J. DIFFERENTIAL GEOMETRY
62 (2002) 143-162

COMPLETE MANIFOLDS WITH POSITIVE SPECTRUM, II

PETER LI & JIAPING WANG

Abstract

In this paper, we continued our investigation of complete manifolds whose spectrum of the Laplacian has an optimal positive lower bound. In particular, we proved a splitting type theorem for n-dimensional manifolds that have a finite volume end. This can be viewed as a study of the equality case of a theorem of Cheng.

0. Introduction

In the authors recent work [9] they proved a splitting type theorem for manifolds whose Ricci curvature is bounded from below by a negative multiple of the lower bound of the spectrum. In particular, their theorem generalized the work of Witten-Yau [14], Cai-Galloway [5], and Wang [13] on conformally compact manifolds to arbitrary complete manifolds. One of the main results (Theorem 2.1) in [9] can be stated as follows.

Theorem 0.1. *Let M^n be a complete Riemannian manifold of dimension $n \geq 3$. Suppose the Ricci curvature of M is bounded by*

$$\mathrm{Ric}_M \geq -(n-1).$$

Let $\lambda_1(M)$ be the largest lower bound of the spectrum of the Laplacian with respect to the metric of M, and assume that

$$\lambda_1(M) \geq (n-2).$$

The first author was partially supported by NSF grant #DMS-0202508 and an Earmarked grant of Hong Kong. The second author was partially supported by NSF grant #DMS-0072181.

Received 11/21/2002.

Then either:

(1) *M has only one end with infinite volume; or*

(2) *$M = \mathbb{R} \times N$ with the warped product metric*

$$ds_M^2 = dt^2 + \cosh^2 t \, ds_N^2,$$

where N is a compact manifold with Ricci curvature bounded from below by

$$\mathrm{Ric}_N \geq -(n-2).$$

In this case, $\lambda_1(M) = n - 2$.

It was also pointed out that this gave a partial description for manifolds whose λ_1 achieves the upper bound given by a theorem of Cheng [3].

Theorem 0.2. *Let M^n be a complete Riemannian manifold of dimension n. Suppose the Ricci curvature of M is bounded from below by*

$$\mathrm{Ric}_M \geq -(n-1).$$

Then

$$\lambda_1(M) \leq \frac{(n-1)^2}{4}.$$

Since $n - 2 \leq \frac{(n-1)^2}{4}$ with equality holds only when $n = 3$, Theorem 0.1 asserts that equality in Theorem 0.2 implies that the manifold must only have one infinite volume end, unless $n = 3$. In this case, the warped product given in Theorem 0.1 is the only exception.

Obviously, an open issue is whether finite volume ends can be ruled out when a manifold satisfies the hypotheses of Theorem 0.1. In [9], the following example was given to demonstrate that finite volume ends can exist in general.

Example 0.3. Let $M^n = \mathbb{R} \times N^{n-1}$ be the complete manifold with the warped product metric

$$ds_M^2 = dt^2 + \exp(2t) \, ds_N^2.$$

If $\{\bar{e}_\alpha\}$ for $\alpha = 2, \ldots, n$ form an orthonormal basis of the tangent space of N with respect the metric on N, then $e_1 = \frac{\partial}{\partial t}$ together with $\{e_\alpha = \exp(t) \, \bar{e}_\alpha\}$ form an orthonormal basis for the tangent space of M. A direct computation shows that the sectional curvatures K_M of M are given by

$$K_M(e_1, e_\alpha) = -1 \quad \text{for all } \ 2 \leq \alpha \leq n,$$

and

$$K_M(e_\alpha, e_\beta) = \exp(-2t)\, K_N(e_\alpha, e_\beta) - 1 \quad \text{for all } \ 2 \leq \alpha, \beta \leq n,$$

where K_N are the sectional curvatures of N. In particular, if the Ricci curvature of N is nonnegative, then

$$\mathrm{Ric}_M \geq -(n-1).$$

Moreover, N is Ricci flat if and only if M is Einstein with

$$\mathrm{Ric}_M = -(n-1).$$

We now claim that $\lambda_1(M) = \frac{(n-1)^2}{4}$ for this example. In fact, let us consider the function

$$f = \exp\left(-\frac{n-1}{2}\, t\right)$$

defined on M. A direct computation shows that

$$\begin{aligned} \Delta f &= \frac{d^2 f}{dt^2} + (n-1)\frac{df}{dt} \\ &= \frac{(n-1)^2}{4}\, f - \frac{(n-1)^2}{2}\, f \\ &= -\frac{(n-1)^2}{4}\, f. \end{aligned}$$

