Complete quantum mechanics: an axiomatic formulation of the Montevideo interpretation

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Abstract

We make a first attempt to axiomatically formulate the Montevideo interpretation of quantum mechanics. In this interpretation environmental decoherence is supplemented with loss of coherence due to the use of realistic clocks to measure time to solve the measurement problem. The resulting formulation is framed entirely in terms of quantum objects without having to invoke the existence of measurable classical quantities like the time in ordinary quantum mechanics. The formulation eliminates any privileged role to the measurement process giving an objective definition of when an event occurs in a system.

I. INTRODUCTION

The usual textbook presentation of the axiomatic formulation of quantum mechanics includes two apparently unconnected problematic issues. The first one is the privileged role of the time variable that is assumed to be a classical variable not represented by a quantum operator. The second is the also privileged role of certain processes called *measurements* where quantum states suffer abrupt changes not described by a unitary evolution, and probabilities are assigned to the values that one may obtain for a physical quantity.

The special role of measurement processes in quantum mechanics requires understanding what distinguishes such processes from the rest of the quantum evolution. This is called the *measurement problem*, which many physicists have alluded to and that ultimately refer to the uniqueness of macroscopic phenomena within a quantum framework that only refers to potentialities. Ghirardi calls this the *problem of macro objectification*.

The orthodox response of the Copenhagen interpretation argues that the objective of quantum mechanics is not to describe *what is* but *what we observe*. The measuring devices are classical objects through which we acquire knowledge of the quantum world. The measurement therefore acquires an epistemological interpretation, referring to processes in which observers acquire knowledge of phenomena. The question about how does quantum mechanics account for events observed in the measurement and the multitude of events that happen every moment in every place giving rise to the defined perception of our experience is left out of the realm of the theory. Those processes belong to a world of objects that our knowledge cannot have access to. As put by d'Espagnat [1], "the (orthodox) quantum formalism is predictive rather than descriptive... [but also] ...the formalism in question is not predictive (probability-wise) of *events*. It is predictive (probability-wise) of *observations*." For him the statements of quantum mechanics are *weakly objective* since they refer to certain human procedures —for instance, of observation—. They are objective because they are true for everyone, "But their form (or context) makes it impossible to take them as descriptions of how the things actually are". Such descriptions were the usual ones in the realm of classical physics, whose statements can be considered as *strongly objective* since one can consider that they inform us about certain attributes of the objects it studies.

If the statements of quantum mechanics can only be weakly objective one must abandon attempts to understand how the passage from quantum potentialities to observed phenom-

ena, from micro to macro, from determinism to randomness, from quantum to classical, takes place. The question of which systems should be treated as classical also becomes not analyzable, an issue that acquires more relevance as more and more macro systems that display quantum behaviors are being constructed by experimentalists.

If one adopts a realist point of view, that is if one assumes the existence of a reality independent of observers, the orthodox description of quantum mechanics is incomplete since it does not tell us which events may occur nor when may they occur. In our view this is a rather extreme point of view that should be reserved only to the case in which one has exhausted all other possibilities for analyzing physically the problem of the production of events. There has been a recent renewed interest among specialists in foundations of quantum mechanics in understanding how an objective description at a macroscopic level compatible with quantum mechanics arises. Several avenues have been proposed to address such a question (for a comprehensive review see [2]).

On the other hand the fact that time is treated unlike any other variable in quantum theory has received much less attention. The usual point of view is that to associate time with a quantum variable is impossible. This is due to the well known Pauli observation that an observable associated with time would be canonically conjugate to the Hamiltonian and it is impossible to have a bounded below operator like the Hamiltonian canonically conjugate to a self adjoint operator. Even if one admits Leibniz' point of view that time is a relational notion and therefore in modern terms described by clocks subject to the laws of quantum mechanics, it is usually thought that this would only complicate the description. The absolute Newtonian view imposed itself not because it was the philosophically correct one but because it was the simplest and yielded highly accurate predictions. A relational treatment is only adopted if its use is inescapable, like in situations where there obviously is no external parameter. An example of this could be quantum cosmology where there are no external clocks nor external apparata to make measurements nor an external observer. As Smolin [3] put it "Can a sensible dynamical theory [of quantum cosmology] be formulated that does not depend on an absolute background space or time? Can quantum mechanics be understood in a way that does not require the existence of a classical Observer outside the system'?" Up to now there have not been formulations of theories of physics that are completely relational without ideal external elements.

The *Montevideo interpretation* [4] of quantum mechanics shows that a relational treat-

ment with quantum clocks allows to solve the measurement problem, therefore providing a solution to both the problems we mentioned above. In this paper we present an axiomatic formulation of the Montevideo interpretation of quantum mechanics where the evolution is described in terms of real clocks. We call it Complete Quantum Mechanics because the formulation does not require us to treat any variable as classical or external. In the axiomatic formulation we establish precisely when and where events occur and what is their nature. Since the formulation arises from an analysis of the problem of time in quantum gravity [5], the proposed description —although presented here in the non-relativistic case only— is formulated in a language that is ready to treat generally covariant theories like general relativity. It can be said that it is a quantum mechanics formulated with an eye towards a quantum theory of gravity.

