Complete Monotonicity of a Difference Between the Exponential and Trigamma Functions and Properties Related to a Modified Bessel Function

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Abstract. In the paper, the authors find necessary and sufficient conditions for a difference between the exponential function $\alpha e^{\beta/t}$, $\alpha, \beta > 0$, and the trigamma function $\psi'(t)$ to be completely monotonic on $(0, \infty)$. While proving the complete monotonicity, the authors discover some properties related to the first order modified Bessel function of the first kind I_1, including inequalities, monotonicity, unimodality, and convexity.

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1. Introduction

Recall from [8, Chapter XIII], [14, Chapter 1] and [15, Chapter IV] that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \tag{1.1}$$

for $x \in I$ and $n \in \{0\} \cup \mathbb{N}$.

The exponential function $e^{1/z}$ for $z \in \mathbb{C}$ with $z \neq 0$ can be expanded into the Laurent series

$$e^{1/z} = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{z^k}, \quad z \neq 0. \tag{1.2}$$

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In [16, Theorem 2.1], derivatives of the function $e^{1/t}$ were inductively and explicitly derived to be

$$\left(e^{1/t}\right)^{(i)} = (-1)^i e^{1/t} \frac{1}{t^{2i}} \sum_{k=0}^{i-1} a_{i,k} t^k \tag{1.3}$$

for $i \in \mathbb{N}$ and $t \neq 0$, where

$$a_{i,k} = \binom{i}{k}\binom{i-1}{k} k! \tag{1.4}$$

for $0 \le k \le i-1$. The series (1.2) and (1.3) both show that the function $e^{1/t}$ is completely monotonic on $(0,\infty)$.

The classical Euler gamma function $\Gamma(z)$ and the digamma function $\psi(z)$ may be defined respectively by

$$\Gamma(z) = \int_0^\infty u^{z-1} e^{-u} \, \mathrm{d} u \tag{1.5}$$

and

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} \tag{1.6}$$

for $\Re z > 0$. It is well known that for $n \in \mathbb{N}$

$$\psi^{(n)}(z) = (-1)^{n+1} \int_0^\infty \frac{u^n}{1-e^{-u}} e^{-zu} \, \mathrm{d} u \tag{1.7}$$

for $\Re z > 0$, see [1, p. 260, 6.4.1], and that

$$\psi^{(n)}(z) = (-1)^{n+1} n! \sum_{k=0}^\infty \frac{1}{(z+k)^{n+1}} \tag{1.8}$$

for $z \in \mathbb{C} \setminus \{0,-1,-2,\dots\}$, see [1, p. 260, 6.4.10]. These formulas show that the trigamma function $\psi'(t)$ is completely monotonic on $(0,\infty)$.

In [11, Lemma 2], the function

$$h(t) = e^{1/t} - \psi'(t), \tag{1.9}$$

a difference between two completely monotonic functions, was stated to be completely monotonic on $(0,\infty)$. But since the complete monotonicity was not used in [4] and there was a mistake in the proof in [11], the statement on complete monotonicity of $h(t)$ was removed from the formally published version [4]. However, the validity of the inequality

$$\psi'(t) < e^{1/t} - 1 \tag{1.10}$$

on $(0,\infty)$ was obtained and applied in [4, Lemma 2].

Recently, the complete monotonicity of $h(t)$ was established in [13, Theorem 1.1].

The first aim of this paper is to provide a simpler proof for [13, Theorem 1.1] and to find a necessary and sufficient condition for the function

$$h_\alpha(t) = \alpha e^{1/t} - \psi'(t) \tag{1.11}$$

to be completely monotonic on $(0,\infty)$. The first main result is formulated as Theorem 3.1 below.

It is natural to consider the more general function

$$h_{\alpha,\beta}(t)=\alpha e^{\beta/t}-\psi'(t) \tag{1.12}$$

for $\alpha,\beta>0$ on $(0,\infty)$. The second aim of this paper is to discover necessary and sufficient conditions for the function $h_{\alpha,\beta}(t)$ to be completely monotonic on $(0,\infty)$. The second main result is formulated as Theorems 4.1 and 6.1 below.