Proposition 0.4. *Let M be a complete Riemannian manifold. If there exists a positive function f defined on M satisfying*

$$\Delta f \leq -\lambda f,$$

then

$$\lambda_1(M) \geq \lambda.$$

Proof. Let $D \subset M$ be a smooth compact subdomain of M and $\lambda_1(D)$ the first Dirichlet eigenvalue on D. Let u be the first eigenfunction satisfying

$$\Delta u = -\lambda_1(D)\, u \quad \text{on } \ D$$

and

$$u = 0 \qquad \text{on} \;\; \partial D.$$

We may assume that $u \geq 0$ on D, and regularity of u asserts that $u > 0$ in the interior of D. Integration by parts yields

$$\begin{aligned}(\lambda_1(D) - \lambda) \int_D u\, f &\geq \int_D u\, \Delta f - \int_D f\, \Delta u \\ &= \int_D u\, \frac{\partial f}{\partial \nu} - \int_D f\, \frac{\partial u}{\partial \nu} \\ &\geq 0,\end{aligned}$$

where ν is the outward unit normal of D. Hence $\lambda_1(D) \geq \lambda$ for any arbitrary compact subdomain $D \subset M$. Since

$$\inf_{D \subset M} \lambda_1(D) = \lambda_1(M),$$

the proposition follows. q.e.d.

Combining Proposition 0.4 together with the upper bound from Theorem 0.2, this implies that $\lambda_1(M) = \frac{(n-1)^2}{4}$ as claimed.

The purpose of this article is to prove that the above example is the only situation when M has a finite volume end if M achieves equality in Cheng's upper bound.

Theorem 0.5. *Let M^n be a complete n-dimensional manifold with $n \geq 4$. Suppose that*

$$\mathrm{Ric}_M \geq -(n-1)$$

and

$$\lambda_1(M) \geq \frac{(n-1)^2}{4}.$$

Then either:

(1) *M has only one end; or*

(2) *$M = \mathbb{R} \times N$ with the warped product metric*

$$ds_M^2 = dt^2 + \exp(2t)\, ds_N^2,$$

where N is a compact manifold with nonnegative Ricci curvature.

In the case when $n = 3$, the two numbers $n-2$ and $\frac{(n-1)^2}{4}$ coincide and we can incorporate Theorem 0.1 into our new result.

Theorem 0.6. *Let M^3 be a complete 3-dimensional manifold. Suppose that*

$$\mathrm{Ric}_M \geq -2$$

and

$$\lambda_1(M) \geq 1.$$

Then either:

(1) *M has only one end; or*

(2) *$M = \mathbb{R} \times N$ with the warped product metric*

$$ds_M^2 = dt^2 + \cosh^2 t\, ds_N^2,$$

where N^2 is a compact manifold with its Gaussian curvature bounded from below by

$$K_N \geq -1; \qquad \textit{or}$$

(3) *$M = \mathbb{R} \times N$ with the warped product metric*

$$ds_M^2 = dt^2 + \exp(2t)\, ds_N^2,$$

where N^2 is a compact manifold with nonnegative Gaussian curvature.

Due to the fact that Theorem 0.1 is not valid in dimension 2, we do not have a splitting theorem for two infinite volume ends. However, our argument for the finite volume end is still valid.

Theorem 0.7. *Let M^2 be a complete 2-dimensional manifold. Suppose that*

$$K_M \geq -1$$

and

$$\lambda_1(M) \geq \frac{1}{4}.$$

Then either:

(1) *M has no finite volume end; or*

(2) *$M = \mathbb{R} \times \mathbb{S}^1$ with the warped product metric*

$$ds_M^2 = dt^2 + \exp(2t)\, d\theta^2.$$

We would like to point out the special case when M^n is a complete hyperbolic manifold given by $\mathbb{H}^n/\Gamma$, where Γ is a torsion-free discrete subgroup of isometries acting on the hyperbolic space form $\mathbb{H}^n$. In this case, a result of Sullivan [11] asserts that if $\delta(\Gamma)$ is the Hausdorff dimension of the limit set of the group Γ, then

$$\lambda_1(M) = \begin{cases} \frac{(n-1)^2}{4}, & \text{if } \delta(\Gamma) \leq \frac{n-1}{2} \\ \delta(\Gamma)(n-1-\delta(\Gamma)), & \text{if } \delta(\Gamma) \geq \frac{n-1}{2}, \end{cases}$$

provided that Γ is geometrically finite, that is, the fundamental domain with respect to Γ has finitely many sides. In the case $n = 2$, however, the geometric finiteness assumption on Γ is not needed and the preceding formula holds true for any complete hyperbolic surface $\mathbb{H}^2/\Gamma$. In dimension 3, Canary [2] showed that an infinite volume, hyperbolic manifold that is topologically tame is geometrically infinite if and only if $\lambda_1(M) = 0$. In any event, if we restrict the Main Theorem to hyperbolic 3-manifolds, then the condition that $\lambda_1(M) = 1$ is equivalent to the condition that $\delta(\Gamma) \leq 1$. Moreover, if Γ is finitely generated and $\delta(\Gamma) = 1$, then Bishop and Taylor [1] proved that the limit set of Γ must be a circle. In this case, Γ is a surface group and M is given by the warped product as in case (2) of Theorem 0.6 when the surface N has constant -1 curvature. When $\delta(\Gamma) < 1$, then the limit set of Γ must be totally disconnected, which corresponds to case (1), and case (3) when N is a flat torus. In this sense, Theorem 0.5 and Theorem 0.6 can be viewed as a generalization of part of these results to manifolds with Ricci curvature bounded from below by $-(n-1)$ and to higher dimensions.