The axiomatic formulation has several goals: a) to give a rigorous definition of what a *real clock* is; b) to list explicitly the hypotheses of the Montevideo interpretation and to show its internal consistency and c) to make explicit the mechanisms for macro objectification and outline a realistic ontology based on this interpretation. The resulting description will be strongly objective in the sense indicated above without referring any time to observers or measurements. It does not attempt to substitute the usual axiomatics in most practical applications, where the use of ideal clocks gives a very precise description. An axiomatic relational formulation necessarily requires systems with enough degrees of freedom to include the micro-systems[1] one studies, the clocks, measuring devices and the environment that is involved in the measurement process.

II. AXIOMS THAT ARE SHARED WITH ORDINARY QUANTUM MECHANICS

Axiom 1: States

The state of a complete physical system $\mathscr{S}$ is described by positive definite self-adjoint operators ρ in a Hilbert space H

We adopt the idea that a state is well defined when it allows to assign correctly probabilities to any property associated with a physical quantity. Particular cases of states are projectors on one-dimensional vector subspaces, in which case the information contained in

[1] the typical systems with few degrees of freedom one usually studies in quantum mechanics

ρ is equivalent to that of a vector in the Hilbert space. The components of the operator ρ in a basis are usually referred to as *density matrix*. The reason we are working with density matrices is that as we will see, when one works with real clocks there is loss of quantum coherence and this is more naturally discussed in terms of density matrices.

The axiomatic we are presenting makes reference to a set of primitive concepts like *system, state, events and the properties that constitute them, and physical quantities* each of them associated with certain mathematical objects of the formalism of ordinary quantum mechanics. All these are defined implicitly in the axioms just like in ordinary axiomatic quantum mechanics one defines system, state, measurement and physical quantities. The first axiom associates certain operators to the states and a Hilbert space to the systems.

Axiom 2: Physical quantities

Any physical quantity $\mathscr{A}$ of $\mathscr{S}$ is described by a self-adjoint operator $\hat{A}$ that acts in H. We will call such operators observables

In most situations, as we will see later, quantities of interest are associated to subsystems of $\mathscr{S}$ and therefore to subspaces of H.

Axiom 3: Properties

The only possible values of a physical quantity $\mathscr{A}$ are the eigenvalues of the corresponding operator $\hat{A}$.

A physical quantity only takes values when an event occurs. If $\mathscr{A}$ has a value A we will say that the event has a property $\mathscr{P}_a$ to which we will associate a projector $\hat{P}_a$ on the eigenspace associated with the corresponding eigenvalue A.

The events that constitute the physical phenomena are the most concrete thing that attains us directly and we cannot ignore. They are what makes the world and what physics has to account for. It is natural that physics, which is an empirical science would take as starting point the events, which are the data from our experience of the world. The word phenomenon comes from the Greek and means something sufficiently apparent to be perceived by our senses. Events are elementary phenomena that we usually associate with a set of properties that we characterize by the numerical values that certain physical quantities

and their associated projectors take. An example of event would be the formation of a dot of silver atoms on a photographic plate of an electron detector or the appearance of droplets in a cloud chamber. In spite of the persistent tendency to think in terms of particles in physics, we only observe events. The trace of a particle in a bubble chamber is just a series of correlated events. Physical properties characterize events. For instance, if we are interested in the position of the dot of silver on the photographic plate, the position will be the physical quantity and the value that it takes in a given experiment will correspond to a property that constitutes the event. Notice that we are not assuming that all events are perceived by our senses.

Axiom 4: Evolution in Newtonian time

In non-relativistic theories there exists an ideal Newtonian time for which the principle of inertia holds. That is, for which free classical particles have a uniform rectilinear motion. Newtonian time imposes an absolute order of events and an absolute notion of simultaneity. Such an absolute time is not an accessible physical quantity. It can only be approximately monitored by physical clocks, which are subject to quantum fluctuations. This next axiom will refer to the particularly simple form of the evolution of operators in Newtonian time, which we will represent by a c-number t. We are here working in the Heisenberg picture in which operators evolve.

The evolution in the ideal Newtonian time of a physical quantity with associated self-adjoint operator $\hat{A}$ is given by the equation

$$i\hbar\frac{d\hat{A}(t)}{dt} = \left[\hat{A}(t), \hat{H}(t)\right] + \frac{\partial\hat{A}(t)}{\partial t}. \tag{1}$$

For instance, in ordinary particle mechanics where one has classical position and momentum given by x and p, an observable associated with the classical quantity $A(r, p, t)$ is quantized by replacing x and p with $\hat{x}$ and $\hat{p}$ and appropriately symmetrizing so that the resulting operator is self-adjoint. The partial derivative refers to the explicit dependence in the parameter t. In ordinary quantum mechanics the Heisenberg and Schrödinger pictures are equivalent and so are here if one is referring to the evolution in terms of the ideal Newtonian time t. If one considers the evolution as described by real clocks there are modifications, as we will subsequently discuss.

III. RELATIONAL AXIOMS

The probability axiom and the reduction axiom radically change their form in the Montevideo interpretation since they now include the observed system and the clock that registers the event, both as quantum mechanical systems. We will consider almost ideal clocks. We expect a generic degree of freedom to approximate an ideal clock when it is weakly interacting, both with other degrees of freedom and with itself. We therefore say that a system contains a decoupled clock when the Hamiltonian may be written in the form,

$$H = H_{\text{clock}} + H_{\text{system}}, \tag{2}$$

where H_{clock} depends only on the coordinates and momentum of the clock and H_{system} is independent of the canonical clock variables. While this situation is, strictly speaking, unphysical, it approximates systems which differ from (2) only by terms that may be treated adiabatically. In correspondence with this we will assume that the quantum state of the complete system is a tensor product of a state for the clock and a state for the system under study, i.e. $\rho = \rho_{\text{cl}} \otimes \rho_{\text{sys}}$.