While discovering necessary and sufficient conditions for the functions $h(t)$, $h_\alpha(t)$, and $h_{\alpha,\beta}(t)$ to be completely monotonic on $(0,\infty)$, some properties such as monotonicity, unimodality, convexity, and inequalities related to the first order modified Bessel function of the first kind $I_1(t)$ are established, where

$$I_\nu(z)=\sum_{k=0}^\infty\frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k+\nu} \tag{1.13}$$

for $\nu\in\mathbb{R}$ and $z\in\mathbb{C}$.

In the final section of this paper, we list several remarks about inequalities related to $I_1(t)$.

2. Lemmas

We need the following lemmas.

Lemma 2.1 ([13, Theorem 1.2])**.** *For $k\in\{0\}\cup\mathbb{N}$ and $z\ne0$, let*

$$H_k(z)=e^{1/z}-\sum_{m=0}^k\frac{1}{m!}\frac{1}{z^m}. \tag{2.1}$$

For $\Re z>0$, the function $H_k(z)$ has the integral representation

$$H_k(z)=\frac{1}{k!(k+1)!}\int_0^\infty {}_1F_2(1;k+1,k+2;t)t^ke^{-zt}\,\mathrm{d}t, \tag{2.2}$$

where the hypergeometric series is given by

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;x)=\sum_{n=0}^\infty\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{x^n}{n!} \tag{2.3}$$

for $b_i\notin\{0,-1,-2,\dots\}$ using the shifted factorial $(a)_0=1$ and

$$(a)_n=a(a+1)\cdots(a+n-1) \tag{2.4}$$

for $n>0$ and any complex number a.

Lemma 2.2 ([15, p. 161, Theorem 12b])**.** *A necessary and sufficient condition that $f(x)$ should be completely monotonic for $0<x<\infty$ is that*

$$f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\mu(t), \tag{2.5}$$

where μ is a positive measure on $[0,\infty)$ such that the integral converges for $0<x<\infty$.

Remark 2.1. Lemma 2.2 means that a function f is completely monotonic on $(0,\infty)$ if and only if it is a Laplace transform of a positive measure μ.

Lemma 2.3 ([3])**.** *Let a_k and b_k for $k\in\{0\}\cup\mathbb{N}$ be real numbers and the power series*

$$A(x)=\sum_{k=0}^{\infty}a_kx^k \quad \textit{and} \quad B(x)=\sum_{k=0}^{\infty}b_kx^k \tag{2.6}$$

be convergent on $(-R,R)$ for some $R>0$. If $b_k>0$ and the ratio $\frac{a_k}{b_k}$ is (strictly) increasing for $k\in\mathbb{N}$, then the function $\frac{A(x)}{B(x)}$ is also (strictly) increasing on $(0,R)$.

Remark 2.2. Lemma 2.3 have several different proofs and various applications. For more information, please refer to the first sentence after [2, p. 582, Lemma 2.1], the papers [9, 12], and closely related references therein. We emphasize that Lemma 2.3 has been generalized in [6, Lemma 2.2].

3. Complete monotonicity of $h_\alpha(t)$

In this section, we will supply a simpler proof for the complete monotonicity of $h(t)$ and find a necessary and sufficient condition for $h_\alpha(t)$ to be completely monotonic on $(0,\infty)$.

Theorem 3.1. *The function $h(t)$ defined by* (1.9) *is completely monotonic on $(0,\infty)$ and*

$$\lim_{t\to\infty}h(t)=1. \tag{3.1}$$

The function $h_\alpha(t)$ defined by (1.11) *is completely monotonic on $(0,\infty)$ if and only if $\alpha\ge1$.*

Proof. From the integral representation (1.7) or the series expansion (1.8), it is obvious that $\lim_{t\to\infty}\psi'(t)=0$. So the limit (3.1) follows immediately.