The authors would like to thank Al Marden for his interest in this work and for pointing out the references for hyperbolic manifolds.

1. Preliminaries

In this section, we will give precise estimates on the volume growth or volume decay of an end of a manifold satisfying the hypotheses of Theorems 0.5-0.7. Recall that an end E is defined to be an unbounded component of $M \setminus D$ for some compact domain D. Without loss of generality, we may assume that $D = B_p(R_0)$ is a geodesic ball centered at some fixed point $p \in M$ with radius $R_0 > 0$. We will denote $V_E(R)$ to be the volume of the set $B_p(R) \cap E$, and $V_E(\infty)$ is simply the volume of E. Also we recall (see [6] and [8]) that an end E is said to be a non-parabolic (or parabolic) end if it admits (or does not admit) a positive Green's function for the Laplacian on E with Neumann boundary condition on ∂E.

Let us first recall Theorem 1.4 of [9] stated for the above class of manifolds.

Theorem 1.1. *Let E be an end of a complete, n-dimensional manifold M satisfying*

$$\lambda_1(M) \geq \frac{(n-1)^2}{4}.$$

Then either:

(1) *E is a parabolic end with finite volume, and it must have exponential volume decay given by*

$$V_E(\infty) - V_E(R) \leq C_1 \exp(-(n-1)R)$$

for $R \geq R_0 + 1$ and some constant $C_1 > 0$ depending on E; or

(2) *E is a non-parabolic end with infinite volume, and it must have exponential volume growth given by*

$$V_E(R) \geq C_2 \exp((n-1)R)$$

for $R \geq R_0 + 1$ and some constant $C_2 > 0$ depending on the end E.

On the other hand, the Bishop volume comparison theorem asserts that if $\mathrm{Ric}_M \geq -(n-1)$ then for any $x \in M$, the ratio of the volumes of geodesic balls $B_x(R_1)$ and $B_x(R_2)$ for $R_1 < R_2$ must satisfy

$$\frac{V_x(R_2)}{V_x(R_1)} \leq \frac{V_{\mathbb{H}^n}(R_2)}{V_{\mathbb{H}^n}(R_1)}, \tag{1.3}$$

where $V_{\mathbb{H}^n}(R)$ is the volume of a geodesic ball of radius R in the n-dimensional hyperbolic space form $\mathbb{H}^n$ of constant -1 curvature. In particular, by taking $x = p$, $R_1 = 0$ and $R_2 = R$ this implies that

$$V_p(R) \leq C_3 \exp((n-1)R) \tag{1.4}$$

for sufficiently large R. On the other hand, if we let $x \in \partial B_p(R)$, $R_1 = 1$ and $R_2 = R + 1$, (1.3) implies

$$\begin{aligned} V_x(1) &\geq C_4\, V_x(R+1) \exp(-(n-1)R) \\ &\geq C_4\, V_p(1) \exp(-(n-1)R). \end{aligned} \tag{1.5}$$

Combining (1.4), (1.5) with Theorem 1.1, we obtain the following corollary.

Corollary 1.2. *Let E be an end of a complete, n-dimensional manifold M satisfying*

$$\mathrm{Ric}_M \geq -(n-1)$$

and

$$\lambda_1(M) \geq \frac{(n-1)^2}{4}.$$

Let $p \in M$ be a fixed point. Then either:

(1) *E is a parabolic end with finite volume, and it must have exponential volume decay given by*

$$C_4 \exp(-(n-1)R) \leq V_E(\infty) - V_E(R) \leq C_1 \exp(-(n-1)R)$$

for $R \geq R_0 + 1$ and some constants $C_1 \geq C_4 > 0$ depending on E; or

(2) *E is a non-parabolic end with infinite volume, and it must have exponential volume growth given by*

$$C_3 \exp((n-1)R) \geq V_E(R) \geq C_2 \exp((n-1)R)$$

for $R \geq R_0 + 1$ and some constants $C_3 \geq C_2 > 0$ depending on the end E.

Note that Theorem 1.1 can be viewed as a refined version of Cheng's Theorem. In fact, if the Ricci curvature of the manifold satisfies

$$\mathrm{Ric}_M \geq -(n-1),$$

then the Bishop volume comparison theorem asserts that

$$\begin{aligned} V_p(R)) &\leq V_{\mathbb{H}^n}(R) \\ &\leq C \exp((n-1)R). \end{aligned}$$

Hence combining with Theorem 1.1, we conclude that $\lambda_1(M) \leq \frac{(n-1)^2}{4}$ as asserted by Cheng.

2. The proofs

Let us first recall that a theorem of Yau [15] asserts that on a manifold with Ricci curvature bounded from below, if f is a positive harmonic function then $|\nabla(\log f)|$ is bounded from above. It turns out that if we are more careful in the proof of the estimate, the upper bound can be made to be sharp for our situation. For this sake, we will provide the argument.