Let $\hat{T}(t)$ be a self-adjoint operator (observable) in the Hilbert space H that describes the physical quantity chosen to measure time by a clock ruled by quantum mechanics and $\hat{Q}^i(t)$ and $\hat{P}^i(t)$ observables associated to quantities $\mathscr{Q}$and $\mathscr{P}$ that commute with $\hat{T}(t)$ and to whose values one wishes to assign probabilities. We assume all variables have continuous spectrum, because clocks normally do, results are easily reworked for variables having discrete spectrum. Let $\hat{P}_{Q_0^i}(t)$ be the projector on the eigenspace of $\hat{Q}^i$ with eigenvalues in the interval of a given width $2\Delta^i$ centered in Q_0^i, that is, $[Q_0^i - \Delta^i, Q_0^i - \Delta^i]$ and analogously the clock variable $\hat{T}$ with its projector $\hat{P}_{T_0}(t)$. In terms of these quantities the probability postulate states that:

Axiom 5: probabilities

The probability that the quantity Q^i of a physical system in a state ρ take a value in a prescribed range of values when the clock in such state takes a value in the interval $[T_0 -$

$\Delta^C, T_0 + \Delta^C]$ *is given by,*

$$\mathcal{P}\left(Q^i \in \left[Q_0^i - \Delta^i, Q_0^i + \Delta^i\right] | T \in \left[T_0 - \Delta^C, T_0 + \Delta^C\right]\right) = \frac{\int_0^\tau dt \mathrm{Tr}\left(\hat{P}_{Q_0^i}(t)\hat{P}_{T_0}(t)\rho\hat{P}_{T_0}(t)\right)}{\int_0^\tau dt \mathrm{Tr}\left(\hat{P}_{T_0}(t)\rho\right)}, \tag{3}$$

where $\hat{P}_{Q_0}(t)$ and $\hat{P}_{T_0}(t)$ are the projectors associated to properties Q and T taking the eigenvalues Q_0 and T_0.

These conditional probabilities are positive and add to one. They refer to the probability of occurrence of events with properties associated with the eigenvalues of the operators involved. Which specific events and when do they occur are issues not determined by this axiom. Notice that a similar construction can be carried out for the P^i quantities, we wrote the expression for the Q^i for concreteness only. The only condition is that the quantities must have vanishing Poisson bracket with $T(t)$.

Note that we are integrating in the ideal Newtonian time t which is taken to be unobservable. The integration interval goes from $t = 0$, instant in which the observable clock T is started, to τ, the maximum Newtonian time for which the clock T operates with the desired precision. The quality of the clock depends on the initial state of the system when the clock is started, the admissible error Δ^C and the total time the clock is used τ. If one wishes to perform subsequent measurements care should be taken to choose the interval Δ^C large enough such that the measurement of the clock variable does not affect too much the accuracy of it.

It is worthwhile expanding on the meaning of the probabilities (3) since there has been some confusion in the literature [6]. Thinking in terms of ordinary quantum mechanics one may interpret that the numerator of (3) is the sum of joint probabilities of O and T for all values of t. This would be incorrect since the events in different t's are not mutually exclusive. The probability (3) corresponds to a physically measurable quantity, and such quantity is actually the only thing one can expect to measure in systems where one does not have direct access to the "ideal" time t. The experimental setup we have in mind is to consider an ensemble of non-interacting systems with two quantum variables each to be measured, O and T. Each system is equipped with a recording device that takes a single snapshot of O and T at a random unknown value of the "ideal" time t. One takes a large number of such systems, launches them all in the same quantum state, "waits for a long time", and concludes the experiment. The recordings taken by the devices are then collected

and analyzed all together. One computes how many times $n(T_j, O_j)$ each reading with a given value $T = T_j, O = O_j$ occurs (to simplify things, for the moment let us assume T, O have discrete spectra; for continuous spectra one would have to consider values in a small finite interval of the value of interest). If one takes each of those values $n(T_j, O_j)$ and divides them by the number of systems in the ensemble, one obtains, in the limit of infinite systems, a joint probability $P(O_j, T_j)$ that is proportional to the numerator of the above expression. The denominator is obtained by counting $n(T_j)$ ignoring the values of O. Notice that this implies a change in the probability axiom with respect to ordinary quantum mechanics. This is what is made explicit in axiom 5.

The previous expression can be straightforwardly extended to the case in which one or both observables involved have discrete spectrum. Since the spectrum may be time dependent it is also convenient to talk about quantities taking values in finite intervals in the discrete case as well.

Although we spelled out the axiom explicitly for the measurement of a single quantity Q^i it is immediately generalizable to the measurement of several commuting operators (functions of the Q^i's and P^i's). The next axiom allows to assign probabilities to histories of events that occur at different instants of time.

