COMPLETE RIEMANNIAN MANIFOLDS AND SOME VECTOR FIELDS

BY

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Introduction and theorems. In this paper we shall always deal with connected Riemannian manifolds with positive definite metric, and suppose that manifolds and quantities are differentiable of class C^∞. Let M be an n-dimensional Riemannian manifold with metric tensor field(1) $g_{\mu\lambda}$. We call a nonconstant scalar field ρ in M a *concircular scalar field*, or simply a *concircular field*, if it satisfies the equation

$$\nabla_\mu \nabla_\lambda \rho = \phi g_{\mu\lambda}, \tag{0.1}$$

where ∇ indicates covariant differentiation with respect to $g_{\mu\lambda}$ and ϕ is a scalar field, called the *characteristic function* of ρ. The term "concircular" comes from the concircular transformation introduced first by K. Yano [17]. A concircular transformation is by definition a conformal transformation preserving geodesic circles.

Concircular scalar fields and transformations appear often in the theory of transformations in Riemannian manifolds, see for instance [3], [14], [15], [17], [18], [20]. Therefore it might be interesting and important to study properties of a concircular scalar field and in particular to determine the structure of manifolds admitting such a field. We shall treat this problem in the first five paragraphs and apply the results to the determination of structure of product Riemannian manifolds admitting a conformal or projective infinitesimal transformation in the last three paragraphs.

We denote the number of isolated stationary points of a concircular scalar field ρ in M by N. After preliminaries are stated in §1, we shall prove in §2 the following

THEOREM 1. *If a complete Riemannian manifold M of dimension $n \geqq 2$ admits a concircular scalar field ρ, then $N \leqq 2$ and M is conformal to one of the following manifolds:*

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(1) As to notations we follow generally S. Ishihara and Y. Tashiro [4], J. A. Schouten [12] and K. Yano [19]. Some of our terminologies are different from Schouten's, in which a gradient vector ρ_λ satisfying (0.1) is called a special concircular vector. We shall use the term "concircular vector field" for an infinitesimal concircular transformation. Greek indices $\kappa, \lambda, \mu, \nu, \omega$ run from 1 to n.

(A) *if* $N=0$, *a direct product* $V\times J$ *of an* $(n-1)$*-dimensional complete Riemannian manifold* V *with an open interval* J *of a straight line,*

(B) *if* $N=1$, *an* n*-dimensional euclidean domain interior to an* $(n-1)$*-dimensional sphere, and consequently an* n*-dimensional hyperbolic space, and*

(C) *if* $N=2$, *an* n*-dimensional spherical space.*

By a spherical or hyperbolic space of curvature κ we mean a simply connected complete Riemannian manifold of positive or negative constant sectional curvature κ respectively.

If its characteristic function ϕ is of the form(2) $\phi=-k\rho+b$ with constant coefficients k and b, a concircular scalar field ρ is called a *special* one and k the *characteristic constant* of ρ. §3 is devoted to studies of special concircular fields, and we shall give the following

THEOREM 2(3). *Let* M *be a complete Riemannian manifold of dimension* $n \geqq 2$ *and suppose it admits a special concircular field* ρ *satisfying the equation*

$$\nabla_\mu \nabla_\lambda \rho = (-k\rho + b) g_{\mu\lambda}. \tag{0.2}$$

Then M *is one of the following manifolds:*

(I,A) *if* $k=b=0$, *the direct product* $V\times I$ *of an* $(n-1)$*-dimensional complete Riemannian manifold* V *with a straight line* I(4),

(I,B) *if* $k=0$ *but* $b\neq 0$, *a euclidean space*(5),

(II,A) *if* $k=-c^2<0$ *and* $N=0$, *a pseudo-hyperbolic space of zero or negative type*(6),

(II,B) *if* $k=-c^2<0$ *and* $N=1$, *a hyperbolic space of curvature* $-c^2$, *and*

(III) *if* $k=c^2>0$, *a spherical space of curvature* c^2,

where c *is a positive constant.*

For brevity, we say a vector field to be *isometric*, *homothetic*, *concircular* and so on according as the generated one-parameter group is isometric, homothetic, concircular and so on, respectively. Moreover, a vector field in M is said to be *complete* if it generates a global one-parameter group of transformations in M. In §§3 and 4, we shall concern ourselves with concircular vector fields and establish the following

(2) The minus sign in the right-hand side is put only for convenience.

(3) From this theorem follow Theorem 3 of [4] and Theorem 2 of [15]. However, in the discussions of [4], we missed case (II, A), which should be added as one of manifolds in the theorems. Fortunately, Theorem 4 of [4] and Theorem 3 of [15] are still true.

(4) This is a special case of de Rham's decomposition theorem [1].

(5) This case is discussed in S. Sasaki and M. Goto [11].

(6) The definition of a pseudo-hyperbolic space will be given in §3. A hyperbolic space is a special one of pseudo-hyperbolic spaces.

THEOREM 3. *If a complete Riemannian manifold M of dimension* $n \geqq 3$ *admits a nonisometric concircular vector field v, then M is one of the following manifolds*: (I) *a locally euclidean manifold*, (II,A) *a pseudo-hyperbolic space*, (II,B) *a hyperbolic space*, (III) *a spherical space, and* (IV) *a pseudo-euclidean space*(7).

As a consequence, we shall also have

THEOREM 4(8). *If a complete Riemannian manifold M of dimension* $n \geqq 3$ *admits a complete nonisometric concircular vector field, then M is a spherical space or a locally euclidean manifold. In the latter case, the vector field is homothetic.*

In §6, we shall consider a conformal vector field in a product Riemannian manifold and obtain the following theorem, the establishment of which is the main purpose of this paper.

