# Complete manifolds with non-negative Ricci curvature and the Caffarelli-Kohn-Nirenberg inequalities 

Manfredo Perdigão do Carmo and Changyu Xia


#### Abstract

In this paper, we prove that complete open Riemannian manifolds with non-negative Ricci curvature of dimension greater than or equal to three in which some Caffarelli-Kohn-Nirenberg type inequalities are satisfied are close to the Euclidean space.


## 1. Introduction

Let $n \geqslant 3$ be an integer and let $a, b$, and $p$ be constants satisfying

$$
\begin{equation*}
-\infty<a<\frac{n-2}{2}, \quad a \leqslant b \leqslant a+1, \quad \text { and } \quad p=\frac{2 n}{n-2+2(b-a)} . \tag{1.1}
\end{equation*}
$$

Denote by $C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ the space of smooth functions with compact support in the $n$-dimensional Euclidean space $\mathbb{R}^{n}$. In [CKN84], among a much more general family of inequalities, Caffarelli, Kohn, and Nirenberg proved the following result. There exists a positive constant $C$ depending only on $a, b$ and $n$ such that

$$
\begin{equation*}
\left(\int_{\mathbb{R}^{n}}|x|^{-b p}|u|^{p} d x\right)^{1 / p} \leqslant C\left(\int_{\mathbb{R}^{n}}|x|^{-2 a}|\nabla u|^{2} d x\right)^{1 / 2}, \tag{1.2}
\end{equation*}
$$

for all $u \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$, where $|x|$ is the Euclidean length of $x \in \mathbb{R}^{n}$. Note that the Caffarelli-KohnNirenberg inequalities contain the classical Sobolev inequality ( $a=b=0$ ) and the Hardy inequality $(a=0, b=1)$ as special cases, which have many important applications (see e.g. [Aub82, Aub98, CKN84, HLP52, Heb96, Heb99, Lie83] and references therein).

Let $K_{a, b}$ be the best constant for the Caffarelli, Kohn, and Nirenberg inequality (1.1), that is

$$
\begin{equation*}
K_{a, b}^{-1}=\inf _{u \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)-\{0\}} \frac{\left(\int_{\mathbb{R}^{n}}|x|^{-2 a}|\nabla u|^{2} d x\right)^{1 / 2}}{\left(\int_{\mathbb{R}^{n}}|x|^{-b p}|u|^{p} d x\right)^{1 / p}} . \tag{1.3}
\end{equation*}
$$

For the Sobolev inequality $(a=b=0)$, it has been proved by Aubin [Aub76] and Talent [Tal76] that

$$
K_{0,0}=\left(\frac{1}{n(n-2)}\right)^{1 / 2}\left(\frac{2 \Gamma(n)}{n \omega_{n} \Gamma^{2}(n / 2)}\right)^{1 / n}
$$

where $\omega_{n}$ is the volume of the unit ball in $\mathbb{R}^{n}$, and that a family of minimizers of (1.3) is given by

$$
u(x)=\left(\lambda+|x|^{2}\right)^{1-n / 2}, \lambda>0 .
$$

In [Lie83], Lieb considered the case $a=0,0<b<1$, and proved that the best constant is

$$
K_{0, b}=\left(\frac{1}{(n-2)(n-b p)}\right)^{1 / 2}\left(\frac{(2-b p) \Gamma((2 n-2 b p) /(2-b p))}{n \omega_{n} \Gamma^{2}((n-b p) /(2-b p))}\right)^{2(n-b p) /(2-b p)}
$$

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## Manifolds with non-negative Ricci curvature

and a family of minimizers is

$$
u(x)=\frac{1}{\left(\lambda+|x|^{2-b p}\right)^{(n-2) /(2-b p)}}, \quad \lambda>0 .
$$

Chou and Chu [CC93] studied the case $a \geqslant 0, a \leqslant b<a+1$, and proved that the best constant is

$$
K_{a, b}=\left(\frac{1}{(n-2 a-2)(n-b p)}\right)^{1 / 2}\left(\frac{(2-b p+2 a) \Gamma((2 n-2 b p) /(2-b p+2 a))}{n \omega_{n} \Gamma^{2}((n-b p) /(2-b p+2 a))}\right)^{2(n-b p) /(2-b p+2 a)},
$$

and that, for $a>0$, all minimizers are non-zero constant multiples of the function

$$
u(x)=\frac{1}{\left(\lambda+|x|^{2-b p+2 a}\right)^{(n-2-2 a) /(2-b p+2 a)}}, \quad \lambda>0 .
$$

For the remaining case, the best constant $K_{a, b}$ and the existence or non-existence of the minimizers have been studied recently in [CW01].

