}^C T(M)$ leaves always the basic 1-form invariant. Therefore it seems to be meaningful to study properties of complete lift ${}^C T(M)$.

The fibre of ${}^C T(M)$ being represented by[4)]

$$x^h = \text{const.}, \qquad p_h = p_h,$$

the integrable distribution tangent to these fibres is spanned by the n independent vectors

$$C_i{}^A = \begin{pmatrix} C_i^h \\ C_i^{\bar{h}} \end{pmatrix} = \begin{pmatrix} 0 \\ \delta_h{}^i \end{pmatrix}. \tag{4.1}$$

Now let us consider the vectors

$$B_i{}^A = \tilde{F}_E{}^A C_i{}^E = \begin{pmatrix} g^{ih} \\ \gamma_{ha} g^{ai} \end{pmatrix}.$$

The n vectors so defined are not in the distribution determined by (4. 1). Since the action of $\tilde{F}$ on any vector is to produce an orthogonal vector, we conclude that the two sets of vectors $B_i{}^A$, $C_i{}^A$ determine two complementary distributions orthogonal to each other. The n vectors $B_i{}^A$ orthogonal to the fibre are called horizontal vectors while the vectors $C_i{}^A$ of the complementary distribution are called vertical vectors. We can easily verify that

$$\tilde{g}_{CB} B_j{}^C B_i{}^B = g^{ji}, \qquad \tilde{g}_{CB} C_j{}^C C_i{}^B = g^{ji}$$

and

$$\tilde{F}_{CB} B_j{}^C B_i{}^B = 0, \qquad \tilde{F}_{CB} C_j{}^C B_i{}^B = g^{ji},$$

$$\tilde{F}_{CB} B_j{}^C C_i{}^B = g^{ji}, \qquad \tilde{F}_{CB} C_j{}^C C_i{}^B = 0.$$

We now refer our tensors to the special frame of reference given by

$$A_\beta{}^A = \begin{pmatrix} B_i{}^A \\ C_i{}^A \end{pmatrix} \tag{4.2}$$

and their inverses

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4) The assertions in the remaining part of this section are analogous to that in tangent bundle, see Yano [7].

$$A^{\alpha}{}_{B}=(B^{h}{}_{B},\ C^{\bar{h}}{}_{B})$$

where

$$\alpha,\ \beta,\ \gamma=1,\ 2,\ \cdots,\ n;\ \bar{1},\ \bar{2},\ \cdots,\ \bar{n}$$

and $B^{h}{}_{B}$ and $C^{\bar{h}}{}_{B}$ are given by

$$B^{h}{}_{B}=(g_{hi},\ 0), \qquad C^{\bar{h}}{}_{B}=(-\gamma_{hi},\ \delta_{h}^{i}).$$

We call this special frams of reference, the adapted frame. Thus for the components

$$\tilde{g}_{\gamma\beta}=\tilde{g}_{CB}A_{\gamma}{}^{C}A_{\beta}{}^{B}, \qquad \tilde{g}^{\beta\alpha}=\tilde{g}^{BA}A^{\beta}{}_{B}A^{\alpha}{}_{A},$$

$$\tilde{F}_{\gamma\beta}=\tilde{F}_{CB}A_{\gamma}{}^{C}A_{\beta}{}^{B}, \qquad \tilde{F}_{\beta}{}^{\alpha}=\tilde{F}_{B}{}^{A}A_{\beta}{}^{B}A^{\alpha}{}_{A}$$

of the tensors $\tilde{g}_{CB}$, $\tilde{g}^{BA}$, $\tilde{F}_{CB}$, $\tilde{F}_{B}{}^{A}$, we can write the following expressions

$$\tilde{g}_{\gamma\beta}=\begin{pmatrix} g^{ji} & 0 \\ 0 & g_{ji} \end{pmatrix}, \qquad \tilde{g}^{\beta\alpha}=\begin{pmatrix} g_{ji} & 0 \\ 0 & g_{ji} \end{pmatrix},$$

$$\tilde{F}_{\gamma\beta}=\begin{pmatrix} 0 & g^{ji} \\ -g^{ji} & 0 \end{pmatrix}, \quad \tilde{F}_{\beta}{}^{\alpha}=\begin{pmatrix} 0 & \delta_{h}^{i} \\ -\delta_{h}^{i} & 0 \end{pmatrix}.$$