THEOREM 5. *Let M be a complete product Riemannian manifold but not a locally euclidean one. If M admits a nonisometric conformal vector field v, then M consists of two irreducible parts,* $M = M_1 \times M_2$. *Moreover, if the dimension of one of the parts is* $\geqq 3$, *then*(9)

(i) *one of the parts is a spherical space and the other a pseudo-hyperbolic space, or*

(ii) *one is a spherical or pseudo-hyperbolic space and the other a straight line or a circle.*

If the pseudo-hyperbolic space is, in particular, hyperbolic in case (i), *the scalar curvatures of the parts are equal to each other to within reversed sign.*

In §7 we shall consider a decomposition of a conformal vector field in a product Riemannian manifold. In §8 we shall investigate a projective vector in a product Riemannian manifold, and prove

THEOREM 6(10). *If a complete, simply connected product Riemannian manifold M of dimension* $n \geqq 2$ *admits a nonaffine projective vector field, then M is a euclidean space.*

1. **Preliminaries.** We shall assume that manifolds treated in this paper are of dimension $n \geqq 3$, unless otherwise stated. Consider a concircular field ρ with characteristic function ϕ:

(7) A pseudo-euclidean space is a complete, conformally euclidean manifold whose definition will be given in §5.

(8) This theorem covers the case of a compact manifold discussed by S. Ishihara [5].

(9) Provided $n \geqq 5$, this assumption is satisfied. Moreover, if the scalar curvature κ of M is constant, the theorem is valid without the assumption.

(10) Cf. Y. Tashiro [14] and S. Tachibana [13].

(1.1) $$\nabla_\mu \nabla_\lambda \rho = \phi g_{\mu\lambda}.$$

If ϕ vanishes identically, the gradient vector field ρ_λ of ρ is parallel and we call ρ a *parallel scalar field*. If ϕ is a constant, then ρ_λ is a concurrent vector field and we call ρ a *concurrent scalar field*.

Applying the Ricci formula to (1.1), it follows that we may put $\partial_\lambda \phi = \psi \rho_\lambda$, ψ being a proportional factor, and hence we have

(1.2) $$K_{\nu\mu\lambda}{}^{\kappa}\rho_k = -\psi(\rho_\nu g_{\mu\lambda} - \rho_\mu g_{\nu\lambda}).$$

By virtue of a theorem due to A. Nijenhuis [9], we obtain

THEOREM 1.1(11). *If there exists a nonconcurrent concircular scalar field in a Riemannian manifold M, then the restricted homogeneous holonomy group of M is the special orthogonal group* SO(n).

The trajectories of the vector field $\rho^\kappa = \rho_\lambda g^{\lambda\kappa}$ are geodesic arcs except at stationary points of ρ, and a geodesic curve in M containing such an arc is called a *ρ-curve*. We can define a family of hypersurfaces by $\rho =$ constant except at stationary points. A connected component of such a hypersurface is called a *ρ-hypersurface*. Passing through an ordinary point P, there exist one ρ-curve and one ρ-hypersurface, which are denoted by $l(P)$ and $V(P)$ respectively. In a neighborhood of an ordinary point, the family of ρ-curves forms the normal congruence of ρ-hypersurfaces. By the same arguments as those in the proof of Theorem 1 in [4], we have

LEMMA 1.2. *Given a concircular field ρ in a Riemannian manifold M, we can choose a local coordinate system (u^κ) in a neighborhood of any ordinary point as follows. The first $n-1$ coordinates $u^1, \cdots, u^{n-1}$ belong to ρ-hypersurfaces, and the last coordinate u^n is equal to the arc length of ρ-curves. The concircular field ρ depends only on u^n. The metric form of M is given by*(12)

(1.3) $$ds^2 = (\rho')^2 f_{ji}(u^h) du^j du^i + (du^n)^2,$$

where the functions $f_{ji}(u^h)$ depend only on $u^1, \cdots, u^{n-1}$, and prime indicates ordinary derivative of ρ with respect to u^n.

Such a coordinate system $(u^\kappa) = (u^h, u^n)$ is called an *adapted* one. The ρ-hypersurfaces in a neighborhood are homothetic to each other and to an $(n-1)$-dimensional manifold V with $\overline{ds}^2 = f_{ji} du^j du^i$ as metric form. With respect to the system (u^κ), the Christoffel symbols

(11) Theorems 2 and 3 of S. I. Goldberg [3] follow immediately from this theorem.

(12) Latin indices h, i, j, k run from 1 to $n-1$.

$$\begin{Bmatrix} \kappa \\ \mu\lambda \end{Bmatrix}$$

of M are given by

$$\begin{aligned} &\begin{Bmatrix} h \\ ji \end{Bmatrix} = \overline{\begin{Bmatrix} h \\ ji \end{Bmatrix}}, \quad \begin{Bmatrix} n \\ ji \end{Bmatrix} = \begin{Bmatrix} n \\ ij \end{Bmatrix} = -\rho'\rho'' f_{ji} = -(\rho''/\rho')g_{ji}, \\ &\begin{Bmatrix} h \\ ni \end{Bmatrix} = \begin{Bmatrix} h \\ in \end{Bmatrix} = (\rho''/\rho')\delta_i^h, \quad \begin{Bmatrix} n \\ ni \end{Bmatrix} = \begin{Bmatrix} n \\ in \end{Bmatrix} = 0, \\ &\begin{Bmatrix} h \\ nn \end{Bmatrix} = \begin{Bmatrix} n \\ nn \end{Bmatrix} = 0, \end{aligned} \tag{1.4}$$

where

$$\overline{\begin{Bmatrix} h \\ ji \end{Bmatrix}}$$

is the Christoffel symbol of the metric tensor f_{ji} in V. The curvature tensor $K_{\nu\mu\lambda}{}^{\kappa}$ of M is given by

$$\begin{aligned} K_{kji}{}^h &= \bar{K}_{kji}{}^h - (\rho'')^2(\delta_k^h f_{ji} - \delta_j^h f_{ki}), \\ K_{nji}{}^n &= -K_{jni}{}^n = -\rho'\rho''' f_{ji}, \\ K_{njn}{}^h &= -K_{jnn}{}^h = (\rho'''/\rho')\delta_j^h, \end{aligned} \tag{1.5}$$

the other components being trivially zero, and the scalar curvature κ is given by

$$\kappa = [(n-2)\bar{\kappa} - (n-2)(\rho'')^2 - 2\rho'\rho''']/n(\rho')^2, \tag{1.6}$$

where $\bar{K}_{kji}{}^h$ and $\bar{\kappa}$ are the curvature tensor and the scalar curvature of V respectively. The gradient vector field ρ_λ has components $\rho'\delta_\lambda^n$, and its length $\|\rho\|$ is given by $|\rho'|$.