Axiom 6: State reduction

When a set of physical quantities (that include the clock) with commuting self-adjoint operators $\hat{A}_1 \dots \hat{A}_n$ take values $A^1 \dots A^n$ in the intervals $[A_0^1 - \Delta^1, A_0^1 + \Delta^1] \dots [A_0^n - \Delta^n, A_0^n + \Delta^n]$ the state of the system can be represented by the normalized quasi-projection of the state ρ associated with the values of the quantities in question,

$$\rho_{\rm red} = \frac{\int_0^\tau dt \hat{P}_{A_0^1}(t) \dots \hat{P}_{A_0^n}(t) \rho \hat{P}_{A_0^n}(t) \dots \hat{P}_{A_0^1}(t)}{{\rm Tr}\left(\int_0^\tau dt \hat{P}_{A_0^1}(t) \dots \hat{P}_{A_0^n}(t) \rho \hat{P}_{A_0^n}(t) \dots \hat{P}_{A_0^1}(t)\right)}. \tag{4}$$

This is a quasi-projection since it is not an exact projector. If one were able to have an ideal clock, that is if the total Hilbert space may be written as the tensor product of the Hilbert space of the clock times the Hilbert space of the rest of the system, then the probability density given by,

$$\mathcal{P}_t(T) = \frac{{\rm Tr}|_{\rm cl}\left(\hat{P}_T(t)\rho_{\rm cl}\right)}{\int_0^\tau dt {\rm Tr}|_{\rm cl}(\hat{P}_T(t)\rho_{\rm cl})} \tag{5}$$

would be a Dirac delta $\mathcal{P}_t(T) = \delta(t-T)$ and (4) would behave as an exact projector when the reduction postulate is used to assign probabilities to histories [7]. $\mathcal{P}_t(T)$ is the probability density that the ideal time take the value t when the physical clock reads T, and is not a directly observable quantity in our framework (since t is not observable) but a mathematical object that appears in intermediate calculations.

This axiom only has epistemological character, it does not say that the state actually undergoes the above mentioned reduction process. In the present theory if the state does or does not undergo reduction is an undecidable proposition, as we will discuss in the next section.

Using the same construction as in ordinary quantum mechanics combining the reduction and the probability axioms one can assign probabilities to histories of events. In [5] we showed in model systems that the resulting probabilities of histories can be used to construct the ordinary particle propagator to leading order in the inaccuracy of the clock. This is true even for generally covariant systems like general relativity, resolving a longstanding issue in the definition of a notion of time for such systems.

Introducing a reduction postulate superficially seems to leave the measurement problem intact. Up to this point, the relational description of evolution does not provide information about when events occur. Notice that one cannot simply say that events happen randomly since generically they lead to a $\rho_{\rm red}$ that is physically distinguishable from ρ and that would completely destroy the predictive power of quantum mechanics. As Bell noted, this would be the situation in ordinary quantum mechanics if we adopted the language of events instead of that of measurements. The main difference in the current axiomatic system, as we will show, is that it allows situations where the events can occur and gives a physical criterion to establish when they occur. The next and final axiom will be crucial for this issue.

IV. AXIOM 7: THE ONTOLOGICAL AXIOM

A. Loss of unitarity due to the use of real clocks

In preparation to formulate the seventh axiom, we would like now to address a new phenomenon: the loss of unitarity of quantum mechanics described with real clocks. Let us

reconsider the conditional probability (3),

$$\mathcal{P}\left(Q^i \in \left[Q_0^i - \Delta^i, Q_0^i + \Delta^i\right] | T \in \left[T_0 - \Delta^C, T_0 + \Delta^C\right]\right) = \frac{\int_0^\tau dt \mathrm{Tr}\left(\hat{P}_{Q_0^i}(t)\hat{P}_{T_0}(t)\rho\hat{P}_{T_0}(t)\right)}{\int_0^\tau dt \mathrm{Tr}\left(\hat{P}_{T_0}(t)\rho\right)}, \tag{6}$$

and make some reasonable assumptions about the clock and the system as we discussed in section III. Going to the Schrödinger picture we define a new density matrix for the system excluding the clock labeled by the physical time T instead of the ideal Newtonian time t,

$$\rho_{\mathrm{sys}}(T) \equiv \int_0^\tau dt \mathcal{P}_t(T)\rho_{\mathrm{sys}}(t) \tag{7}$$

where $\mathcal{P}_t(T)$ was defined in (5). In terms of these density matrices the conditional probability can be rewritten as,

$$\mathcal{P}\left(Q^i \in \left[Q_0^i - \Delta^i, Q^i + \Delta^i\right] | T \in \left[T_0 - \Delta^C, T + \Delta^C\right]\right) = \frac{\mathrm{Tr}|_{\mathrm{sys}}\left(\hat{P}^S_{Q_0^i}\rho_{\mathrm{sys}}(T)\right)}{\mathrm{Tr}|_{\mathrm{sys}}\left(\rho_{\mathrm{sys}}(T)\right)}, \tag{8}$$

where $\hat{P}^S_{Q_0^i}$ is the projector in the Schrödinger picture. We therefore see that we have recovered the ordinary definition of probability of measuring Q^i at time T in usual quantum mechanics. This shows the usefulness of the definition (7). Within such definition one can immediately see the root of the loss of unitarity when one uses real clocks to describe quantum mechanics. The density matrix in the right hand side of (7) evolves unitarily in the ideal time t. However, due to the presence of the probability $\mathcal{P}_t(T)$ the left hand side does not evolve unitarily. If one starts with a pure state, in the right hand side it will remain pure, but in the left hand side after some time has evolved one will end up with a superposition of pure states due to the integral. Only if the probability $\mathcal{P}_t(T)$ were a Dirac delta one would have a unitary evolution. That would mean that one has a clock that correlates perfectly with t, which is not possible with a real clock.