From the recurrence formula

$$\psi^{(n)}(z+1)=\psi^{(n)}(z)+(-1)^n\frac{n!}{z^{n+1}}, \quad n\ge0, \tag{3.2}$$

see [1, p. 260, 6.4.7], we see that

$$\begin{aligned}
h(t)-h(t+1)&=e^{1/t}-e^{1/(t+1)}+\psi'(t+1)-\psi'(t)\\
&=e^{1/t}-e^{1/(t+1)}-\frac{1}{t^2}\\
&=-\frac{1}{t^2}+\sum_{k=1}^{\infty}\frac{1}{k!}\left[\frac{1}{t^k}-\frac{1}{(t+1)^k}\right]\\
&=\frac{1}{6t^3(t+1)^3}+\sum_{k=4}^{\infty}\frac{1}{k!}\left[\frac{1}{t^k}-\frac{1}{(t+1)^k}\right]
\end{aligned}$$

$$= \frac{1}{6t^3(t+1)^3} + \sum_{k=4}^{\infty} \frac{1}{(k-1)!} \int_0^1 \frac{\mathrm{d}x}{(t+x)^{k+1}}.$$

Since $\frac{1}{t^3(t+1)^3}$ and $\frac{1}{(t+x)^k}$ are completely monotonic with respect to $t \in (0, \infty)$ for $k \ge 1$ and $x \ge 0$, so is the difference $h(t) - h(t+1)$. Accordingly,

$$h(t) - h(t+n+1) = \sum_{k=0}^{n} [h(t+k) - h(t+k+1)]$$

is completely monotonic on $(0, \infty)$, and so is the pointwise limit

$$\lim_{n\to\infty} [h(t) - h(t+n+1)] = h(t) - 1.$$

If the function $h_\alpha(t)$ is completely monotonic on $(0, \infty)$, then its first derivative should be non-positive, that is,

$$-\frac{\alpha}{t^2} e^{1/t} - \psi''(t) \le 0,$$

which can be rearranged as

$$\alpha \ge -\frac{t^2 \psi''(t)}{e^{1/t}} \to -\lim_{t\to\infty} [t^2 \psi''(t)] = 1$$

as $t \to \infty$, where the limit

$$\lim_{x\to\infty} \left[(-1)^{k+1} x^k \psi^{(k)}(x)\right] = (k-1)!, \tag{3.3}$$

see [10, p. 81, (41)], is used. As a result, the condition $\alpha \ge 1$ is necessary.

Since

$$\alpha e^{1/t} - \psi'(t) = (\alpha - 1)e^{1/t} + h(t)$$

and both of the functions $e^{1/t}$ and $h(t)$ are completely monotonic on $(0, \infty)$, the condition $\alpha \ge 1$ is also sufficient. Theorem 3.1 is thus proved. □

4. Complete monotonicity of $h_{\alpha,\beta}(t)$

In this section, with the help of Theorem 3.1, we will discover necessary and sufficient conditions for $h_{\alpha,\beta}(t)$ to be completely monotonic on $(0, \infty)$.

Theorem 4.1. *For $\alpha, \beta > 0$,*

1. *the function $h_{1,\beta}(t)$ is completely monotonic on $(0, \infty)$ if and only if $\beta \ge 1$;*
2. *if $\beta \ge 1$ and $\alpha\beta \ge 1$, the function $h_{\alpha,\beta}(t)$ is completely monotonic on $(0, \infty)$;*
3. *a necessary condition for the function $h_{\alpha,\beta}(t)$ to be completely monotonic on $(0, \infty)$ is $\alpha\beta \ge 1$;*
4. *if $0 < \beta < 1$, the condition*

$$\alpha\beta \ge \max_{u\in(0,\infty)} F_\beta(u) > 1, \tag{4.1}$$

where

$$F_\beta(u) = \frac{u}{1 - e^{-u}} \frac{\sqrt{\beta u}}{I_1\left(2\sqrt{\beta u}\right)} \tag{4.2}$$

on $(0,\infty)$ *with*

$$\lim_{u\to0^+} F_\beta(u)=1 \quad \text{and} \quad \lim_{u\to\infty} F_\beta(u)=0 \tag{4.3}$$

for all $\beta>0$*, is necessary and sufficient for* $h_{\alpha,\beta}(t)$ *to be completely monotonic on* $(0,\infty)$*.*