Let O be an isolated stationary point of ρ, if there is any, and W a spherical neighborhood of O, which contains no stationary point except O and any point of which can be joined to O by a unique geodesic arc. The characteristic function ϕ is a function of ρ in W and differentiable in ρ in the open domain $W - O$. Along any geodesic curve in W, the equation (1.1) is reduced to the ordinary differential equation

$$\frac{d^2\rho}{ds^2} = \phi(\rho), \tag{1.7}$$

s being the arc length of the geodesic. Every geodesic issuing from O is a ρ-curve, and every Riemannian hypersphere with center O in W is a ρ-hypersurface and an $(n-1)$-dimensional spherical space(13). Hence the metric $\overline{ds}^2 = f_{ji}du^j du^i$ of V is of positive constant scalar curvature $\bar{\kappa}$ and we put $\bar{\kappa} = \bar{c}^2$. If the arc length s of the ρ-curves issuing from O is measured from O and s tends to zero, then

(13) See Lemmas 1 and 5 of [4]. The proof of Lemma 1 in [4] was not exact in some point, which can be modified by the same reasoning as in the proof of Lemma 2.1 in this paper.

$\rho'(s)/s$ tends to a nonzero finite value equal to $\phi(O)$, in other words, $\rho'(s)$ is of the same order as s and differentiable at $s = 0$. Moreover it follows from (1.6) that

$$|\phi(O)| = |\rho''(0)| = \bar{c}. \tag{1.8}$$

2. **Complete manifolds admitting a concircular field.** *A* ρ-hypersurface V is connected and closed in M. For any point P of V, we denote by $P(s)$ the point lying on the ρ-curve $l(P)$ at distance s from P. Since ρ and the derivative $\rho'(s)$ along ρ-curves are constant on V, the field ρ takes same value $\rho(s)$ at the points $P(s)$ corresponding to all points P of V, by the uniqueness of solution of (1.7). If one of the ρ-curves has no stationary point, then all of them have no stationary point. Since M is complete and the ρ-curves are geodesic, any ρ-curve l is defined for all s belonging to the whole interval $(-\infty, +\infty)$. In this case we put $I = (-\infty, +\infty)$. If one of the ρ-curves meets a stationary point at distance s_1 from V at first and no stationary point in the opposite direction, then so do all of them. In this case we put $I = (s_1, +\infty)$. In the case where one of the ρ-curves, and consequently all of them, meet at first two stationary points at distances s_1 and s_2 in the opposite directions, we put $I = (s_1, s_2)$. The derivative $\rho'(s)$ does not vanish on the arc corresponding to I of a ρ-curve, and ρ is a monotone function of s. Therefore the arc is diffeomorphic to the interval I in either case. The points $P(s)$ with same value $s \in I$ all lie in a connected ρ-hypersurface, and hence we have a diffeomorphism v of the product $V \times I$ into M. In the first case where the ρ-curves from points of V have no stationary point, the image $v(V \times I)$ is obviously open and closed in M and therefore v is an onto-diffeomorphism. By means of Lemma 1.2, the metric form of M is expressed in the form

$$ds^2 = (\rho'(u^n))^2 \overline{ds}^2 + (du^n)^2, \qquad u^n \in I, \tag{2.1}$$

in $v(V \times I)$, where $\overline{ds}^2$ is the metric form of V.

Lemma 2.1. *If M is complete, then a stationary point of a concircular field ρ is isolated.*

Proof. If one of the ρ-curves issuing from points of V meets at first a stationary point O at distance s_1, then so do all of them. Since $\lim_{s \to s_1} \rho'(s) = 0$, all the ρ-curves meet each other at the stationary point O. Describe the Riemannian hypersphere V' with center O and radius s_1. It is easily seen that the intersection $V \cap V'$ is open in V', and V coincides with V' and hence O is isolated.

Q.E.D.

By means of the above discussions and by use of normal coordinate systems in spherical neighborhoods of stationary points, if there are any, we can state

LEMMA 2.2[14]. *If a complete Riemannian manifold M admits a concircular field ρ, then the number N of stationary points of ρ is $\leqq 2$ and M is diffeomorphic to one of the following manifolds*:

(A) *the direct product $V\times I$ of an $(n-1)$-dimensional complete Riemannian manifold V with a straight line I, if $N=0$,*

(B) *a euclidean space, if $N=1$, and*

(C) *a sphere, if $N=2$.*

The metric form of M is globally given by (2.1) except at stationary points, and the interval of u^n can be taken as $(0,+\infty)$ in case (B) and $(0,s_2)$ in case (C), s_2 being the distance between two stationary points.

Proof of Theorem 1. Case (A). Along the ρ-curves, we define a parameter r by

$$r(s) = \int_0^s \frac{ds}{|\rho'(s)|}, \qquad s\in I = (-\infty,+\infty), \tag{2.2}$$

which is a monotonely increasing function of s. Put $r_1=\lim_{s\to-\infty}r(s)$ and $r_2=\lim_{s\to+\infty}r(s)$, which may be the minus or plus infinities, and let J be the interval (r_1,r_2). Since we have $dr=ds/|\rho'(s)|$, the metric form of M is equal to

$$ds^2 = \rho'(u^n)^2(\overline{ds}^2 + dr^2), \qquad r\in J, \tag{2.3}$$

in the whole manifold M. Thus M is conformal to the direct product $V\times J$.