Lemma 2.1. *Let M^n be a complete manifold with Ricci curvature bounded from below by*

$$\mathrm{Ric}_M \geq -(n-1).$$

If f is a positive harmonic function defined on the geodesic ball $B_p(2R) \subset M$, then there exists a constant C depending on n such that

$$|\nabla(\log f)|(x) \leq (n-1) + C\,R^{-1}$$

for all $x \in B_p(R)$. In particular, for a positive harmonic function u on M,

$$|\nabla(\log u)| \leq n-1$$

on M.

Proof. If we set $h = \log f$, then a direct computation shows that h satisfies the equation

$$\Delta h = -|\nabla h|^2.$$

Using the Bochner formula, we compute

$$\begin{aligned}\Delta|\nabla h|^2 &= 2h_{ij}^2 + 2\mathrm{Ric}_M(\nabla h, \nabla h) + 2\langle \nabla h, \nabla \Delta h\rangle \\ &\geq 2h_{ij}^2 - 2(n-1)|\nabla h|^2 - 2\langle \nabla h, \nabla|\nabla h|^2\rangle.\end{aligned} \tag{2.1}$$

Choosing an orthonormal frame $\{e_1, \ldots, e_n\}$ at a point so that $|\nabla h|\, e_1 = \nabla h$, we can write

$$\begin{aligned}|\nabla|\nabla h|^2|^2 &= 4\sum_{i=1}^n \left(\sum_{j=1}^n h_j\, h_{ji}\right)^2 \\ &= 4h_1^2 \sum_{j=1}^n h_{1j}^2 \\ &= 4|\nabla h|^2 \sum_{j=1}^n h_{1j}^2.\end{aligned} \tag{2.2}$$

On the other hand,

$$
\begin{aligned}
(2.3) \qquad h_{ij}^2 &\geq h_{11}^2 + 2\sum_{\alpha=2}^n h_{1\alpha}^2 + \sum_{\alpha=2}^n h_{\alpha\alpha}^2 \\
&\geq h_{11}^2 + 2\sum_{\alpha=2}^n h_{1\alpha}^2 + \frac{(\sum_{\alpha=2}^n h_{\alpha\alpha})^2}{n-1} \\
&\geq h_{11}^2 + 2\sum_{\alpha=2}^n h_{1\alpha}^2 + \frac{(\Delta h - h_{11})^2}{n-1} \\
&\geq h_{11}^2 + 2\sum_{\alpha=2}^n h_{1\alpha}^2 + \frac{(|\nabla h|^2 + h_{11})^2}{n-1} \\
&\geq \frac{n}{n-1}\sum_{j=1}^n h_{1j}^2 + \frac{1}{n-1}|\nabla h|^4 + \frac{2}{n-1}|\nabla h|^2\, h_{11}.
\end{aligned}
$$

However, using the identity

$$
\begin{aligned}
2h_1\, h_{11} &= e_1(|\nabla h|^2) \\
&= \langle \nabla |\nabla h|^2, \nabla h\rangle\, |\nabla h|^{-1}
\end{aligned}
$$

we conclude that

$$2|\nabla h|^2\, h_{11} = \langle \nabla |\nabla h|^2, \nabla h\rangle.$$

Substituting into (2.3), we obtain

$$h_{ij}^2 \geq \frac{n}{n-1}\sum_{j=1}^n h_{1j}^2 + \frac{1}{n-1}|\nabla h|^4 + \frac{1}{n-1}\langle \nabla|\nabla h|^2, \nabla h\rangle.$$

Combining with (2.1) and (2.2) yields

$$
\begin{aligned}
(2.4) \qquad \Delta|\nabla h|^2 \geq{}& \frac{n}{2(n-1)}\, |\nabla|\nabla h|^2|^2\, |\nabla h|^{-2} - 2(n-1)|\nabla h|^2 \\
&- \frac{2(n-2)}{n-1}\, \langle \nabla h, \nabla|\nabla h|^2\rangle + \frac{2}{n-1}\, |\nabla h|^4.
\end{aligned}
$$

Let ϕ be a nonnegative cut-off function. If we set $G = \phi\, |\nabla h|^2$, then using (2.4) we have

$$
\begin{aligned}
\Delta G &= (\Delta\phi)\,|\nabla h|^2 + 2\langle\nabla\phi, \nabla|\nabla h|^2\rangle + \phi\,\Delta(|\nabla h|^2) \\
&\geq \frac{\Delta\phi}{\phi}\,G + 2\phi^{-1}\,\langle\nabla\phi,\nabla G\rangle - 2|\nabla\phi|^2\,\phi^{-2}\,G \\
&\quad + \frac{n}{2(n-1)}\,\phi\,|\nabla|\nabla h|^2|^2\,|\nabla h|^{-2} \\
&\quad - 2(n-1)G - \frac{2(n-2)}{n-1}\,\langle\nabla h,\nabla G\rangle \\
&\quad + \frac{2(n-2)}{n-1}\,|\nabla h|^2\,\langle\nabla h,\nabla\phi\rangle + \frac{2}{n-1}\,\phi\,|\nabla h|^4.
\end{aligned}
\tag{2.5}
$$