Proof. Combining the integral representations (1.7) for $n=1$ and (2.2) for $k=0$ leads to

$$\begin{aligned} h_{\alpha,\beta}(z)&=\alpha\biggl[1+\int_0^\infty \frac{I_1(2\sqrt{u}\,)}{\sqrt{u}}e^{-zu/\beta}\,\mathrm{d}\,u\biggr]-\int_0^\infty \frac{u}{1-e^{-u}}e^{-zu}\,\mathrm{d}\,u\\ &=\alpha+\int_0^\infty\biggl[\alpha\sqrt{\frac{\beta}{u}}\,I_1(2\sqrt{\beta u}\,)-\frac{u}{1-e^{-u}}\biggr]e^{-zu}\,\mathrm{d}\,u\\ &=\alpha+\int_0^\infty\biggl[\alpha\beta\frac{I_1(2\sqrt{\beta u}\,)}{\sqrt{\beta u}}-\frac{u}{1-e^{-u}}\biggr]e^{-zu}\,\mathrm{d}\,u \end{aligned} \tag{4.4}$$

for $\Re z>0$.

Since, by Theorem 3.1, the function $h(t)=h_{1,1}(t)$ is known to be completely monotonic on $(0,\infty)$, it follows from Lemma 2.2 that the inequality

$$\frac{I_1(2\sqrt{u}\,)}{\sqrt{u}}\ge\frac{u}{1-e^{-u}} \tag{4.5}$$

holds true for $u>0$.

An easy calculation gives

$$\frac{I_1(2\sqrt{u}\,)}{\sqrt{u}}=1+\frac{u}{2}+\frac{u^2}{12}+\frac{u^3}{144}+o(u^3)$$

and

$$\frac{u}{1-e^{-u}}=1+\frac{u}{2}+\frac{u^2}{12}-\frac{u^4}{720}+o(u^4)$$

for $u\to0$. Consequently,

$$\alpha\beta\frac{I_1(2\sqrt{\beta u}\,)}{\sqrt{\beta u}}-\frac{u}{1-e^{-u}}=\alpha\beta-1+\frac{\alpha\beta^2-1}{2}u+\frac{\alpha\beta^3-1}{12}u^2+o(u^2) \tag{4.6}$$

for $u\to0$. By (4.5), (4.6), and the fact that the function $\frac{I_1(2s)}{s}$ is strictly increasing on $(0,\infty)$, it follows that the inequality

$$\frac{I_1(2\sqrt{\beta u}\,)}{\sqrt{\beta u}}\ge\frac{u}{1-e^{-u}} \tag{4.7}$$

is valid on $(0,\infty)$ if and only if $\beta\ge1$. As a result, by Lemma 2.2, the function $h_{1,\beta}(t)$ is completely monotonic on $(0,\infty)$ if and only if $\beta\ge1$.

If $\beta\ge1$ and $\alpha\beta\ge1$, then

$$\alpha\beta\frac{I_1(2\sqrt{\beta u}\,)}{\sqrt{\beta u}}\ge\frac{I_1(2\sqrt{u}\,)}{\sqrt{u}}\ge\frac{u}{1-e^{-u}} \tag{4.8}$$

is valid on $(0,\infty)$, so, by Lemma 2.2 once again, the function $h_{\alpha,\beta}(t)$ is completely monotonic on $(0,\infty)$ for $\beta\ge1$ and $\alpha\beta\ge1$.

By (4.6) or by a straightforward computation, it is easy to obtain that

$$\lim_{u\to 0^+}\left[\alpha\beta\frac{I_1\big(2\sqrt{\beta u}\,\big)}{\sqrt{\beta u}}-\frac{u}{1-e^{-u}}\right]=\alpha\beta-1. \tag{4.9}$$

So $\alpha\beta\ge 1$ is a necessary condition for the function $h_{\alpha,\beta}(t)$ to be completely monotonic on $(0,\infty)$.