In this paper, we study complete manifolds with non-negative Ricci curvature in which some Caffarelli-Kohn-Nirenberg inequalities are satisfied. Now we fix some notation. For an integer $n \geqslant 3$, we will from now on let $a$ and $b$ be constants satisfying

$$
\begin{equation*}
0 \leqslant a<\frac{n-2}{2}, \quad a \leqslant b<a+1, \tag{1.4}
\end{equation*}
$$

and set

$$
\begin{equation*}
p=\frac{2 n}{n-2+2(b-a)} . \tag{1.5}
\end{equation*}
$$

For a Riemannian manifold $M$, we let $d v$ be the Riemannian volume element on $M$, denote by $\nabla$ the gradient operator, $C_{0}^{\infty}(M)$ the space of smooth functions on $M$ with compact support, $B(x, r)$ the geodesic ball with center $x \in M$ and radius $r$, and $\operatorname{vol}[B(p, r)]$ the volume of $B(p, r)$.

Our purpose is to prove the following result.
Theorem 1.1. Let $C \geqslant K_{a, b}$ be a constant and $M$ be an $n$-dimensional ( $n \geqslant 3$ ) complete open Riemannian manifold with non-negative Ricci curvature. Fix a point $x_{0} \in M$ and denote by $\rho$ the distance function on $M$ from $x_{0}$. Assume that, for any $u \in C_{0}^{\infty}(M)$, we have

$$
\begin{equation*}
\left(\int_{M} \rho^{-b p}|u|^{p} d v\right)^{1 / p} \leqslant C\left(\int_{M} \rho^{-2 a}|\nabla u|^{2} d v\right)^{1 / 2} . \tag{1.6}
\end{equation*}
$$

Then for any $x \in M$, we have

$$
\begin{equation*}
\operatorname{vol}[B(x, r)] \geqslant\left(C^{-1} K_{a, b}\right)^{n /(1+a-b)} V_{0}(r), \quad \forall r>0 \tag{1.7}
\end{equation*}
$$

where $V_{0}(r)$ is the volume of the $r$-ball in $\mathbb{R}^{n}$.
In the special case that $a=b=0$, the above theorem has been proved in [Xia01].
The theorem has several consequences for manifolds with non-negative Ricci curvature.
The Bishop-Gromov comparison theorem (cf. [BC64, Cha93, GLP81]) implies that, if $M$ is an $n$-dimensional complete Riemannian manifold with non-negative Ricci curvature, then for any $x \in M, \operatorname{vol}[B(x, r)] \leqslant V_{0}(r)$, with equality holding if and only if $B(x, r)$ is isometric to an $r$-ball in $\mathbb{R}^{n}$. Combining this fact and Theorem 1.1, one immediately gets the following rigidity theorem.

Corollary 1.2. An n-dimensional ( $n \geqslant 3$ ) complete open Riemannian manifold $M$ with nonnegative Ricci curvature in which the inequality

$$
\left(\int_{M} \rho^{-b p}|u|^{p} d v\right)^{1 / p} \leqslant K_{a, b}\left(\int_{M} \rho^{-2 a}|\nabla u|^{2} d v\right)^{1 / 2}, \quad \forall u \in C_{0}^{\infty}(M)
$$

is satisfied, is isometric to $\mathbb{R}^{n}$.

## M. P. do Carmo and C. Xia

When $a=b=0$, Corollary 1.2 is the main theorem in [Led99].
A theorem of Cheeger and Colding [CC97] states that given integer $n \geqslant 2$ there exists a constant $\delta(n)>0$ such that any $n$-dimensional complete Riemannian manifold with non-negative Ricci curvature and $\operatorname{vol}[B(x, r)] \geqslant(1-\delta(n)) V_{0}(r)$ for some $p \in M$ and all $r>0$ is diffeomorphic to $\mathbb{R}^{n}$. Thus combining the Cheeger-Colding theorem and Theorem 1.1, one deduces the following topological rigidity for manifolds with non-negative Ricci curvature.