On the other hand, writing

$$
\begin{aligned}
|\nabla G|^2 &= |\nabla(\phi\,|\nabla h|^2)|^2 \\
&= |\nabla\phi|^2\,|\nabla h|^4 + 2\phi\,|\nabla h|^2\,\langle\nabla\phi,\nabla|\nabla h|^2\rangle + \phi^2\,|\nabla|\nabla h|^2|^2 \\
&= |\nabla\phi|^2\,\phi^{-2}\,G^2 + 2G\,\phi^{-1}\,\langle\nabla\phi,\nabla G\rangle \\
&\quad - 2G^2\,|\nabla\phi|^2\,\phi^{-2} + \phi^2\,|\nabla|\nabla h|^2|^2,
\end{aligned}
$$

we have

$$
\begin{aligned}
&\phi\,|\nabla|\nabla h|^2|^2\,|\nabla h|^{-2} \\
&= |\nabla G|^2\,G^{-1} - |\nabla\phi|^2\,\phi^{-2}\,G - 2\phi^{-1}\,\langle\nabla\phi,\nabla G\rangle + 2G\,|\nabla\phi|^2\,\phi^{-2}.
\end{aligned}
$$

Also, for any $\delta > 0$, the Schwarz inequality implies that

$$
2|\nabla h|^2\,\langle\nabla h,\nabla\phi\rangle \geq -\delta\phi\,|\nabla h|^4 - \frac{1}{\delta}|\nabla\phi|^2\,\phi^{-2}\,G.
$$

Hence (2.5) becomes

$$
\begin{aligned}
\Delta G \geq{}& \frac{\Delta\phi}{\phi}\,G + 2\phi^{-1}\,\langle\nabla\phi,\nabla G\rangle - \left(\frac{3n-4}{2(n-1)} + \frac{n-2}{\delta(n-1)}\right)|\nabla\phi|^2\,\phi^{-2}\,G \\
&+ \frac{n}{2(n-1)}\,|\nabla G|^2\,G^{-1} - \frac{n}{(n-1)}\,\phi^{-1}\,\langle\nabla\phi,\nabla G\rangle - 2(n-1)G \\
&- \frac{2(n-2)}{n-1}\langle\nabla h,\nabla G\rangle + \left(\frac{2}{n-1} - \frac{\delta(n-2)}{n-1}\right)\,\phi^{-1}\,G^2
\end{aligned}
$$

by taking $\delta(n-1) < 2$. At the maximum point x_0 of G, the maximum principle asserts that

$$
\Delta G(x_0) \leq 0
$$

and

$$\nabla G(x_0) = 0.$$

Hence

$$(2.6) \qquad 0 \geq (\Delta\phi)\, G + \left(\frac{n}{2(n-1)} - \frac{2\delta+1}{\delta}\right) |\nabla\phi|^2\, \phi^{-1}\, G \\ - 2(n-1)\phi\, G + \left(\frac{2}{n-1} - \delta\right) G^2.$$

Let us choose $\phi(x) = \phi(r(x))$ to be a function of the distance r to the fixed point p with the property that

$$\begin{aligned} \phi = 1 \quad &\text{on} \;\; B_p(R), \\ \phi = 0 \quad &\text{on} \;\; M \setminus B_p(2R), \\ -CR^{-1}\,\phi^{\frac{1}{2}} \leq \phi' \leq 0 \quad &\text{on} \;\; B_p(2R) \setminus B_p(R), \end{aligned}$$

and

$$|\phi''| \leq C\, R^{-2} \quad \text{on} \;\; B_p(2R) \setminus B_p(R).$$

Then the Laplacian comparison theorem asserts that

$$\begin{aligned} \Delta\phi &= \phi'\,\Delta r + \phi'' \\ &\geq -C_1(R^{-1} + R^{-2}), \end{aligned}$$

and also

$$|\nabla\phi|^2\,\phi^{-1} \leq C_2\, R^{-2}.$$

Hence (2.6) yields

$$C_1\, R^{-1} + C_3 \left(\frac{2\delta+1}{\delta}\right) R^{-2} + 2(n-1) \geq \left(\frac{2}{n-1} - \delta\right) G.$$

Since this estimate is valid at x_0 the maximum point of G, it is therefore valid for all points $x \in B_p(2R)$. In particular, when restricted on $B_p(R)$ and by taking $\delta = (R+n-1)^{-1}$, we have

$$C_1\, R^{-1} + C_3(2R^{-2} + R^{-1}) + 2(n-1) \geq \left(\frac{2}{n-1} - (R+n-1)^{-1}\right) |\nabla h|^2.$$

Hence we conclude that

$$C_4\, R^{-1} + (n-1)^2 \geq |\nabla h|^2$$

and

$$|\nabla h| \leq (n-1) + C_5\, R^{-1}.$$

q.e.d.