Now let us introduce the notations

$$D_{\beta}f=A_{\beta}{}^{B}\partial_{B}f, \qquad \omega^{\alpha}=A^{\alpha}{}_{B}dx^{B},$$

which will give, for the various types of indices,

$$D_{i}f=g^{ia}\partial_{a}f+\gamma_{ba}g^{ai}\partial_{\bar{b}}f, \qquad D_{\bar{i}}f=\partial_{\bar{i}}f,$$

$$\omega^{h}=g_{ha}dx^{a}, \qquad \omega^{\bar{h}}=-\gamma_{ha}dx^{a}+dp_{h}.$$

We shall need the components of the non-holonomic object which is important when we use a frame of reference such as (4. 2) which is not the natural one associated with the coordinate system. They are

$$\Omega_{\gamma\beta}{}^{\alpha}=(D_{\gamma}A_{\beta}{}^{A}-D_{\beta}A_{\gamma}{}^{A})A^{\alpha}{}_{A},$$

the only non-vanishing components of which will be

$$\Omega_{ji}{}^{h}=-g^{ja}\Gamma_{ah}^{i}+g^{ai}\Gamma_{ah}^{j}, \qquad \Omega_{ji}{}^{\bar{h}}=g^{jb}g^{ia}R_{bah}{}^{c}p_{c},$$

(4. 4)

$$\Omega_{j\bar{i}}{}^{\bar{h}}=-\Gamma_{ha}^{i}g^{aj},$$

If $\tilde{\Gamma}_{CB}{}^{A}$ denote the three index symbols of Christoffel with respect to the $\tilde{g}_{CB}$, the corresponding coefficients with respect to the adapted frame just introduced are given by

$$\tilde{\Gamma}_{\gamma\beta}^{\alpha}=(D_{\gamma}A_{\beta}{}^{A}+\tilde{\Gamma}_{CB}^{A}A_{\gamma}{}^{C}A_{\beta}{}^{B})A^{\alpha}{}_{A},$$

$$\tilde{\Gamma}_{\gamma\beta}^{\alpha}-\tilde{\Gamma}_{\beta\gamma}^{\alpha}=\Omega_{\gamma\beta}^{\alpha}.$$

The covariant derivative of $\tilde{g}_{\gamma\beta}$ is given by

$$\tilde{\nabla}_{\delta}\tilde{g}_{\gamma\beta}=D_{\delta}\tilde{g}_{\gamma\beta}-\tilde{\Gamma}_{\delta\gamma}^{\varepsilon}\tilde{g}_{\varepsilon\beta}-\tilde{\Gamma}_{\delta\beta}^{\varepsilon}\tilde{g}_{\gamma\varepsilon}=0$$

from which we can deduce

$$\tilde{\Gamma}_{\gamma\beta}^{\alpha}=\frac{1}{2}\tilde{g}^{\alpha\varepsilon}(D_{\gamma}\tilde{g}_{\beta\varepsilon}+D_{\beta}\tilde{g}_{\gamma\varepsilon}-D_{\varepsilon}\tilde{g}_{\gamma\beta})$$

$$+\frac{1}{2}(\Omega_{\gamma\beta}^{\alpha}+\Omega^{\alpha}{}_{\gamma\beta}+\Omega^{\alpha}{}_{\beta\gamma}),$$

where we have put

$$\Omega^{\alpha}{}_{\gamma\beta}=\tilde{g}^{\alpha\varepsilon}\tilde{g}_{\delta\beta}\Omega_{\varepsilon\beta}^{\delta}.$$