Corollary 1.3. Given integer $n \geqslant 3$, there exists a positive constant $\epsilon=\epsilon(n, a, b)$ such that any $n$-dimensional $(n \geqslant 3)$ complete non-compact Riemannian manifold $M$ with non-negative Ricci curvature in which the inequality

$$
\left(\int_{M} \rho^{-b p}|u|^{p} d v\right)^{1 / p} \leqslant\left(K_{a, b}+\epsilon\right)\left(\int_{M} \rho^{-2 a}|\nabla u|^{2} d v\right)^{1 / 2}, \quad \forall u \in C_{0}^{\infty}(M)
$$

is satisfied, is diffeomorphic to $\mathbb{R}^{n}$.
A theorem due to Li [Li86] and Anderson [And90] states that, if $M$ is an $n$-dimensional complete manifold with non-negative Ricci curvature in which the inequality $\operatorname{vol}[B(p, r)] \geqslant \alpha V_{0}(r)$ holds for some constant $\alpha>0$ and all $r>0$, the fundamental group $\pi_{1}(M)$ is finite and ${ }^{\#} \pi_{1}(M) \leqslant 1 / \alpha$. Thus from the Li-Anderson theorem and Theorem 1.1 we have the following corollary.

Corollary 1.4. Let $C \geqslant K_{a, b}$ be a constant and $M$ be an $n$-dimensional ( $n \geqslant 3$ ) complete open Riemannian manifold with non-negative Ricci curvature. Assume that, for any $u \in C_{0}^{\infty}(M)$, we have

$$
\begin{equation*}
\left(\int_{M} \rho^{-b p}|u|^{p} d v\right)^{1 / p} \leqslant C\left(\int_{M} \rho^{-2 a}|\nabla u|^{2} d v\right)^{1 / 2} \tag{1.8}
\end{equation*}
$$

Then $M$ has finite fundamental group and the order of $\pi_{1}(M)$ is bounded above by $\left(K_{a, b}^{-1} C\right)^{n /(1+a-b)}$.
One can find some related results about the topology of complete manifolds with non-negative Ricci curvature, for example, in [AG90, And90, CX00, Col98, Li86, OSY00, Ots89, SS97, She93, She96, SS01, Sor00, Xia99].

## 2. A Proof of Theorem 1.1

First notice the following fact. The Bishop-Gromov comparison theorem (cf. [BC64, Cha93, GLP81]) tells us that for any $p \in M$ the function $\operatorname{vol}[B(p, r)] / V_{0}(r)$ is decreasing and so the limit

$$
\lim _{r \rightarrow+\infty} \frac{\operatorname{vol}[B(p, r)]}{V_{0}(r)}
$$

exists. Also one can easily check that the above limit does not depend on the choice of $p$. It then follows that if (1.7) holds for some point $p_{0} \in M$, then it is satisfied for all $x \in M$. Now we are going to show that (1.7) holds at the point $x_{0}$.

Set

$$
\begin{equation*}
w=2 a-b p+2, \quad q=\frac{(n-2 a-2) p}{2 a-b p+2}=\frac{2 p}{p-2}, \tag{2.1}
\end{equation*}
$$

and, for any $\lambda>0$, let

$$
\begin{equation*}
F(\lambda)=\frac{p-2}{p+2} \int_{M} \frac{d v}{\rho^{b p}\left(\lambda+\rho^{w}\right)^{q-1}} . \tag{2.2}
\end{equation*}
$$

Then, for $\lambda>0$, we have from the Fubini theorem (cf. [SY94]) that

$$
F(\lambda)=\frac{p-2}{p+2} \int_{0}^{+\infty} \operatorname{vol}\left\{x: \frac{1}{\rho^{b p}\left(\lambda+\rho^{w}\right)^{q-1}}>s\right\} d s
$$