We are now ready to prove Theorem 0.5.

Proof of Theorem 0.5. Suppose that M satisfies the hypothesis of Theorem 0.5. Then by Theorem 0.1 we know that M must have only one infinite volume end because the warped product with the metric given by

$$ds_M^2 = dt^2 + \cosh^2 t\, ds_N^2$$

has $\lambda_1(M) = (n-2)$. Let us now assume that M has a finite volume end. Since $\lambda_1(M) > 0$, M must also have an infinite volume end. By choosing the compact set D appropriately, we may assume that $M \setminus D$ has one infinite volume, non-parabolic end E_1 and one finite volume, parabolic end E_2.

Recall that the theory of Li-Tam [6] (also see [8]) asserts that there exists a positive harmonic function f satisfying the following properties:

(1) $\inf_{\partial E_1(R)} f \to 0 \quad \text{as} \quad R \to \infty$;

(2) $\sup_{\partial E_2(R)} f \to \infty \quad \text{as} \quad R \to \infty$; and

(3) f is bounded and has finite Dirichlet integral on E_1.

By integrating the gradient estimate of Lemma 2.1 along geodesics, we conclude that f must satisfy the growth estimate

$$f(x) \leq C\, \exp((n-1)r(x)),$$

where $r(x)$ is the geodesic distance from x to a fixed point $p \in M$. In particular, when restricted on E_2, together with the volume estimate of Theorem 1.1, we conclude that

$$\int_{E_2(R)} f \leq C\, R. \tag{2.7}$$

On the other hand, it was proved in Lemma 1.1 in [9] that on E_1, the function f must satisfy the decay estimate

$$\int_{E_1(R+1)\setminus E_1(R)} f^2 \leq C\, \exp(-(n-1)R)$$

for R sufficiently large. In particular, the Schwarz inequality implies that

$$\int_{E_1(R+1)\setminus E_1(R)} f \leq C \exp\left(-\frac{(n-1)}{2}R\right) V_{E_1}^{\frac{1}{2}}(R+1).$$

Combining this with the volume estimate of Corollary 1.2, we conclude that

$$\int_{E_1(R+1)\setminus E_1(R)} f \leq C$$

for some constant C independent of R. In particular,

$$\int_{E_1(R)} f \leq C\,R.$$

Together with (2.7) we conclude that

$$\int_{B_p(R)} f \leq C\,R. \tag{2.8}$$

Let us now consider the function $g = f^{\frac{1}{2}}$. Direct computation and Lemma 2.1 imply that

$$\begin{aligned} \Delta g &= -\frac{1}{4} f^{-\frac{3}{2}}\,|\nabla f|^2 \\ &\geq -\frac{(n-1)^2}{4} g. \end{aligned} \tag{2.9}$$

If ϕ is a nonnegative compactly supported function, then

$$\begin{aligned} \int_M |\nabla(\phi\, g)|^2 &= \int_M |\nabla \phi|^2 g^2 - \int_M \phi^2\, g\, \Delta g \\ &= \int_M |\nabla \phi|^2\, g^2 + \frac{(n-1)^2}{4} \int_M \phi^2\, g^2 \\ &\quad - \int_M \phi^2\, g\left(\Delta g + \frac{(n-1)^2}{4} g\right). \end{aligned}$$

On the other hand, using the assumption that $\lambda_1(M) \geq \frac{(n-1)^2}{4}$, we conclude that

$$\int_M \phi^2\, g\left(\Delta g + \frac{(n-1)^2}{4} g\right) \leq \int_M |\nabla \phi|^2\, g^2.$$

Choosing ϕ to be

$$\phi = \begin{cases} 1 & \text{on } \ B_p(R) \\ \frac{2R-r}{R} & \text{on } \ B_p(2R) \setminus B_p(R) \\ 0 & \text{on } \ M \setminus B_p(2R), \end{cases}$$

we conclude that

$$\int_M |\nabla \phi|^2 \, g^2 = R^{-2} \int_{B_p(2R)\setminus B_p(R)} g^2.$$

Inequality (2.8) implies that

$$\int_M |\nabla \phi|^2 \, g^2 \to 0$$

as $R \to \infty$. Hence inequality (2.9) used in the above argument is an equality. In particular,

$$|\nabla f| = (n-1) f$$

and

$$|\nabla h|^2 = (n-1)^2. \tag{2.10}$$

Also we must have equality for (2.4) since $\Delta |\nabla h|^2$ and $\nabla |\nabla h|^2$ both vanish. Hence the inequalities used to prove (2.4) must all be equalities. More specifically,

$$h_{1j} = 0 \qquad \text{for all } \ 1 \le j \le n,$$

and

$$\begin{aligned} h_{\alpha\beta} &= -\frac{\delta_{\alpha\beta}}{n-1} |\nabla h|^2 \\ &= -(n-1)\delta_{\alpha\beta} \qquad \text{for all } \ 2 \le \alpha, \beta \le n. \end{aligned}$$

Since e_1 is the unit normal to the level set of h, the second fundamental form II of the level set is given by

$$\begin{aligned} h_{\alpha\beta} &= \mathrm{II}_{\alpha\beta} \, h_1 \\ &= (n-1)\mathrm{II}_{\alpha\beta}. \end{aligned}$$

Hence, $\mathrm{II}_{\alpha\beta}$ must be the matrix $(-\delta_{\alpha\beta})$.