We therefore see that the result of Axiom 5 is to have a theory that looks like ordinary quantum mechanics but in terms of the physical time T. The only difference is that the evolution in terms of the physical time is only approximately unitary. If one assumes that the clock is very good the probability $\mathcal{P}_t(T)$ will be a Dirac delta with small corrections,

$$\mathcal{P}_t(T) = \delta(T-t) + a(T)\delta'(T-t) + b(T)\delta''(T-t) + \ldots \tag{9}$$

and one can show that in such a case the density matrix evolves according to the equation,

$$i\hbar\frac{\partial\rho}{\partial T} = [H,\rho] + \frac{\partial b(T)}{\partial T}[H,[H,\rho]] \tag{10}$$

so we see that to leading order we get the ordinary Schrödinger evolution and the first corrective term has to do with the rate of spread of the width of the probability $\mathcal{P}_t(T)$ plus higher order corrections. Another way of putting it is that it is determined by how inaccurate the physical clock becomes over time. The effect can therefore be reduced by choosing clocks that remain as accurate as possible over time. However, there exist fundamental physical limitations as to how accurate one can keep a clock over time. There are several arguments in the literature [8] that suggest that the best accuracy one can achieve with a clock is given by $\delta T \sim T^a T_{\text{Planck}}^{1-a}$ and $T_{\text{Planck}} = 10^{-44}s$ is Planck's time. The estimates for a vary but several authors claim it is $1/3$. From the point of view of the purposes of this paper, it suffices to say that δT is a growing function of T. Then unitarity is inevitably lost.

The loss of coherence due to imperfect clocks makes the off diagonal elements of the density matrix of a quantum system in the energy eigen-basis to decrease exponentially. For $a = 1/3$, the exponent for the mn-th matrix element is given by $\omega_{mn}^2 T_{\text{Planck}}^{4/3} T^{2/3}$, where $\omega_{mn} = E_{mn}/\hbar$ is the difference of energy between levels m and n divided by $\hbar$ (the Bohr frequency between n and m). One could see this effect in the lab in reasonable times (hours) only if one were handling "macroscopic" quantum states corresponding to about 10^{13} atoms in coherence. The direct observation of this effect is therefore beyond our current experimental capabilities. However, it has profound implications at a foundational level, as this new formulation of quantum mechanics we are presenting attests to.

B. Undecidability

The loss of unitarity due to the inaccuracies of real clocks has implications for the usual explanation of the measurement process through environmental decoherence. The results of such program can be summarized as follows: consider a system $\mathscr{S}$ interacting with an environment $\mathscr{E}$ with a total Hamiltonian $H = H_{SA} + H_E + H_{int}$ with H_{SA} the Hamiltonian of the micro-system, which may include a measuring apparatus, H_E that of the environment and H_{int} the interaction Hamiltonian between the system and the environment. The effect of such interaction is an attenuation of the interference terms in the reduced density matrix of the system $\mathscr{S}$, obtained by partially tracing of the degrees of freedom of the environment. This effect happens in the so-called "pointer basis", determined by the Hamiltonian, as has been discussed in some detail in [9]. The interpretation of this attenuation is as follows:

when one carries out local measurements on the system $\mathscr{S}$ it will behave classically, any expectation value will be equal to the case in which the system has suffered a state reduction and we cannot see the typical interference terms of quantum superpositions. Since interactions with the environment are almost inevitable, this is the reason the everyday world we experience behaves classically and quantum effects can only be seen in very controlled circumstances in the lab. This is therefore portrayed as a solution to the measurement problem.

There exist three limitations that have been pointed out in the literature that may preclude everyone from accepting that environmental decoherence is a solution to the measurement problem. The first two limitations are related to the fact that the evolution for the total system $\mathscr{S}$ plus $\mathscr{E}$ is unitary. The first limitation is the possibility of *revivals*. That is, for a closed total system one could wait for a long time and see the quantum coherence in the system $\mathscr{S}$ plus the measuring apparatus reappear. The use of real clocks prevents this from happening, since waiting for very long actually increases the loss of coherence due to the clocks. The second limitation, suggested in [10], argues that one could perhaps construct global observables that depend on variables in the system and the environment that would suffer different changes in their expectation values if a reduction takes place or not. A detailed analysis [11] in model systems shows that one is prevented from measuring such observables when one takes into account the loss of coherence due to real clocks. The third limitation to viewing the use of environmental decoherence as a solution to the measurement problem is that "nothing happens", that is, there is no criterion given for telling when a measurement takes place. The fact that the reduced matrix of the open subsystem composed by the micro-system and the measurement device takes a diagonal form does not change the interpretation for the state as a superpositions of options. This is what Bell has called "the and/or problem" alluding to the lack of justification for assuming that a transition from superposed options to alternative options takes place. We will resolve this in our approach by providing a criterion for when an event takes place.

Returning to the first objection, one may ask how many degrees of freedom does one need to consider for the exponential decrease to kill the possibility of revivals? A criterion would be that the magnitude of the off diagonal term in the revivals be smaller than the magnitude of the off diagonal terms in the intermediate region between revivals. If that were the case the revival would be less than the "background noise" in regions where there is

no revival. The magnitude of the interference terms in the density matrix were studied by Zurek [12] in a simple model with two levels where the environment is characterized as N particles, and goes as $\rho_{+-} \sim 1/2^{N/2}$ with N the number of particles. The time for revivals to occur goes as $T \sim N!$. This implies, at least in this particular example, that if one has more than hundreds of particles in the environment the loss of coherence will make impossible the observation of revivals. In realistic environments the number of degrees of freedom is of course vastly higher.