## Manifolds with non-negative Ricci curvature

Making the variable change $s=1\left(h^{b p}\left(\lambda+h^{w}\right)^{q-1}\right)$ in the above equality, one concludes that

$$
\begin{align*}
F(\lambda) & =\frac{p-2}{p+2} \int_{0}^{+\infty} \operatorname{vol}\{x: \rho(x)<h\} \frac{\left(b p \lambda+(b p+(q-1) w) h^{w}\right)}{h^{b p+1}\left(\lambda+h^{w}\right)^{q}} d h \\
& =\frac{p-2}{p+2} \int_{0}^{+\infty} \operatorname{vol}\left[B\left(x_{0}, h\right)\right] \frac{\left(b p \lambda+(b p+(q-1) w) h^{w}\right)}{h^{b p+1}\left(\lambda+h^{w}\right)^{q}} d h . \tag{2.3}
\end{align*}
$$

Since the Bishop-Gromov comparison theorem implies that $\operatorname{vol}\left[B\left(x_{0}, h\right)\right] \leqslant \omega_{n} h^{n}$, we have

$$
F(\lambda) \leqslant \frac{\omega_{n}(p-2)}{p+2} \int_{0}^{+\infty}\left(b p \lambda+(b p+(q-1) w) h^{w}\right) h^{n-b p-1}\left(\lambda+h^{w}\right)^{-q} d h
$$

On the other hand, one can deduce from (1.4), (1.5), and (2.1) that

$$
n-b p-1>-1, \quad n-b p-1+w(1-q)<-1 .
$$

It then follows that $0 \leqslant F(\lambda)<+\infty, \forall \lambda>0$, and that $F$ is differentiable. Also, we have

$$
\begin{equation*}
F^{\prime}(\lambda)=-\int_{M} \frac{d v}{\rho^{b p}\left(\lambda+\rho^{w}\right)^{q}} . \tag{2.4}
\end{equation*}
$$

Consider the function $H_{0}:(0,+\infty) \rightarrow \mathbb{R}$ defined by

$$
H_{0}(\lambda)=\frac{p-2}{p+2} \int_{\mathbb{R}^{n}} \frac{d x}{|x|^{b p}\left(\lambda+|x|^{w}\right)^{q-1}} .
$$

Recall that when $M=\mathbb{R}^{n}$ and $C=K_{a, b}$, the extremal functions in the inequality (1.6) are the functions $u_{\lambda}:=\left(\lambda+|x|^{w}\right)^{-q / p}, \lambda>0$. That is, we have

$$
\begin{aligned}
\left(-H_{0}^{\prime}(\lambda)\right)^{2 / p} & =\left(\int_{\mathbb{R}^{n}} \frac{d x}{|x|^{b p}\left(\lambda+|x|^{w}\right)^{q}}\right)^{2 / p} \\
& =\left(\frac{K_{a, b w}}{p}\right)^{2} \int_{\mathbb{R}^{n}} \frac{d x}{|x|^{2(1+a-w)}\left(\lambda+|x|^{w}\right)^{2+2 q / p}} \\
& =\left(\frac{K_{a, b} q w}{p}\right)^{2} \int_{\mathbb{R}^{n}} \frac{d x}{|x|^{b p-w}\left(\lambda+|x|^{w}\right)^{q}} \\
& =\left(\frac{K_{a, b q w}}{p}\right)^{2}\left(H_{0}^{\prime}(\lambda)+\frac{p+2}{p-2} H_{0}(\lambda)\right) .
\end{aligned}
$$

Substituting $H_{0}(\lambda)=H_{0}(1) \lambda^{-2 /(p-2)}$ into the above equation, one gets

$$
\begin{align*}
H_{0}(1) & =\frac{p-2}{p+2} \int_{\mathbb{R}^{n}} \frac{d x}{|x|^{b p}\left(1+|x|^{w}\right)^{q-1}} \\
& =2^{2 /(p-2)}(p-2)\left((n-2 a-2)^{2} K_{a, b}^{2}\right)^{-p /(p-2)} . \tag{2.5}
\end{align*}
$$