Case (B). Taking a value $s_1\in(0,+\infty)$ and using the positive constant $\bar{c}=\bar{\kappa}^{1/2}$, we define along the ρ-curve a parameter r by

$$r(s) = \exp \bar{c}\int_{s_1}^s \frac{ds}{|\rho'(s)|}, \qquad 0<s<+\infty, \tag{2.4}$$

which is a monotonely increasing function of s. By the fact that $\rho'(s)$ is of the same order as s and by (1.8), we can verify that $r(s)$ is of the same order as s when s tends to zero, and hence $\lim_{s\to 0}r(s)=0$. Putting $r_2=\lim_{s\to\infty}r(s)\leqq+\infty$, the parameter r varies in the interval $[0,r_2)$ as s varies in $[0,+\infty)$. Since $dr/r=\bar{c}\,ds/|\rho'(s)|$, the metric form of M is equal to

$$ds^2 = \left(\frac{\rho'(u^n)}{\bar{c}r(u^n)}\right)^2[\bar{c}^2r^2\overline{ds}^2 + dr^2], \qquad 0<r<r_2, \tag{2.5}$$

in $M-O$. Since $\bar{c}\overline{ds}$ is the metric of an $(n-1)$-dimensional unit sphere, the expression in the brackets of (2.5) is the polar form of an n-dimensional euclidean metric. On the other hand, taking account of the order of $\rho'(s)$ and $r(s)$, we can see that the coefficient $\rho'(s)/r(s)$ is not equal to zero but it is differentiable at O.

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(14) Cf. T. Maebashi [8].

Thus M is conformal to the euclidean domain interior to an $(n-1)$-sphere with radius r_2.

Case (C). By (1.8), we have $|\phi(O)| = |\phi(O')| = \bar{c}$ at the stationary points O and O'. Putting $s_2 = 2s_1$ for the distance between O and O', along the ρ-curves, we define a parameter θ by

$$\theta(s) = 2\arctan\exp\bar{c}\int_{s_1}^{s}\frac{ds}{|\rho'(s)|}, \qquad 0 < s < s_2, \tag{2.6}$$

which is a monotonely increasing function of s and has values $\lim_{s\to 0}\theta(s) = 0$, $\theta(s_1) = \pi/2$, $\lim_{s\to s_2}\theta(s) = \pi$. Hence θ varies in the closed interval $[0,\pi]$ as s varies in $[0,s_2]$. Since $d\theta/\sin\theta = \bar{c}\,ds/|\rho'(s)|$, the metric form of M is given by

$$ds^2 = \left(\frac{\rho'(u^n)}{\bar{c}\sin\theta(u^n)}\right)^2[(\sin\theta)^2\bar{c}^2\overline{ds}^2 + d\theta^2], \qquad 0 < \theta < \pi, \tag{2.7}$$

in $M - O - O'$. The expression in the brackets is the polar form of the metric of an n-dimensional unit sphere. Noticing the order of $\rho'(s)$ and $\theta(s)$, we can verify that the factor $\rho'(s)/\sin\theta(s)$ is not equal to zero but differentiable at both the stationary points O and O'. Therefore M is conformal to an n-dimensional spherical space. Q.E.D.

In the case of a two-dimensional Riemannian manifold M, we can develop arguments in just the same way as in the theory of surfaces in an ordinary euclidean space. By means of Gauss' theorem(15), the metric form of M is given by

$$ds^2 = (\rho'(u^2))^2(du^1)^2 + (du^2)^2. \tag{2.8}$$

Therefore the results in this paragraph and Theorem 1 are also valid for two-dimensional manifolds. In particular, we notice that the Gaussian curvature of M is given by

$$\kappa = K_{1221}/g_{11} = -\rho'''/\rho'. \tag{2.9}$$

3. **Special concircular scalar field.** Now we consider a special concircular scalar field ρ satisfying the equation

$$\nabla_\mu\nabla_\lambda\rho = (-k\rho + b)g_{\mu\lambda}. \tag{3.1}$$

Along any geodesic curve l with arc length s, this equation is reduced to the ordinary differential equation

$$\frac{d^2\rho}{ds^2} + k\rho = b. \tag{3.2}$$

According to the signature of the characteristic constant k, we put

$$k = 0 \ \ \text{(I)}, \ -c^2 \ \ \text{(II)}, \ \ c^2 \ \ \text{(III)}, \tag{3.3}$$

(15) See, for instance, [2, p. 174].

c being a positive constant. Choosing suitably the arc length s, the solution of (3.2) is given by

$$\rho(s) = \begin{cases} \text{(I,A)} & as \ (b=0), \\ \text{(I,B)} & \tfrac{1}{2}bs^2 + a \ (b \neq 0), \\ (\text{II},\text{A}_0) & a \exp cs - b/c^2, \\ (\text{II},\text{A}_-) & a \sinh cs - b/c^2, \\ \text{(II,B)} & a \cosh cs - b/c^2, \\ \text{(III)} & a \cos cs + b/c^2, \end{cases} \tag{3.4}$$

a being an arbitrary constant. If l is in particular a ρ-curve, then the length $\|\rho\|$ of the gradient vector field ρ_λ is given by

$$\|\rho\| = |\rho'| = \begin{cases} \text{(I,A)} & |a|, \\ \text{(I,B)} & |bs|, \\ (\text{II},\text{A}_0) & |ac \exp cs|, \\ (\text{II},\text{A}_-) & |ac \cosh cs|, \\ \text{(II,B)} & |ac \sinh cs|, \\ \text{(III)} & |ac \sin cs| \end{cases} \tag{3.5}$$

along l, in the respective cases. If M is complete, then there is no stationary point in cases (I,A), (II,A_0) or (II,A_-), one corresponding to $s=0$ in cases (I,B) or (II,B), and two corresponding to $s=0$ and $s=\pi/c$ in case (III), respectively.

Proof of Theorem 2. In the cases belonging to case (A), $N=0$, we choose the arc length s of ρ-curves such that the points corresponding to $s=0$ lie on the same ρ-hypersurface, and then the coefficient a is same for all ρ-curves. Taking s as the nth coordinate u^n, it follows from (2.1) that the metric form of M is given by

$$ds^2 = \begin{cases} \text{(I,A)} & a^2\overline{ds}^2 + (du^n)^2, \\ (\text{II},\text{A}_0) & (ac \exp cu^n)^2\overline{ds}^2 + (du^n)^2, \\ (\text{II},\text{A}_-) & (ac \cosh cu^n)^2\overline{ds}^2 + (du^n)^2. \end{cases} \qquad u^n \in I, \tag{3.6}$$

A complete Riemannian manifold, which is topologically a product $V \times I$ and has the metric form (II,A_0) or (II,A_-) of (3.6), is called a *pseudo-hyperbolic space of zero* or *negative type* respectively.