Moreover, (2.10) also implies that if we set $t = \frac{h}{n-1}$, then t must be the distance function between the level sets of f, hence also for h. The fact that $\mathrm{II}_{\alpha\beta} = (-\delta_{\alpha\beta})$ implies that the metric on M can be written as

$$ds_M^2 = dt^2 + \exp(-2t)\, ds_N^2.$$

Since M has two ends, N must be compact. A direct computation shows that the condition $\mathrm{Ric}_M \geq -(n-1)$ implies that $\mathrm{Ric}_N \geq 0$. This proves the theorem. q.e.d.

Theorem 0.6 and Theorem 0.7 follow from the same argument as above by incorporating Theorem 0.1.

3. Dimension two

Due to the fact that Theorem 0.1 is invalid in dimension 2, we do not have a rigidity theorem for the case when M^2 has two infinite volume ends. In particular, it is a natural question to ask if the warped product given by

$$M = \mathbb{R} \times \mathbb{S}^1$$

with metric

$$ds_M^2 = dt^2 + \cosh^2 t\ d\theta^2$$

is rigid. In fact, a direct computation shows that M has constant -1 curvature. Moreover, we will show that $\lambda_1(M) = \frac{1}{4}$, which achieves the sharp estimate given by Cheng's theorem. In this section, we will show that there are deformations of M so that the metrics have curvature bounded from below by -1 and $\lambda_1 = \frac{1}{4}$. In fact, one can even choose the deformations to be among warped product metrics. This will indicate that there are no rigidity theorems in dimension 2 for two infinite volume ends.

It is convenient for us to consider the general warped product metric on $M = \mathbb{R} \times \mathbb{S}^1$ given by

$$ds_M^2 = dt^2 + f^2(t)\, d\theta^2. \tag{3.1}$$

A direct computation shows that the Gaussian curvature of M is given by

$$K = \frac{-f''}{f}.$$

Hence the condition that $K \geq -1$ is equivalent to the inequality

$$f'' \leq f. \tag{3.2}$$

The Laplacian for the metric (3.1) when restricted to a function of t alone is given by

$$\Delta u = \frac{d^2u}{dt^2} + \frac{f'}{f}\,\frac{du}{dt}.$$

In order to prove that the metric (3.1) has $\lambda_1 \geq \frac{1}{4}$, Proposition 0.4 implies that it suffices to show that there exists a positive function u such that

$$\Delta u \leq -\frac{1}{4}u.$$

On the other hand, if f satisfies (3.2), then by Cheng's theorem $\lambda_1 \leq \frac{1}{4}$, hence $\lambda_1 = \frac{1}{4}$.

If we set $u(t) = f^{-\frac{1}{2}}(t)$, then

$$\Delta u + \frac{1}{4}u = -\frac{1}{2}f^{-\frac{3}{2}}\,f'' + \frac{1}{4}f^{-\frac{5}{2}}\,(f')^2 + \frac{1}{4}f^{-\frac{1}{2}}.$$

The condition

$$\Delta u \leq -\frac{1}{4}u$$

is then equivalent to

$$2ff'' - (f')^2 - f^2 \geq 0. \tag{3.3}$$

Obviously the function $f = \cosh t$ satisfies both inequalities (3.2) and (3.3). We now claim that there are perturbations of $\cosh t$ that satisfy both inequalities also. This will confirm our claim stated in the beginning of this section.

To see this, let us set

$$f = \cosh t + g.$$

Inequalities (3.2) and (3.3) become

$$g'' \leq g \tag{3.4}$$

and

$$1 + 2g''\,\cosh t - 2g'\,\sinh t + 2g\,g'' - (g')^2 - g^2 \geq 0. \tag{3.5}$$

Let us choose g to be

$$g = \alpha \cosh^{-\beta} t.$$

Direct differentiation gives

$$g' = -\alpha\beta \cosh^{-\beta-1} t \, \sinh t$$

and

$$g'' = \alpha\beta(\beta+1)\cosh^{-\beta-2} t \, \sinh^2 t - \alpha\beta \cosh^{-\beta} t.$$