By a simple approximation procedure, we can apply (1.6) to $\left(\lambda+\rho^{w}\right)^{-q / p}$ for every $\lambda>0$ to get

$$
\begin{aligned}
\left(\int_{M} \frac{d v}{\rho^{b p}\left(\lambda+\rho^{w}\right)^{q}}\right)^{2 / p} & \leqslant\left(\frac{q w C}{p}\right)^{2} \int_{M} \frac{d v}{\rho^{2(1+a-w)}\left(\lambda+\rho^{w}\right)^{2+2 q / p}} \\
& =\left(\frac{q w C}{p}\right)^{2} \int_{M} \frac{d v}{\rho^{b p-w}\left(\lambda+\rho^{w}\right)^{q}}
\end{aligned}
$$

Let $l=(p / q w C)^{2}$; then the above inequality becomes

$$
\begin{equation*}
l\left(-F^{\prime}(\lambda)\right)^{2 / p}-\lambda F^{\prime}(\lambda) \leqslant \frac{p+2}{p-2} F(\lambda) . \tag{2.6}
\end{equation*}
$$

M. P. do Carmo and C. Xia

The idea now is to compare the solutions of (2.6) to the solutions $H$ of the following differential equality:

$$
\begin{equation*}
l\left(-H^{\prime}(\lambda)\right)^{2 / p}-\lambda H^{\prime}(\lambda)=\frac{p+2}{p-2} H(\lambda) \tag{2.7}
\end{equation*}
$$

One can easily check that $H_{1}(\lambda)$ given by

$$
\begin{equation*}
H_{1}(\lambda):=A \lambda^{-2 /(p-2)} \tag{2.8}
\end{equation*}
$$

is a particular solution of (2.7), where

$$
\begin{align*}
A & =2^{2 /(p-2)}(p-2)\left(\frac{l}{p}\right)^{p /(p-2)} \\
& =2^{2 /(p-2)}(p-2)\left((n-2 a-2)^{2} p C^{2}\right)^{-p /(p-2)} \\
& =\left(C^{-1} K_{a, b}\right)^{2 p /(p-2)} \cdot 2^{2 /(p-2)}(p-2) \cdot\left((n-2 a-2)^{2} p K_{a, b}^{2}\right)^{-p /(p-2)} \\
& =\left(C^{-1} K_{a, b}\right)^{2 p /(p-2)} \cdot \frac{p-2}{p+2} \int_{\mathbb{R}^{n}} \frac{d x}{|x|^{b p}\left(1+|x|^{w}\right)^{q-1}} \\
& =\left(C^{-1} K_{a, b}\right)^{n /(1+a-b)} \cdot \frac{p-2}{p+2} \int_{\mathbb{R}^{n}} \frac{d x}{|x|^{b p}\left(1+|x|^{w}\right)^{q-1}} . \tag{2.9}
\end{align*}
$$

Observe that

$$
\begin{align*}
H_{1}(\lambda) & =\left(C^{-1} K_{a, b}\right)^{n /(1+a-b)} \cdot \lambda^{-2 /(p-2)} \cdot \frac{p-2}{p+2} \int_{\mathbb{R}^{n}} \frac{d x}{|x|^{b p}\left(1+|x|^{w}\right)^{q-1}} \\
& =\left(C^{-1} K_{a, b}\right)^{n /(1+a-b)} H_{0}(\lambda) . \tag{2.10}
\end{align*}
$$

Before we can conclude the proof of Theorem 1.1, we shall need the following two lemmas.
Lemma 2.1. If for some $\lambda_{0}>0, F\left(\lambda_{0}\right)<H_{1}\left(\lambda_{0}\right)$, then $F(\lambda)<H_{1}(\lambda) \forall \lambda \in\left(0, \lambda_{0}\right]$.
Proof. Suppose that Lemma 2.1 is false. Set

$$
\lambda_{1}=\sup \left\{\lambda<\lambda_{0} ; F(\lambda)=H_{1}(\lambda)\right\} .
$$

For each $\lambda>0$, the function $\phi_{\lambda}:[0,+\infty) \rightarrow \mathbb{R}$ defined by

$$
\phi_{\lambda}(s)=l s^{2 / p}+\lambda s
$$

is increasing. By (2.6), we have

$$
\phi_{\lambda}\left(-F^{\prime}(\lambda)\right) \leqslant \frac{p+2}{p-2} F(\lambda),
$$

which gives

$$
-F^{\prime}(\lambda) \leqslant \phi_{\lambda}^{-1}\left(\frac{p+2}{p-2} F(\lambda)\right) .
$$