In the cases belonging to (B), $N=1$, we have

$$\bar{c} = |\phi(O)| = \begin{cases} \text{(I,B)} & |b|, \\ \text{(II,B)} & |a|c^2 \end{cases} \tag{3.7}$$

by (1.8). The parameter r defined by (2.4) is equal to

$$r(s) = \begin{cases} (\mathrm{I,B}) & s/B, \\ (\mathrm{II,B}) & (2/B)\tanh(cs/2), \end{cases} \qquad 0 \leqq s < \infty, \tag{3.8}$$

where B is a positive constant and the factor 2 in (II, B) is put only for convenience. The interval of $r(s)$ is $[0, +\infty)$ in case (I, B) or $[0, 2/B)$ in case (II, B). By (3.4) and (3.8), we have

$$\rho'(s) = \begin{cases} (\mathrm{I,B}) & Bbr, \\ (\mathrm{II,B}) & Bacr/(1 - Br^2/4). \end{cases} \tag{3.9}$$

By (2.5), (3.7) and (3.9), the metric form of M is given by

$$ds^2 = \begin{cases} (\mathrm{I,B}) & B^2(r^2\bar{c}^2\overline{ds}^2 + dr^2), \\ (\mathrm{II,B}) & \dfrac{B^2}{c^2}\dfrac{1}{(1-B^2r^2/4)^2}(r^2\bar{c}^2\overline{ds}^2 + dr^2). \end{cases} \tag{3.10}$$

Since the expressions in parentheses are the polar forms of a euclidean metric, the first metric form itself is also euclidean. The second expression without the constant factor B^2/c^2 is a hyperbolic metric of curvature $-B^2$ and consequently the second itself is a hyperbolic one of curvature $-c^2$.

In case (III) belonging to (C), we have $\bar{c} = |\phi(O)| = |a| c^2$ by (1.8). The parameter θ defined by (2.6) with $s_1 = \pi/2c$ is equal to $\theta = cs$, and we have $\rho'(s) = -ac\sin\theta$. Therefore the metric form of M is given by

$$ds^2 = (1/c^2)[(\sin\theta)^2\bar{c}^2\overline{ds}^2 + d\theta^2], \tag{3.11}$$

which is a spherical metric of curvature c^2. Q.E.D.

It is to be noticed that the arbitrary constant a in (3.4) has no effect on the structure of the manifolds and the constant b has also no effect in case (I, B).

In the case of a two-dimensional manifold M, the equation (3.1) is also reduced to (3.2) in an adapted coordinate system. Hence, by (2.9), the Gaussian curvature κ is equal to the characteristic constant k and the manifold M is of constant curvature 0 in case (I), $-c^2$ in case (II) or c^2 in case (III). Since a complete one-dimensional manifold is a straight line or a circle, M is diffeomorphic to a euclidean plane or a cylinder in the cases belonging to (A). Theorem 2 is therefore sharpened in such a way that M is a plane or a cylinder with euclidean metric in case (I,A) or with a hyperbolic metric in case (II,A).

From the equations (1.5), we can easily prove the following.

LEMMA 3.1. *In order that a pseudo-hyperbolic space M is locally hyperbolic, i.e., of negative constant sectional curvature, it is necessary and sufficient that the ρ-hypersurfaces are locally euclidean if M is of zero type in case* (II,A_0), *or locally hyperbolic if M is of negative type in case* (II,A_-).

The gradient vector field ρ^κ of a special concircular field ρ is parallel in case (I,A) or concurrent in case (I,B), that is, it is a special homothetic vector field. If M is complete, then the vector field is complete [10], [21]. It is complete in case (III), too, because M is compact. Conversely we can prove the following.

LEMMA 3.2. *If M is complete and the gradient vector field ρ^κ of a special concircular field ρ in M is complete, then the characteristic constant k should not be negative and M is one of the manifolds of* (I,A), (I,B) *and* (III) *of Theorem* 2.

Proof. The canonical parameter t of the vector field ρ^κ is defined by the equation

$$ds/dt = \rho'(s) \tag{3.12}$$

along a trajectory of ρ^κ, which is a ρ-curve. We may suppose $a = 1$ in case (I,A), $b = 1$ in case (I,B) and $a = 1/c$ in cases (II) and (III). By integrating (3.12) substituted from (3.5) and expressing ρ as function of t, we have

$$\rho = \begin{cases} \text{(I,A)} & t - t_0, \\ \text{(I,B)} & (1/2)\exp 2(t - t_0) + a, \\ (\text{II},\text{A}_0) & -1/[c^2(t - t_0)] - b/c^2, \\ (\text{II},\text{A}_-) & (1/c)\cot c(t - t_0) - b/c^2, \\ \text{(II,B)} & -(1/c)\tan c(t - t_0) - b/c^2, \\ \text{(III)} & -(1/c)\tanh c(t - t_0) + b/c^2, \end{cases} \tag{3.13}$$

t_0 being an arbitrary constant. Hence ρ has a singularity corresponding to $t = t_0$ in case (II,A_0) or (II,A_-) or to $t = t_0 + \pi/2c$ in case (II,B) on any ρ-curve. Q.E.D.

4. **Concircular vector field having non-euclidean ρ-hypersurfaces.** We denote by $£_v$ Lie differentiation with respect to a vector field v. A conformal vector field, or an infinitesimal conformal transformation, $v = (v^\kappa)$ is characterized by the equation

$$£_v g_{\mu\lambda} = \nabla_\mu v_\lambda + \nabla_\lambda v_\mu = 2\rho g_{\mu\lambda}, \tag{4.1}$$

ρ being a scalar field, called the associated scalar field with v. A conformal vector field v is *homothetic* if ρ is a constant, and *concircular* if it is concircular [18], [5],

$$\nabla_\mu \nabla_\lambda \rho = \phi g_{\mu\lambda}. \tag{4.2}$$

In an Einstein manifold M with Ricci tensor $K_{\mu\lambda} = (n-1)\kappa g_{\mu\lambda}$, a conformal vector field v is concircular and its associated scalar field ρ is a special concircular one having κ as characteristic constant [20],

$$\nabla_\mu \rho_\lambda = -\kappa\rho g_{\mu\lambda}. \tag{4.3}$$

In particular, if M is locally euclidean, ρ is parallel. We know a theorem due to S. Kobayashi [6] and K. Yano and T. Nagano [21], which will play an important role in our discussions.