At this point it is worthwhile emphasizing the robustness of this result in practical terms. One could, for instance, question how reliable the fundamental limits for the inaccuracy of clocks we are considering are. Some authors have characterized the fundamental limit as too optimistically large, arguing that the real fundamental limit should be larger of the order of Planck time itself. In view of this it is interesting to notice that if one posits a much more conservative estimate of the error of a clock, for instance $\delta T \sim T^{\epsilon} T_{\text{Planck}}^{1-\epsilon}$, for any small value of ϵ the only modification would be to change the number of particles $N_0 \sim 100$ to at least $N \sim N_0/(3\epsilon)$. So the only real requirement is that the inaccuracy of the clock increase with the time measured, a very reasonable property for any realistic clock.

Using a real clock introduces a fundamental difference. Whereas in the usual formalism the state of the complete system will evolve unitarily, here it will lose coherence without the possibility of recovering it in another part of the system. This brings us to the idea of *undecidability*. If a system suffers an interaction such that one cannot distinguish by any empirical means if a unitary evolution or a reduction took place we will claim that an *event took place*. This provides a criterion for the production of events, as we had anticipated. A detailed implementation for the criterion can be carried out by looking at the off diagonal elements of the density matrix as we did when we discussed the revivals. Notice that for a quantum micro-system in isolation, events would not occur. However for a quantum system interacting with an environment, events will be plentiful. The same goes for a system being measured by a macroscopic measuring device.

C. Axiom 7: The ontological axiom

The analysis of the previous section shows that contrary to what happens in quantum mechanics with an ideal clock, in the relational pictures the possibility to determine (not just

in practice but in principle) if a system has suffered a state reduction or evolved unitarily decreases exponentially with the number of degrees of freedom of the system. That is, it requires of ensembles with a number of identical macroscopic systems exponential in the number of degrees of freedom of the total system including environment and measuring apparatus. One cannot therefore argue —as is done in the case of ordinary environmental decoherence— that the problem moves on to the complete system that retains the complete initial quantum information. The existence of this phenomenon in systems that interact with an environment implies, as follows from the above analysis, that in processes where there does not exist an unlimited capability of preparing the initial state of the system it will be empirically undecidable if there has been or has not been a reduction of the state. By empirically undecidable we mean that no experimental setup will give different results for both situations. In particular, the expectation values of any observable of $\mathscr{S}$ will be identical in both cases.

This leads to the following ontological axiom that gives sufficient physical conditions for the production of an event. We lay it out for variables with continuous spectrum but it is readily generalizable to other cases. The axiom reads:

Given a closed system $\mathscr{S}$ with its associated Hilbert space H and a physical quantity $\mathscr{A}$ represented by an observable $\hat{A}$ in H with a decomposition of the identity allowing to write $\hat{A}(t) = \sum_n a_n \hat{P}_{a_n}(t)$, an event occurs when it becomes impossible to distinguish empirically in a certain instant in which the clock reads in an interval $2\Delta^C$ centered in T_0 between the initial state[2] *of $\mathscr{S}$ modified by the clock reading,*

$$\rho_{\rm mod} = \frac{\int_0^\tau dt \hat{P}_{T_0}(t)\rho\hat{P}_{T_0}(t)}{\int_0^\tau {\rm Tr}\left(\hat{P}_{T_0}(t)\rho\right)} \tag{11}$$

and,

$$\rho_{\rm e} = \frac{\int_0^\tau dt \sum_n \hat{P}_{a_n}(t)\hat{P}_{T_0}(t)\rho\hat{P}_{T_0}(t)\hat{P}_{a_n}(t)}{\int_0^\tau dt {\rm Tr}\left(\hat{P}_{T_0}(t)\rho\right)} \tag{12}$$

The event associated with the physical quantity $\mathscr{A}$ taking the value a_n occurs with a probability given by axiom 5. Such event will have a property associated with the projector $\hat{P}_{a_n}(t)$ with relative probability $\mathcal{P}_t(T_0)$. Notice that $\rho_{\rm e}$ is the density matrix that one would have

[2] Notice that we are in the Heisenberg representation. In the Schrödinger representation it would be the density matrix at time t modified.

after a traditional wavefunction collapse.

We are assuming that we have a good clock that works with certain degree of accuracy for a period of Newtonian time $\tau \gg T_0$. With this hypothesis the above construction is independent of tau. It is not possible to assign a single property to the observation of a_n since the clock does not allow to identify a single projector due to the ambiguity in the value of the ideal time t in which the property occurs. In realistic situations, with good clocks, such ambiguity does not have practical consequences since the variation of $\hat{P}_{a_n}(t)$ in the interval $\left[T_0 - \Delta^C, T_0 + \Delta^C\right]$ will be negligible.

As explained above, an event occurs when one cannot experimentally[3] distinguish between the modified density matrix $\rho_{\rm mod}$ and the one given by $\rho_{\rm e}$. This situation arises typically in systems that interact with an environment with a large number of degrees of freedom. When this happens the physical quantity characterized by A will take a definite value. As we have emphasized, $\mathscr{S}$ includes the micro-system and the environment with which it has interacted.