On the other hand, (2.7) implies that

$$
-H_{1}^{\prime}(\lambda)=\phi_{\lambda}^{-1}\left(\frac{p+2}{p-2} H_{1}(\lambda)\right) .
$$

Thus, on the subset $\left\{s \mid F(s) \leqslant H_{1}(s)\right\}$, we have

$$
F^{\prime}(\lambda)-H_{1}^{\prime}(\lambda) \geqslant \phi_{\lambda}^{-1}\left(\frac{p+2}{p-2} H_{1}(\lambda)\right)-\phi_{\lambda}^{-1}\left(\frac{p+2}{p-2} F(\lambda)\right) .
$$

Since $\left.\left(F-H_{1}\right)\right|_{\left[\lambda_{1}, \lambda_{0}\right]} \leqslant 0$, we conclude therefore that $\left(F-H_{1}\right)^{\prime} \leqslant 0$ on $\left[\lambda_{1}, \lambda_{0}\right]$. Consequently, one gets

$$
0=\left(F-H_{1}\right)\left(\lambda_{1}\right) \leqslant\left(F-H_{1}\right)\left(\lambda_{0}\right)<0 .
$$

This is a contradiction and completes the proof of Lemma 2.1.

Lemma 2.2. We have

$$
\begin{equation*}
\liminf _{\lambda \rightarrow 0} \frac{F(\lambda)}{H_{0}(\lambda)} \geqslant 1 \tag{2.11}
\end{equation*}
$$

Proof. Fix a small $\epsilon>0$. Since

$$
\lim _{u \rightarrow 0} \frac{\operatorname{vol}\left[B\left(x_{0}, u\right)\right]}{V_{0}(u)}=1
$$

there exists a $\delta>0$ such that $\operatorname{vol}\left[B\left(x_{0}, h\right)\right] \geqslant(1-\epsilon) V_{0}(h), \forall h \leqslant \delta$.
It then follows from (2.3) that

$$
\begin{aligned}
F(\lambda) & \geqslant \frac{p-2}{p+2}(1-\epsilon) \int_{0}^{\delta} V_{0}(h) \frac{\left(b p \lambda+(b p+(q-1) w) h^{w}\right)}{h^{b p+1}\left(\lambda+h^{w}\right)^{q}} d h \\
& =\frac{p-2}{p+2}(1-\epsilon) \lambda^{[(n+b p) / w]+1-q} \int_{0}^{\delta / \lambda^{1 / w}} V_{0}(s) \frac{\left(b p+(b p+(q-1) w) s^{w}\right)}{s^{b p+1}\left(1+s^{w}\right)^{q}} d s \\
& =\frac{p-2}{p+2}(1-\epsilon) \lambda^{-2 /(p-2)} \int_{0}^{\delta / \lambda^{1 / w}} V_{0}(s) \frac{\left(b p+(b p+(q-1) w) s^{w}\right)}{s^{b p+1}\left(1+s^{w}\right)^{q}} d s .
\end{aligned}
$$

On the other hand, it is easy to see that

$$
H_{0}(\lambda)=\frac{p-2}{p+2} \lambda^{-2 /(p-2)} \int_{0}^{+\infty} V_{0}(s) \frac{\left(b p+(b p+(q-1) w) s^{w}\right)}{s^{b p+1}\left(1+s^{w}\right)^{q}} d s
$$

We conclude therefore that

$$
\liminf _{\lambda \rightarrow 0} \frac{F(\lambda)}{H_{0}(\lambda)} \geqslant 1-\epsilon
$$

Letting $\epsilon \rightarrow 0$, one gets

$$
\begin{equation*}
\liminf _{\lambda \rightarrow 0} \frac{F(\lambda)}{H_{0}(\lambda)} \geqslant 1 \tag{2.12}
\end{equation*}
$$