THEOREM 4.1. *If a complete Riemannian manifold M of dimension $n \geqq 2$ admits a nonisometric homothetic vector field, then M is locally euclidean.*

Now we consider a concircular vector field v in a manifold M of dimension $n \geqq 3$. If the associated scalar field ρ is not constant, then there is an adapted coordinate system (u^κ) where the metric form of M is given by (1.3). By use of (1.4), the equation (4.1) is written separately as

$$
\begin{aligned}
\bar\nabla_j v_i + \bar\nabla_i v_j &= 2(\rho\rho'^2 - \rho'\rho'' v_n) f_{ji},\\
\partial_n v_i + \partial_i v_n &= 2(\rho''/\rho') v_i,\\
\partial_n v_n &= \rho,
\end{aligned} \tag{4.4}
$$

$\bar\nabla$ denoting covariant differentiation with respect to

$$\overline{\begin{Bmatrix} h \\ ji \end{Bmatrix}}$$

belonging to ρ-hypersurfaces and prime indicating ordinary derivatives with respect to the nth coordinate u^n. The components v^h restricted on each ρ-hypersurface V together define a vector field on V, which will be called the *restriction* of v on V and denoted by $\bar v$. The covariant components, denoted by ξ_i, of $\bar v$ with respect to f_{ji} are related to v_i by $v_i = \rho'^2 \xi_i$, and we have from (4.4) equations

$$
\begin{aligned}
\bar\nabla_j \xi_i + \bar\nabla_i \xi_j &= 2[\rho - (\rho''/\rho') v_n] f_{ji},\\
\partial_n \xi_i + (1/\rho')^2 \partial_i v_n &= 0.
\end{aligned} \tag{4.5}
$$

The first equation shows that the restriction $\bar v$ on each ρ-hypersurface V is conformal with respect to the metric $\overline{ds}$.

LEMMA 4.2. *Let M be a complete Riemannian manifold. If v is a concircular vector field with parallel scalar field ρ, then v is isometric unless M is locally euclidean.*

Proof. By virtue of Theorem 2 (I,A), the manifold M is the direct product of a complete Riemannian manifold V with a straight line I. It follows from $(4.5)_1$(16) that we have

$$\bar\nabla_j \xi_i + \bar\nabla_i \xi_j = 2\rho f_{ji} \tag{4.6}$$

(16) Parentheses suffixed with numbers such as $(\ \cdot\ )_1$, $(\ \cdot\ )_2$ and so on indicate to refer the first, second and so on of the equations $(\ \cdot\ )$.

on each ρ-hypersurface. Since ρ is independent of u^h belonging to V, the restriction $\bar{v}$ is a homothetic vector field in each ρ-hypersurface. However, by means of Theorem 4.1 and completeness of V, $\bar{v}$ is isometric and ρ vanishes in each ρ-hypersurface unless V is locally euclidean. Hence ρ vanishes identically unless $M = V \times I$ is locally euclidean. Q.E.D.

LEMMA 4.3. *If a complete Riemannian manifold M admits a nonhomothetic concircular vector field v with associated scalar field ρ and the ρ-hypersurfaces are not locally euclidean, then ρ is a special concircular field with nonzero characteristic constant.*

Proof. Differentiating $(4.5)_1$ in u^n, taking account of $4.4)_3$ and eliminating v_n, we obtain

$$\bar{\nabla}_j \eta_i + \bar{\nabla}_i \eta_j = 2(\rho\rho''' - \rho'\rho'')f_{ji}, \tag{4.7}$$

where we have put

$$\eta_i = (\rho'\rho''' - \rho''^2)\xi_i/\rho' - \rho''\partial_n \xi_i.$$

The $n-1$ functions η_i on each ρ-hypersurface V together define a vector field in V, which is isometric unless V is locally euclidean by the same argument as that in the proof of Lemma 4.2. Thus we have $\rho\rho''' - \rho'\rho'' = 0$, or $\rho'' = -k\rho$, k being a constant. Since the stationary point of a concircular field in a complete manifold is isolated by Lemma 2.1, the equation holds along the whole of any ρ-curve and it is also rewritten as the tensor equation

$$\nabla_\mu \nabla_\lambda \rho = -k\rho g_{\mu\lambda} \tag{4.8}$$

in a general coordinate system. Thus ρ is a special concircular field with $b = 0$ in (3.1). If $k = 0$, then ρ is parallel and the vector field v is reduced to an isometric one by virtue of Lemma 4.2. Q.E.D.

We notice that manifolds of constant scalar curvature κ belong to the above case even if the ρ-hypersurfaces are locally euclidean, because we can easily obtain (4.3) for such manifolds, too, from formulas of Lie derivatives.

LEMMA 4.4([17]). *If its associated scalar field ρ is a special one with $k \neq 0$, the concircular vector field v is decomposed into*

$$v^\kappa = w^\kappa - \rho^\kappa / k, \tag{4.9}$$

where w^k is an isometric vector field and ρ^k is the gradient vector field of ρ. If M is complete and v is complete, then so is ρ^κ.

Proof. Putting $w^\kappa = v^\kappa + \rho^\kappa/k$, it follows from (4.1) and (4.8) that

([17]) Since a conformal vector field in an Einstein manifold is concircular, this is a generalization of A. Lichnerowicz' theorem [7].

$\pounds_w g_{\mu\lambda} = \nabla_\mu w_\lambda + \nabla_\lambda w_\mu = 0$ and hence w is isometric. The last part is clear because an isometric vector field in a complete manifold is complete [10]. Q.E.D.

5. **Concircular vector field having locally euclidean ρ-hypersurfaces.** The ρ-hypersurfaces may be locally euclidean only in the cases where the associated scalar field ρ is constant in M or it has no stationary point. In the former case the vector field is homothetic and M itself is locally euclidean.