If we take $0 \le \beta \le 1$, then

$$\begin{aligned} g'' - g &= \alpha\beta(\beta+1)\cosh^{-\beta-2} t \, \sinh^2 t - \alpha(\beta+1)\cosh^{-\beta} t \\ &= -\alpha(\beta+1)\cosh^{-\beta-2} t (1 + (1-\beta)\sinh^2 t) \\ &\le 0, \end{aligned}$$

and (3.4) is satisfied. Moreover,

$$\begin{aligned} &1 + 2g'' \cosh t - 2g' \sinh t + 2g\, g'' - (g')^2 - g^2 \\ &= 1 + 2\alpha\beta(\beta+1)\cosh^{-\beta-1} t \, \sinh^2 t \\ &\quad - 2\alpha\beta \cosh^{-\beta+1} t + 2\alpha\beta \cosh^{-\beta-1} t \, \sinh^2 t \\ &\quad + 2\alpha^2\beta(\beta+1)\cosh^{-2\beta-2} t \, \sinh^2 t \\ &\quad - 2\alpha^2\beta \cosh^{-2\beta} t - \alpha^2\beta^2 \cosh^{-2\beta-2} t \, \sinh^2 t \\ &\quad - \alpha^2 \cosh^{-2\beta} t. \\ &= 1 + 2\alpha\beta(\beta+2)\cosh^{-\beta-1} t \, \sinh^2 t - 2\alpha\beta \cosh^{-\beta+1} t \\ &\quad + \alpha^2\beta(\beta+2)\cosh^{-2\beta-2} t \, \sinh^2 t - \alpha^2(2\beta+1)\cosh^{-2\beta} t \\ &= 1 + 2\alpha\beta(\beta+1)\cosh^{-\beta-1} t \, \sinh^2 t - 2\alpha\beta \cosh^{-\beta-1} t \\ &\quad + \alpha^2\beta(\beta+1)\cosh^{-2\beta-2} t \, \sinh^2 t - \alpha^2(2\beta+1)\cosh^{-2\beta-2} t \\ &\ge 1 - 2\alpha\beta - \alpha^2(2\beta+1). \end{aligned}$$

Clearly, for any $0 \le \beta \le 1$, we can choose $0 < \alpha \le (2\beta+1)^{-1}$ so that the right-hand side is nonnegative. This verifies (3.5). Hence metrics of the form

$$ds_M^2 = dt^2 + (\cosh t + \alpha \cosh^{-\beta} t)^2 \, d\theta^2,$$

with $0 \le \beta \le 1$ and $0 < \alpha \le (2\beta+1)^{-1}$, also has $K_M \ge -1$ and $\lambda_1(M) = \frac{1}{4}$.

References

[1] C. Bishop & P. Jones, *Hausdorff dimension and Kleinian groups*, Acta Math. **179** (1997) 1–39, MR 98k:22043, Zbl 0921.30032.

[2] R. Canary, *On the Laplacian and the geometry of hyperbolic* 3*-manifolds*, J. Differential Geom. **36** (1992) 349–367, MR 93g:57016, Zbl 0763.53040.

[3] S.Y. Cheng, *Eigenvalue comparison theorems and its geometric application*, Math. Z. **143** (1975) 289–297, MR 51 #14170, Zbl 0329.53035.

[4] S.Y. Cheng & S.-T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** (1975) 333–354, MR 52 #6608, Zbl 0312.53031.

[5] M. Cai & G.J. Galloway, *Boundaries of zero scalar curvature in the ADS/CFT correspondence*, Adv. Theor. Math. Phys. **3** (1999) 1769–1783, MR 2002k:53080, Zbl 0978.53084.

[6] P. Li, *Curvature and function theory on Riemannian manifolds*, in 'Survey in Differential Geometry' (In Honor of Atiyah, Bott, Hirzebruch, and Singer), International Press, Cambridge, **VII** (2000) 71–111, MR 1 919 432.

[7] P. Li & L.F. Tam, *The heat equation and harmonic maps of complete manifolds*, Invent. Math. **105** (1991) 1–46, MR 93e:58039, Zbl 0748.58006.

[8] P. Li & L.F. Tam, *Harmonic functions and the structure of complete manifolds*, J. Differential Geom. **35** (1992) 359–383, MR 93b:53033, Zbl 0768.53018.

[9] P. Li & J. Wang, *Complete manifolds with positive spectrum*, J. Differential Geom. **58** (2001) 501–534, MR 2003e:58046.

[10] P. Li & S.-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. **156** (1986) 153–201, MR 87f:58156, Zbl 0611.58045.

[11] D. Sullivan, *Related aspects of positivity in Riemannian geometry*, J. Differential Geom. **25** (1987) 327–351, MR 88d:58132, Zbl 0615.53029.

[12] C.J. Sung, L.F. Tam & J. Wang, *Spaces of harmonic functions*, J. London Math. Soc. **61** (2000) 789–806, MR 2001i:31013, Zbl 0963.31004.

[13] X. Wang, *On conformally compact Einstein manifolds*, Math. Res. Let. **8** (2001) 671–688, MR 2003d:53075.

[14] E. Witten & S.-T. Yau, *Connectedness of the boundary in the AdS/CFT correspondence*, Adv. Theor. Math. Phys. **3** (1999) 1635–1655, MR 2002b:53071, Zbl 0978.53085.

[15] S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975) 201–228, MR 55 #4042, Zbl 0291.31002.

University of California
Irvine, CA 92697-3875

University of Minnesota
Minneapolis, MN 55455