This completes the proof of Lemma 2.2.
Now we continue on the proof of Theorem 1.1. We separate the proof into two cases.
Case 1: $C>K_{a, b}$. In this case, it follows from (2.10) and Lemma 2.2 that

$$
\begin{align*}
\liminf _{\lambda \rightarrow 0} \frac{F(\lambda)}{H_{1}(\lambda)} & =\left(\frac{C}{K_{a, b}}\right)^{n /(1+a-b)} \liminf _{\lambda \rightarrow 0} \frac{F(\lambda)}{H_{0}(\lambda)} \\
& \geqslant\left(\frac{C}{K_{a, b}}\right)^{n /(1+a-b)}>1 \tag{2.13}
\end{align*}
$$

which, combining with Lemma 2.1, implies that

$$
\begin{equation*}
F(\lambda) \geqslant H_{1}(\lambda), \quad \forall \lambda>0 . \tag{2.14}
\end{equation*}
$$

That is, for any $\lambda>0$, we have

$$
\begin{equation*}
\int_{0}^{+\infty}\left(\operatorname{vol}\left[B\left(x_{0}, s\right)\right]-\left(C^{-1} K_{a, b}\right)^{n /(1+a-b)} V_{0}(s)\right) \frac{b p \lambda+(b p+(q-1) w) s^{w}}{s^{b p+1}\left(\lambda+s^{w}\right)^{q}} d s \geqslant 0 \tag{2.15}
\end{equation*}
$$

Recall that the Bishop-Gromov comparison theorem says that the function $\left|B\left(x_{0}, s\right)\right| / V_{0}(s)$ is decreasing. Set $d=\left(C^{-1} K(n, q)\right)^{n /(1+a-b)}$ and assume that

$$
\lim _{s \rightarrow+\infty} \frac{\left|B\left(x_{0}, s\right)\right|}{V_{0}(s)}=d_{0}
$$

## M. P. do Carmo and C. Xia

The proof of Theorem 1.1 will be completed if we can show that $d_{0} \geqslant d$. We prove this fact by contradiction. Thus suppose that $d_{0}=d-\epsilon_{0}$, for some $\epsilon_{0}>0$. Then there exists an $N_{0}>0$ such that

$$
\begin{equation*}
\frac{\operatorname{vol}\left[B\left(x_{0}, s\right)\right]}{V_{0}(s)} \leqslant d-\frac{\epsilon_{0}}{2}, \quad \forall s \geqslant N_{0} . \tag{2.16}
\end{equation*}
$$