In the latter case, the manifold M is diffeomorphic to the product $V \times I$ of a locally euclidean manifold V with a straight line I, and its metric form is globally given by (2.1) in which $\overline{ds}^2 = f_{ji}du^j du^i$ is the locally euclidean metric of V. By the change of parameters defined by (2.2), the metric form is also expressed as

$$ds^2 = \rho'(r)(\overline{ds}^2 + dr^2), \tag{5.1}$$

and M is therefore conformal to the direct product $\bar{M} = V \times J$ of V with an interval $J = (r_1, r_2)$. The product $\bar{M}$ is locally euclidean.

In the following, prime indicates the derivatives of ρ with respect to the parameter enclosed in parentheses. Along ρ-curves with arc length s, we have $\rho'(r) = \rho'(s)^2$. In a local coordinate system (u^h, r), the Christoffel symbol of M has components

$$\begin{Bmatrix} h \\ ji \end{Bmatrix} = \overline{\begin{Bmatrix} h \\ ji \end{Bmatrix}}, \quad \begin{Bmatrix} h \\ ni \end{Bmatrix} = \frac{1}{2}\frac{\rho''(r)}{\rho'(r)}\delta_i^h, \quad \begin{Bmatrix} h \\ nn \end{Bmatrix} = 0,$$
$$\begin{Bmatrix} n \\ ji \end{Bmatrix} = -\frac{1}{2}\frac{\rho''(r)}{\rho'(r)}f_{ji}, \quad \begin{Bmatrix} n \\ ni \end{Bmatrix} = 0, \quad \begin{Bmatrix} n \\ nn \end{Bmatrix} = \frac{1}{2}\frac{\rho''(r)}{\rho'(r)}, \tag{5.2}$$

where

$$\overline{\begin{Bmatrix} h \\ ji \end{Bmatrix}}$$

is the Christoffel symbol composed from the euclidean metric of V. The covariant components, denoted by ξ_λ, of the vector field v with respect to the euclidean metric of $\bar{M}$ are $\xi_i = f_{ih}v^h$ and $\xi_n = v^n$, which are related with v_λ by the equation

$$v_\lambda = \rho'(r)\xi_\lambda. \tag{5.3}$$

In the system (u^h, r) the equation (4.1) is decomposed into

$$\begin{aligned} \nabla_j v_i + \nabla_i v_j &= \bar{\nabla}_j v_i + \bar{\nabla}_i v_j + \frac{\rho''(r)}{\rho'(r)} f_{ji} v_n = 2\rho\rho'(r) f_{ji}, \\ \nabla_n v_i + \nabla_i v_n &= \partial_n v_i + \partial_i v_n - \frac{\rho''(r)}{\rho'(r)} v_i = 0, \\ \nabla_n v_n &= \partial_n v_n - \frac{1}{2}\frac{\rho''(r)}{\rho'(r)} v_n = \rho\rho'(r). \end{aligned} \tag{5.4}$$

Substituting (5.3) into these equations, we have

$$(5.5)\qquad \begin{aligned} \bar\nabla_j\xi_i + \bar\nabla_i\xi_j &= 2[\rho - (\rho''(r)/2\rho'(r))\xi_n]f_{ji},\\ \partial_n\xi_i + \partial_i\xi_n &= 0,\\ \partial_n\xi_n &= \rho - (\rho''(r)/2\rho'(r))\xi_n. \end{aligned}$$

These equations mean that the vector field v is conformal as a vector field in the locally euclidean manifold $\bar M$, and hence its associated scalar field is parallel:

$$(5.6)\qquad \bar\nabla_\mu\bar\nabla_\lambda[\rho(r) - (\rho''(r)/2\rho'(r))\xi_n] = 0,$$

where $\bar\nabla$ indicates covariant differentiation in $\bar M$ and its part belonging to V coincides with that of V. Putting $\lambda = i$, $\mu = j$ in (5.6), we have $\bar\nabla_j\bar\nabla_i\xi_n = 0$ unless

$$(5.7)\qquad \rho''(r) = 0.$$

Differentiating covariantly $(5.5)_2$ in $\bar M$, we have $\partial_n\nabla_j\xi_i = 0$ and, from the derivative of $(5.5)_1$ in r,

$$\rho'(r)[2\rho'(r)^2 - \rho\rho''(r)] = [\rho'(r)\rho'''(r) - (3/2)\rho''(r)^2]\xi_n.$$

This equation means that ξ_n depends only on r, unless

$$(5.8)\qquad \rho'(r)\rho'''(r) - (3/2)\rho''(r)^2 = 2\rho'(r)^2 - \rho\rho''(r) = 0.$$

Using again $(5.5)_2$, ξ_i are independent of r, and we see from $(5.5)_1$ that the coefficient of the right-hand side is equal to a constant, say

$$(5.9)\qquad \rho - (\rho''(r)/2\rho'(r))\xi_n = A.$$

Thus the vector field v is homothetic as a vector field in $\bar M$.

We say here a point $P \in J$ to be of the *first* or *second kind* according as (5.7) or (5.8) holds at P, and other point to be of the *third kind*. The sets of points of the first or the second kind are closed and the set of points of the third kind is open in J. Since $\rho'(r)$ does not vanish anywhere in J, the set of points of the first kind does not intersect with that of points of the second kind.

If the set of points of the first kind contains an interval, then we have there $\rho'(r) = a^2$, or

$$(5.10)\qquad \rho(r) = a^2 r + b,$$

a being a positive constant and b an arbitrary constant. Similarly, if the set of points of the second kind contains an interval, then we have there $\rho'(r) = c^2\rho^2$, or

$$(5.11)\qquad \rho(r) = -1/(c^2 r + d),$$

c being a positive constant and d an arbitrary constant. In an interval of points of the third kind, we have from $(5.5)_3$ and (5.9)

$$(5.12)\qquad \xi_n = Ar + B,$$

and, substituting (5.12) again into (5.9),

$$(5.13)\qquad (Ar+B)\rho''(r) = 2\rho\rho'(r) - 2A\rho'(r)$$

or

$$(5.14)\qquad (Ar+B)\rho'(r) = \rho^2 - A\rho + C,$$

B and C being constants.