When $\beta<1$, the complete monotonicity of $h_{\alpha,\beta}(t)$ is equivalent to the inequality

$$\alpha\beta\ge\frac{u}{1-e^{-u}}\frac{\sqrt{\beta u}}{I_1\big(2\sqrt{\beta u}\,\big)}$$

for all $u\in(0,\infty)$. It is not difficult to see that the function $\frac{u}{1-e^{-u}}$ is increasing on $(0,\infty)$, with

$$\lim_{u\to 0^+}\frac{u}{1-e^{-u}}=1\quad\text{and}\quad\lim_{u\to\infty}\frac{u}{1-e^{-u}}=\infty,$$

and that the function $\frac{\sqrt{\beta u}}{I_1(2\sqrt{\beta u}\,)}$ is decreasing on $(0,\infty)$ for any given number $\beta>0$, with

$$\lim_{u\to 0^+}\frac{\sqrt{\beta u}}{I_1\big(2\sqrt{\beta u}\,\big)}=1\quad\text{and}\quad\lim_{u\to\infty}\frac{\sqrt{\beta u}}{I_1\big(2\sqrt{\beta u}\,\big)}=0.$$

On the other hand, the function $F_\beta(u)$ may be rearranged as

$$F_\beta(u)=\frac{e^u}{e^u-1}\frac{u\sqrt{\beta u}}{\sum_{k=0}^\infty\frac{1}{k!(k+1)!}\big(\sqrt{\beta u}\,\big)^{2k+1}}=\frac{e^u}{e^u-1}\frac{u}{\sum_{k=0}^\infty\frac{\beta^k}{k!(k+1)!}u^k}$$

and it tends to 0 as $u\to\infty$. Accordingly, the condition (4.1) and the limits in (4.3) are obtained. The proof of Theorem 4.1 is complete. □

5. Monotonicity and unimodality of $F_\beta(u)$

In this section, we will find monotonicity, unimodality, and convexity of the function $F_\beta(u)$. Some of these properties will be used in Theorem 6.1 below to recover necessary and sufficient conditions for $h_{\alpha,\beta}(t)$ to be completely monotonic on $(0,\infty)$.

Theorem 5.1. *When $\beta\ge1$, the function $F_\beta(u)$ defined by* (4.2) *is decreasing on $(0,\infty)$; when $0<\beta<1$, it is unimodal and its reciprocal $\frac{1}{F_\beta(u)}$ is convex on $(0,\infty)$.*

Proof. For simplicity, we consider the reciprocal

$$\frac{1}{F_\beta(u)}=\frac{1-e^{-u}}{u}\frac{I_1\big(2\sqrt{\beta u}\,\big)}{\sqrt{\beta u}}=\frac{1}{e^u}\frac{e^u-1}{u}\frac{I_1\big(2\sqrt{\beta u}\,\big)}{\sqrt{\beta u}}\triangleq G_\beta(u). \tag{5.1}$$

By (1.13) and the power series expansion of e^u, we have

$$\frac{e^u-1}{u}\frac{I_1\big(2\sqrt{\beta u}\,\big)}{\sqrt{\beta u}}=\sum_{n=0}^\infty\frac{u^n}{(n+1)!}\sum_{n=0}^\infty\frac{\beta^n}{n!(n+1)!}u^n=\sum_{n=0}^\infty a_nu^n,$$

where

$$a_n=\sum_{k=0}^{n}\frac{\beta^k}{k!(k+1)!(n+1-k)!}=\frac{1}{(n+1)!}\sum_{k=0}^{n}\binom{n+1}{k}\frac{\beta^k}{(k+1)!}.$$

Hence,

$$G_\beta(u)=\frac{\sum_{n=0}^\infty a_nu^n}{\sum_{n=0}^\infty u^n/n!}.$$

When $\beta\ge1$, let $c_n=n!a_n$ for $n\in\{0\}\cup\mathbb{N}$. It is clear that $c_0=1$, $c_1=\frac{1+\beta}{2}$, and $c_2=\frac13+\frac{\beta}{2}+\frac{\beta^2}{6}$ satisfy $c_0\le c_1\le c_2$. For $n\ge2$,

$$\begin{aligned}c_{n+1}-c_n&=\frac{1}{n+2}\sum_{k=0}^{n+1}\binom{n+2}{k}\frac{\beta^k}{(k+1)!}-\frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k}\frac{\beta^k}{(k+1)!}\\&=\frac{\beta^{n+1}}{(n+2)!}+\sum_{k=2}^{n}\binom{n}{k-2}\frac{\beta^k}{k(k+1)!}-\frac{1}{(n+2)(n+1)}\\&\ge\frac{\beta^2}{12}-\frac{1}{(n+2)(n+1)}\\&\ge0.\end{aligned}$$

In other words, when $\beta\ge1$, the sequence $c_n=n!a_n$ is increasing. From this and Lemma 2.3, it follows that, when $\beta\ge1$, the function $G_\beta(u)$ is increasing on $(0,\infty)$. Equivalently, when $\beta\ge1$, the function $F_\beta(u)$ is decreasing on $(0,\infty)$.