By introducing (2.16) into (2.15), one derives for every $\lambda>0$ that

$$
\begin{aligned}
& 0 \leqslant \int_{0}^{+\infty}\left(\frac{\operatorname{vol}\left[B\left(x_{0}, s\right)\right]}{V_{0}(s)}-d\right) \frac{s^{n}\left(b p \lambda+(b p+(q-1) w) s^{w}\right)}{s^{b p+1}\left(\lambda+s^{w}\right)^{q}} d s \\
& \leqslant \int_{0}^{N_{0}} \frac{\operatorname{vol}\left[B\left(x_{0}, s\right)\right]}{V_{0}(s)} \frac{s^{n}\left(b p \lambda+(b p+(q-1) w) s^{w}\right)}{s^{b p+1}\left(\lambda+s^{w}\right)^{q}} d s \\
& +\int_{N_{0}}^{+\infty}\left(d-\frac{\epsilon_{0}}{2}\right) \frac{s^{n}\left(b p \lambda+(b p+(q-1) w) s^{w}\right)}{s^{b p+1}\left(\lambda+s^{w}\right)^{q}} d s \\
& -d \int_{0}^{+\infty} \frac{s^{n}\left(b p \lambda+(b p+(q-1) w) s^{w}\right)}{s^{b p+1}\left(\lambda+s^{w}\right)^{q}} d s \\
& \leqslant \int_{0}^{N_{0}} \frac{s^{n}\left(b p \lambda+(b p+(q-1) w) s^{w}\right)}{s^{b p+1}\left(\lambda+s^{w}\right)^{q}} d s \\
& +\int_{N_{0}}^{+\infty}\left(d-\frac{\epsilon_{0}}{2}\right) \frac{s^{n}\left(b p \lambda+(b p+(q-1) w) s^{w}\right)}{s^{b p+1}\left(\lambda+s^{w}\right)^{q}} d s \\
& -d \int_{0}^{+\infty} \frac{s^{n}\left(b p \lambda+(b p+(q-1) w) s^{w}\right)}{s^{b p+1}\left(\lambda+s^{w}\right)^{q}} d s \\
& =\int_{0}^{N_{0}}\left(1-d+\frac{\epsilon_{0}}{2}\right) \frac{s^{n}\left(b p \lambda+(b p+(q-1) w) s^{w}\right)}{s^{b p+1}\left(\lambda+s^{w}\right)^{q}} d s \\
& -\frac{\epsilon_{0}}{2 \omega_{n}} \int_{0}^{+\infty} \frac{V_{0}(s)\left(b p \lambda+(b p+(q-1) w) s^{w}\right)}{s^{b p+1}\left(\lambda+s^{w}\right)^{q}} d s \\
& =\int_{0}^{N_{0}}\left(1-d+\frac{\epsilon_{0}}{2}\right) \frac{s^{n}\left(b p \lambda+(b p+(q-1) w) s^{w}\right)}{s^{b p+1}\left(\lambda+s^{w}\right)^{q}} d s \\
& -\frac{\epsilon_{0}}{2 \omega_{n}} \cdot \frac{p+2}{p-2} \cdot H_{0}(\lambda) \\
& \leqslant\left(1-d+\frac{\epsilon_{0}}{2}\right) \lambda^{-q} \int_{0}^{N_{0}}\left(b p \lambda s^{n-b p-1}+(b p+(q-1) w) s^{n+w-b p-1}\right) d s \\
& -\frac{\epsilon_{0}}{2 \omega_{n}} \cdot \frac{p+2}{p-2} \cdot \lambda^{-2 /(p-2)} \cdot H_{0}(1) \\
& =\left(1-d+\frac{\epsilon_{0}}{2}\right) \lambda^{-q}\left(\frac{\lambda b p N_{0}^{n-b p}}{n-b p}+\frac{(b p+(q-1) w) N_{0}^{n+w-b p}}{n+w-b p}\right) \\
& -\frac{\epsilon_{0}(p+2) H_{0}(1)}{2 \omega_{n}(p-2)} \cdot \lambda^{-2 /(p-2)},
\end{aligned}
$$

which implies for any $\lambda>0$ that

$$
\frac{\epsilon_{0}(p+2) H_{0}(1)}{2 \omega_{n}(p-2)\left(1-d+\epsilon_{0} / 2\right)} \leqslant \lambda^{2 /(p-2)-q}\left(\frac{\lambda b p N_{0}^{n-b p}}{n-b p}+\frac{(b p+(q-1) w) N_{0}^{n+w-b p}}{n+w-b p}\right) .
$$

Letting $\lambda \rightarrow+\infty$ in the above inequality and observing that $2 /(p-2)-q+1<0$, one obtains the desired contradiction. Thus $d_{0} \geqslant d$. This completes the proof of Theorem 1.1 in the case that $C>K_{a, b}$.

## Manifolds with non-negative Ricci curvature

Case 2: $C=K_{a, b}$. In this case, we have for any fixed $\delta>0$ that

$$
\left(\int_{M} \rho^{-b p}|u|^{p} d v\right)^{1 / p} \leqslant\left(K_{a, b}+\delta\right)\left(\int_{M} \rho^{-2 a}|\nabla u|^{2} d v\right)^{1 / 2} .
$$

Thus for any $x \in M$ we have from case 1 that

$$
\operatorname{vol}[B(x, r)] \geqslant\left(\frac{K_{a, b}}{K_{a, b}+\delta}\right)^{n /(1+a-b)} V_{0}(r), \quad \forall r>0
$$

Letting $\delta \rightarrow 0$, one obtains that

$$
\operatorname{vol}[B(x, r)] \geqslant V_{0}(r), \quad \forall r>0 .
$$

This completes the proof of Theorem 1.1 for the case that $C=K_{a, b}$.

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## Manifolds with non-negative Ricci curvature

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Manfredo Perdigão do Carmo manfredo@impa.br
Institute de Matematica Pura e Aplicada, Estrada Dona Castorina 110, Jardim Botanico, 22460-320
Rio de Janeiro RJ, Brazil

Changyu Xia xia@mat.unb.br
Departamento de Matemática-IE, Fundação Universidade de Brasília, Campus Universitário, 70910-900 Brasília DF, Brazil


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