The particular values of $\tilde{\Gamma}_{\gamma\beta}^{\alpha}$ for different indices, on taking account of (4. 3) and (4. 4), are found to be

$$\tilde{\Gamma}_{ji}^{h}=-g^{ja}\Gamma_{ah}^{i},\qquad \tilde{\Gamma}_{j\bar{i}}{}^{h}=\frac{1}{2}g^{jc}g^{ib}R_{hcb}{}^{a}p_{a},$$

$$\tilde{\Gamma}_{\bar{j}i}{}^{h}=\frac{1}{2}g^{ic}g^{jb}R_{hcb}{}^{a}p_{a},\qquad \tilde{\Gamma}_{\bar{j}\bar{i}}{}^{h}=0,\qquad \tilde{\Gamma}_{ji}{}^{\bar{h}}=\frac{1}{2}g^{jc}g^{ib}R_{cbh}{}^{a}p_{a},$$

$$\tilde{\Gamma}_{j\bar{i}}{}^{\bar{h}}=-g^{ja}\Gamma_{ah}{}^{i},\qquad \tilde{\Gamma}_{\bar{j}i}{}^{\bar{h}}=\tilde{\Gamma}_{\bar{j}\bar{i}}{}^{\bar{h}}=0.$$

Consider a vector field X in M. The complete lift X^{C} of X has components

$$\tilde{X}^{\alpha}=\begin{pmatrix} X^{h} \\ -p_{i}\tilde{\nabla}_{h}X^{i} \end{pmatrix}$$

with respect to the adapted frame. The covariant derivative of the complete lift is given by

$$\tilde{\nabla}_{\beta}\tilde{X}^{\alpha}=\begin{pmatrix} g^{ja}\nabla_{a}X_{h}-\frac{1}{2}g^{jc}R_{hcb}{}^{a}p_{a}p_{d}\nabla^{b}X^{d} & \frac{1}{2}g^{jb}R_{hcb}{}^{a}p_{a}X^{c} \\ -(\nabla^{j}\nabla_{h}X^{a})p_{a}+\frac{1}{2}g^{jc}R_{cbh}{}^{a}p_{a}X^{b} & -\nabla_{h}X^{j} \end{pmatrix}.$$

Thus we have

THEOREM 4. 1. *The complete lift to ${}^{C}T(M)$ of a vector field in M is parallel if and only if the vector field is parallel in M.*

The Lie derivative of $\tilde{g}$ with respect to the complete lift X^C of X is given by

$$(\mathcal{L}(X^C\tilde{g}))_{\gamma\beta}=\begin{pmatrix} \nabla^j X^i+\nabla^i X^j & -(\nabla^j\nabla^i X^a)p_a+g^{jc}g^{ib}R_{cdb}{}^a p_a X^d \\ (\nabla^i\nabla^j X^a)p_a-g^{jc}g^{ib}R_{bdc}{}^a p_a X^d & 0 \end{pmatrix},$$

from which we have

THEOREM 4. 2. *The complete lift to ${}^{C}T(M)$ of a vector field in M is a Killing vector field if and only if the vector field is a Killing vector field and have vanishing second covariant derivative in M.*

The rotation for the complete lift X^C are

$$\tilde{\nabla}_\beta\tilde{X}_\alpha-\tilde{\nabla}_\alpha\tilde{X}_\beta=\begin{pmatrix} \nabla^j X^i-\nabla^i X^j+g^{jc}g^{ib}R_{cbd}{}^a p_a p_c\nabla^a X^e & -(\nabla^j\nabla^i X^a)p_a \\ (\nabla^i\nabla^j X^a)p_a & 0 \end{pmatrix}$$

and

$$\tilde{\nabla}_\beta\tilde{X}^\beta=0$$

whence it follows that if in a symmetric Riemannian space M, X is closed and have vanishing second derivative, then the complete lift X^C is harmonic.

N. B. We hope the above facts of the complete lift to ${}^{C}T(M)$ will be compared with that of the complete lift to $T(M)$. See Yano [7].

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