A direct computation yields

$$G_\beta'(u)=\frac{1}{e^u}\left\{\left[\frac{e^u-1}{u}\frac{I_1\bigl(2\sqrt{\beta u}\bigr)}{\sqrt{\beta u}}\right]'-\frac{e^u-1}{u}\frac{I_1\bigl(2\sqrt{\beta u}\bigr)}{\sqrt{\beta u}}\right\}=\frac{\sum_{n=0}^\infty b_nu^n}{\sum_{n=0}^\infty u^n/n!},$$

where $b_n=(n+1)a_{n+1}-a_n$ satisfy

$$b_0=a_1-a_0=\frac{\beta-1}{2},\quad b_1=2a_2-a_1=\frac{\beta^2-1}{6},$$

and for $n\ge2$

$$\begin{aligned}b_n&=\frac{1}{n!}\left[\sum_{k=0}^{n+1}\frac{1}{n+2}\binom{n+2}{k}\frac{\beta^k}{(k+1)!}-\sum_{k=0}^{n}\frac{1}{n+1}\binom{n+1}{k}\frac{\beta^k}{(k+1)!}\right]\\&=\frac{1}{n!}\left[\frac{\beta^{n+1}}{(n+2)!}-\frac{1}{(n+1)(n+2)}+\sum_{k=2}^{n}\binom{n}{k-2}\frac{\beta^k}{k(k+1)!}\right].\end{aligned}$$

When $0<\beta<1$, let $C_n=n!b_n$. Then

$$C_0=\frac{\beta-1}{2}<C_1=\frac{\beta^2-1}{6}$$

and for $n \ge 1$

$$C_{n+1} - C_n = \frac{\beta^{n+1}}{(n+2)!}\left(\frac{\beta}{n+3} - 1 + \frac{n}{2}\right) + \frac{2}{(n+1)(n+2)(n+3)} + \sum_{k=3}^{n}\binom{n}{k-3}\frac{\beta^k}{k(k+1)!}.$$

Therefore,

$$C_2 - C_1 = \frac{\beta^2(\beta-2)}{24} + \frac{1}{12} > 0$$

and for $n \ge 2$

$$C_{n+1} - C_n > \frac{\beta^{n+1}}{(n+2)!}\frac{\beta}{n+3} > 0.$$

Consequently, when $0 < \beta < 1$, the sequence $C_n = n!b_n$ is strictly increasing. By Lemma 2.3 again, it follows that $G'_\beta(u)$ is strictly increasing, and so $G_\beta(u) = \frac{1}{F_\beta(u)}$ is convex, on $(0,\infty)$ for $0 < \beta < 1$. Since

$$b_n \ge -\frac{1}{(n+2)!} + \frac{\beta^3}{72(n-1)!}$$

for $n \ge 3$, we get for such n

$$\begin{aligned}\sum_{n=0}^{\infty} b_n u^n &\ge b_0 + b_1 u + b_2 u^2 - \sum_{n=3}^{\infty}\frac{u^n}{(n+2)!} + \frac{\beta^3}{72}\sum_{n=3}^{\infty}\frac{u^n}{(n-1)!} \\ &> b_0 + b_1 u + b_2 u^2 - \frac{e^u}{u^2} + \frac{\beta^3 u}{72}(e^u - 1 - u),\end{aligned}$$

thus

$$G'_\beta(u) > \frac{b_0 + b_1 u + b_2 u^2}{e^u} - \frac{1}{u^2} + \frac{\beta^3 u}{72}\left(1 - \frac{1+u}{e^u}\right) \to \infty$$

as $u \to \infty$. On the other hand, when $0 < \beta < 1$,

$$G'_\beta(0) = b_0 = \frac{\beta - 1}{2} < 0.$$

As a result, when $0 < \beta < 1$, by the above proved monotonicity of $G'_\beta(u)$, the derivative $G'_\beta(u)$ has a unique zero, and so the positive function $G_\beta(u)$ has a unique minimum, on $(0,\infty)$. In other words, when $0 < \beta < 1$, the positive function $F_\beta(u)$ has a unique maximum on $(0,\infty)$. Theorem 5.1 is proved. □