In general many observables will satisfy the above condition, and therefore many properties of the system will actualize. To illustrate this point we will consider a simplified situation. Let us assume that after the process of decoherence has been completed, the only Hamiltonian present is that of the clock and that the system does not evolve, so that we have time independent projectors,

$$\rho_e = \sum_n P_{a_n} \left(\frac{\int_0^\tau dt P_{T_0}(t) \rho P_{T_0}(t)}{\int_0^\tau dt \mathrm{Tr}\left(\hat{P}_{T_0}(t)\rho\right)} \right) P_{a_n} \equiv \sum_n P_{a_n} \rho(T_0) P_{a_n}, \tag{13}$$

and it should be noted that $\rho(T_0)$ is the density matrix of the *complete system*, in the Schrödinger picture labeled by the real clock time T_0. We will show that the condition for an observable B to also actualize is that its projectors' eigen-spaces include the eigen-spaces of $A's$ projectors. That is,

$$P_{b_n} P_{a_n} |\psi\rangle = P_{a_n} |\psi\rangle \tag{14}$$

and

$$P_{b_m} P_{a_n} |\psi\rangle = 0; \quad m \neq n. \tag{15}$$

[3] To be mathematically precise, given the states $\rho_{\rm mod}$ and $\rho_{\rm e}$ and any property of $\mathscr{S}$ given by a projector $\hat{P}$ one has that $|\mathrm{Tr}\left(P\left(\rho_{\rm mod} - \rho_{\rm e}\right)\right)| < \epsilon$ with $\epsilon = \exp(-\alpha N)$. α is a positive constant and N the number of particles in the system.

When the above conditions are satisfied we will say that the projector P_{a_n} includes P_{b_n}, and that the property corresponding to the first includes the second, $\mathscr{P}_{b_n} \subset \mathscr{P}_{a_n}$.

Let us assume that we have undecidability,

$$\rho_e = \sum_n P_{a_n} \rho(T_0) P_{a_n}, \tag{16}$$

then we will see that for observable B the undecidability condition is also satisfied.

Using the closure relationship we have that,

$$\sum_n P_{b_n} \rho(T_0) P_{b_n} = \sum_n P_{b_n} \left(\sum_k P_{a_k} \right) \rho(T_0) \left(\sum_l P_{a_l} \right) P_{b_n}, \tag{17}$$

and together with (15) imply,

$$\sum_n P_{b_n} \rho(T_0) P_{b_n} = \sum_n P_{b_n} P_{a_n} \rho(T_0) P_{a_n} P_{b_n}. \tag{18}$$

Now using (14) we have that,

$$\sum_n P_{b_n} \rho(T_0) P_{b_n} = \sum_n P_{a_n} \rho(T_0) P_{a_n} = \rho_e, \tag{19}$$

and therefore B is also undecidable.

We will call "essential property" the one that includes all properties that actualize, that is, all properties whose projectors satisfy the undecidability condition. This "essential property" contains the information of every physical quantity that the system acquires.

Let us see how this works more explicitly in a simple example. We will consider a system composed of only three spins, and the clock. Let us assume that the initial state for the spins is

$$\rho(0) = \frac{|c_1|^2}{2} \left(|++-\rangle + |+-+\rangle \right) \left(\langle++-| + \langle+-+| \right) + |c_2|^2 \left(|-++\rangle \right) \left(\langle-++| \right) \tag{20}$$

$$+ \frac{c_1 c_2^*}{\sqrt{2}} \left(|++-\rangle + |+-+\rangle \right) \left(\langle-++| \right) + \frac{c_1^* c_2}{\sqrt{2}} \left(\langle-++| \right) \left(\langle++-| + \langle+-+| \right). \tag{21}$$

Suppose that the evolution is such that an event occurs[4], with essential properties characterized by,

$$P_{a_1} = \left(|++-\rangle + |+-+\rangle \right) \left(\langle++-| + \langle+-+| \right), \tag{22}$$

[4] for small systems like the one we are considering events will not occur in general, since there is no undecidability.

and

$$P_{a_2} = (|-++\rangle)\,(\langle -++|)\,. \tag{23}$$

As we noticed before, if for instance the property given by P_{a_1} is attained, it gives all the information about the physical quantities the system has. We can now consider the compatible property associated with the projector,

$$P_{\text{up}} = |+\rangle\langle +| \otimes I_2 \otimes I_3, \tag{24}$$

which corresponds to "spin 1 is up". And we could also consider the compatible property associated to

$$P_{\text{2opposite3}} = I_1 \otimes (|+-\rangle + |-+\rangle)\,(\langle +-| + \langle -+|) \tag{25}$$

which corresponds to "spins 2 and 3 are opposite". Both P_{up} and $P_{\text{2opposite3}}$ satisfy condition (14), so these properties will actualize.

The projectors compatible with the essential properties determine the properties that can be associated to different subsystems. So, in the case of the property corresponding to P_{a_1} being acquired by the system, we can ask whether spin 1 is up or not, we can ask whether spins 2 and 3 are opposite or not, but we cannot ask whether spin 2 is up or not, because this last property is incompatible and is therefore not acquired by the subsystem.

Usually the essential property acquired by the system is complicated and not experimentally accessible, but we are interested in properties acquired by the subsystems when events occur.

The ontological axiom completes the formulation of the complete quantum mechanics. It eliminates the need to give special treatment to measurements and observers and gives rise to an objective description completely independent of cognizant beings.

The reader may question what is the situation in an actual measurement in the lab. There we have the possibility of forcing the occurrence of events by designing a measuring apparatus/environment combination that interacts with the system under study in such a way that the pointer basis corresponds to eigenstates of the observable one desires to measure. The effects discussed above occur and an event takes place. The measurements discussed in quantum mechanics textbooks therefore reduce to finding the correct Hamiltonians so that the properties that actualize their values correspond to the observables that one wishes to measure in each case.