6. Recovery of complete monotonicity of $h_{\alpha,\beta}(t)$

In this section, with the aid of Theorem 5.1, we will simply recover necessary and sufficient conditions in Theorem 4.1.

Theorem 6.1. *For $\alpha, \beta > 0$, the function $h_{\alpha,\beta}(t)$ defined by (1.12) is completely monotonic on $(0,\infty)$ if and only if the condition (4.1) holds true.*

Proof. By Lemma 2.2 and the integral representation (4.4), we easily see that the condition (4.1) is necessary and sufficient. Theorem 6.1 is thus proved. □

Corollary 6.1. *When $\beta \ge 1$, the function $h_{\alpha,\beta}(t)$ defined by (1.12) is completely monotonic on $(0,\infty)$ if and only if $\alpha\beta \ge 1$.*

Proof. This follows from monotonicity of $F_\beta(u)$ established in Theorem 5.1, the first limit in (4.3), and Theorem 6.1. □

7. Inequalities for the modified Bessel function

By observing the proof of Theorem 4.1 or by Theorem 5.1, we conclude the following inequalities which give new lower bounds for I_1.

Theorem 7.1. *The inequalities*

$$\alpha I_1(x) > \frac{(x/2)^3}{1-e^{-(x/2)^2}} \tag{7.1}$$

and

$$I_1(x) \ge \frac{\frac{1}{\beta}\left(\frac{x}{2}\right)^3}{1-\exp\left[-\frac{1}{\beta}\left(\frac{x}{2}\right)^2\right]} \tag{7.2}$$

are valid on $(0,\infty)$ if and only if $\alpha \ge 1$ and $\beta \ge 1$.

Proof. The inequality (7.2) follows from replacing u by $\frac{1}{\beta}\left(\frac{x}{2}\right)^2$ in (4.7) and further simplifying.

The inequality (7.1) follows from combining the second part of Theorem 3.1 with the formula (4.4) for $h_{\alpha,1} = h_\alpha$ and Lemma 2.2.

The inequality (7.2) can also be deduced from Theorem 5.1. The proof of Theorem 7.1 is complete. □

8. Remarks

In this section, we would like to compare the inequalities (7.1) and (7.2) with other known ones.

Remark 8.1. When $\beta = 1$, the inequality (7.2) is quite good for $0 < x < 2$, but for large x the right-hand side of (7.2) increases like $\frac{x^3}{8}$, while $I_1(x)$ increases faster than any power of x.

Remark 8.2. We refer to the following known inequalities

$$I_1(x) > \frac{1-x/2}{2+x} x e^x, \tag{8.1}$$

cf. [7, (6.23)], and

$$I_1(x) > \frac{x}{2}\left(1+\frac{x^2}{j_{1,1}^2}\right)^{j_{1,1}^2/8}, \tag{8.2}$$

cf. [5, (3.20)], for $x > 0$, where $j_{1,1} = 3.83\dots$ is the first zero of J_1 and

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{k!\Gamma(\nu+k+1)} \tag{8.3}$$

for $\nu \in \mathbb{R}$ and $z \in \mathbb{C}$ is the Bessel function of the first kind. Looking at graphs of the lower bounds in (7.2), (8.1), and (8.2) reveals that the lower bound in (7.2) for $\beta = 1$ is the largest of the three for $0 < x < 5$.

Remark 8.3. The inequality (7.1) and the second part of Theorem 3.1 are equivalent to each other, so are the inequality (7.2) and the necessary and sufficient condition $\beta \ge 1$ in Theorem 4.1 for $h_{1,\beta}(t)$ to be completely monotonic on $(0, \infty)$.

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