V. THE ROLE OF STATES: DO THEY DESCRIBE SYSTEMS OR ENSEMBLES?

What happens with the states? As we observed, it is not empirically decidable what happens with the states when an event occurs. Although the interpretation is compatible with a state of the universe given once and for all, for practical purposes we will not have predictive power if we do not know all the actualizations of events prior to the moment of interest. Due to this it will be convenient and possible from the epistemological point of view to postulate that a reduction takes place after the observation of an event. As Omnès points out: "reduction is not in itself a physical effect but a convenient way of speaking" [13]. More precisely, in the construction presented in this paper it is not physically decidable if the reduction of the state takes place or not. This is precisely the condition, as established in axiom 7, for events to occur.

If it were the case that the state undergoes an effective reduction process every time an event occurs, then the state can be associated at all times to an individual system and knowledge of the state represents the maximum information available to make predictions about future behaviors.

If one adopts the opposite point of view and assumes that the state remains unchanged during the processes in which events occur, the state —which would be none other than the initial state of the universe— would describe ensembles of systems in which in every member of the ensemble events of different nature would occur. In this case in order to have complete information about the future behavior of the universe would require not only knowledge of the state but all the events that have occurred previously to the instant in which one wishes to have the information. It is important to notice here that the proposed formulation would be complete without axiom number 6. It only has the purpose of resolving the ambiguity noted above in order to use the information added by the occurrence of the event in future predictions. Axiom 6 is therefore, as we have mentioned, of epistemological character. It allows to actualize the information available after each measurement.

We have limited ourselves to closed systems. The systems have to be general enough to include the various subsystems involved in the occurrence of the events of interest. Some subsystems are agents that initiate the process, like the electron in the double-slit experiment. Others are recipients of the action, like the photographic plate in that experiment. The total systems will only allow a complete description of *some* processes that lead to events

in $\mathscr{S}$. We are able to describe events in which the system $\mathscr{S}$ contains as subsystems. the quantum micro-system, the environment and perhaps a measuring device. There might be situations in which subsystems of $\mathscr{S}$ act or are acted upon by subsystems not included in $\mathscr{S}$. Events and states have a primary ontological status whereas the systems considered here have circumstantial character and are considered as long as they support the events and states of interest.

VI. CONCLUSIONS

We have presented an axiomatic formulation of the Montevideo interpretation of quantum mechanics. In this interpretation environmental decoherence is supplemented with a fundamental mechanism of loss of coherence due to the inaccuracy in tracking time that real clocks introduce to produce a resolution to the measurement problem and a characterization of when events occur. The resulting construction is completely formulated in terms of quantum mechanical objects, without having to refer to any classical preferred quantity. That is the reason we dubbed it complete quantum mechanics. More work is needed in order to fill some gaps related with proofs of undecidability in more general contexts and including adiabatic couplings between the system and the clock.

The formulation is naturally geared towards dealing with generally covariant theories like quantum general relativity. It may also have implications for how the quantum to classical transition in cosmological perturbations in the inflationary period take place.

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[1] B. d'Espagnat, "On physics and Philosophy", Princeton University Press, Princeton (2006).

[2] A. Bassi and G. C. Ghirardi, Phys. Rep. **379**, 257 (2003) [arXiv:quant-ph/0302164].

[3] L. Smolin, "Space and time in the quantum universe" in "Conceptual Problems of Quantum Gravity", A. Ashtekar, J. Stachel (editors), Birkhäuser, Boston (1988)

[4] R. Gambini and J. Pullin, J. Phys. Conf. Ser. **174**, 012003 (2009) [arXiv:0905.4402 [quant-ph]].

[5] R. Gambini, R. A. Porto, J. Pullin and S. Torterolo, Phys. Rev. D **79**, 041501 (2009) [arXiv:0809.4235 [gr-qc]].

[6] C. Anastopoulos and B. L. Hu, J. Phys. Conf. Ser. **67**, 012012 (2007) [arXiv:0803.3447 [gr-qc]].

[7] R. Gambini, R. Porto and J. Pullin, Gen. Rel. Grav. **39**, 1143 (2007) [arXiv:gr-qc/0603090].

[8] F. Károlhyázy, A. Frenkel, B. Lukács in "Quantum concepts in space and time" R. Penrose and C. Isham, editors, Oxford University Press, Oxford (1986); Y. J. Ng and H. van Dam, Annals N. Y. Acad. Sci. **755**, 579 (1995) [arXiv:hep-th/9406110]; Mod. Phys. Lett. A **9**, 335 (1994); G. Amelino-Camelia, Measurability Of Space-Time Distances In The Semiclassical 3415 (1994) [arXiv:gr-qc/9603014]; S. Lloyd, J. Ng, Scientific American, November (2004).

[9] J. Paz, W. Zurek, Phys. Rev. Lett. **82**, 5181 (1999).

[10] B. d'Espagnat "Veiled reality", Addison Wesley, New York (1995).

[11] R. Gambini, L. P. G. Pintos and J. Pullin, Found. Phys. 40, 93 (2010) [arXiv:0905.4222 [quant-ph]].

[12] W. Zurek, Phys. Rev. **D26**, 1862 (1982).

[13] R. Omnés, "The interpretation of quantum mechanics", Princeton University Press, Princeton, NJ (1994).