When an interval of points of the first or the second kind is in contact with an interval of points of the third kind, we may suppose that the contact point is corresponding to $r = 0$. By substituting successive derivatives of (5.10) or (5.11) at $r = 0$ into (5.13) and (5.14), we can easily prove the following

LEMMA 5.1. *The set of points of the first or the second kind coincides with the whole interval J or has no interior point. The case of $C = 0$ in* (5.14) *is reduced to the second kind.*

Therefore the equation (5.10) holds in the whole interval J in the case of the first kind, and (5.11) in the case of the second kind. Also, in the case of the third kind, ξ_n depends only on r in the whole manifold and we have (5.14) everywhere.

For the first kind, we have $\rho'(s) = a$, that is, ρ is parallel and M itself is locally euclidean. For the second kind, we have $\rho'(s) = c\rho$, or $\rho = \exp cs$ by choosing suitably the parameter s. This is a special one of case $(\mathrm{II}, \mathrm{A}_0)$ in Theorem 2, where ρ-hypersurfaces are locally euclidean. By virtue of Lemma 3.1, M is locally hyperbolic of curvature $-c^2$.

For the third kind, we have $C \neq 0$. If $A = 0$ in (5.9), then we have $\xi_n = B$ and $B\rho'(s)^2 = \rho^2 + C$. Since ρ has no stationary point, ρ is a special concircular one given by $(\mathrm{II}, \mathrm{A}_-)$ of (3.4), and we have $\rho = a \sinh cs$, where $B = 1/c^2$ and $C = a^2$. Therefore M is a pseudo-hyperbolic space of negative type with locally euclidean ρ-hypersurfaces.

If $A \neq 0$, then we may choose the parameter r such as $\xi_n = Ar$. The equation (5.14) is reduced to

$$(5.15)\qquad Ar\rho'(r) = (\rho - A/2)^2 + D,$$

where $D = C - A^2$. Since $v^h = f^{hi}\xi_i$ are independent of r, the orthogonal projection of the trajectories of v of $\bar{M}$ into the interval J is given by the differential equation

$$(5.16)\qquad dr/dt = v^n = Ar$$

with respect to the canonical parameter t of the vector field v. Hence we have

$$(5.17)\qquad \log r = At$$

with initial condition $t = 0$ for $r = 1$. Putting $t_1 = (\log r_1)/A$ and $t_2 = (\log r_2)/A$, the interval $L = (t_1, t_2)$ corresponds to $J = (r_1, r_2)$. Substituting (5.16) into (5.15), we have the equation

$$(5.18)\qquad \rho'(t) = (\rho - A/2)^2 + D,$$

whose solution is given by

$$(5.19)\qquad \rho - A/2 = \begin{cases} -1/(t-t_0) & \text{if } D = 0, \\ c\tan c(t-t_0) & \text{if } D = c^2, \\ c\coth c(t-t_0) & \text{if } D = -c^2 \text{ and } |\rho > c, \\ c\tanh c(t-t_0) & \text{if } D = -c^2 \text{ and } |\rho| < c, \end{cases}$$

c being a positive constant and t_0 an arbitrary constant. From these expressions, we can obtain four kinds of expressions of $\rho'(r)$ and see that the manifold M is conformal to the locally euclidean manifold $\bar{M} = V \times J$ and the metric form is given by (5.1) with one of the expressions of $\rho'(r)$. We call such conformally euclidean and complete manifolds *pseudo-euclidean spaces.*

Theorem 3 follows from Theorem 2, Lemma 4.3 and the above discussions.

Proof of Theorem 4. In the first four cases of Theorem 3, the theorem follows from Theorem 4.1, Lemmas 3.2, 4.2, 4.4, and the above discussions. Hence we have only to prove that a concircular vector field inducing a pseudo-euclidean structure in a complete manifold cannot be complete. If a concircular vector field v is complete in M, then so is it in $\bar{M}$ because $\bar{M}$ has the same underlying manifold as M. In order that the equation (5.17) is valid for the whole interval $L = (-\infty, +\infty)$ of the canonical parameter t, the interval J of r must be the open half interval $(0, +\infty)$. Then t may be regarded as an ordinate of $\bar{M}$, and ρ is given by (5.19). In the first three cases, ρ has a singular point at $t = t_0$, $t_0 + \pi/2c$ or t_0 respectively. In the fourth case, we proceed as follows: From (5.15)–(5.19), we have

$$(ds/dt)^2 = -A(\exp At)c^2 \operatorname{sech}^2 c(t-t_0).$$

Since $\rho'(r) = \rho'(s)^2$ is positive, A should be negative and $\exp At/2 < 1$ for $t \geqq 0$. Thus we have

$$\begin{aligned} s - s_0 &< c(-A)^{1/2} \int_0^t \operatorname{sech} c(t-t_0)dt \\ &= (-A)^{1/2} \arctan\sinh c(t-t_0), \end{aligned}$$

s_0 being a constant. Hence, as t tends to the infinity, $s - s_0$ tends to a finite value less than $(-A)^{1/2}\pi/2$. This contradicts to the existence of the homeomorphisms $I = (-\infty, +\infty) \leftrightarrow J = (0, +\infty) \leftrightarrow L = (-\infty, +\infty)$. Q.E.D.

6. **Conformal vector field in a product Riemannian manifold.** We consider a product Riemannian manifold $M = M_1 \times M_2 \times \cdots \times M_p$ of a number of Riemannian manifolds M_t $(t = 1, \cdots, p)$. We call each manifold M_t or its isometric diffeomorphe in M a *part* of M. The part diffeomorphic to M_t, passing through a point P, is denoted by $M_t(P)$. The orthogonal projection onto each part $M_t(P)$ of a vector field v restricted on $M_t(P)$ is a vector field in $M_t(P)$, which will be called the *restriction* of v on $M_t(P)$ and denoted by $v_{(t)}$.

By the use of a separate coordinate system, we proved in [16] that, if a product

Riemannian manifold M has at least three parts, then a conformal vector field v in M is a concircular one with parallel associated scalar field. Therefore it follows from Lemma 4.2 that

THEOREM 6.1. *If a complete product Riemannian manifold M has at least three parts, then a conformal vector field in M is isometric, unless M is locally euclidean.*

Proof of Theorem 5. By virtue of the above theorem, our consideration may be confined with a product Riemannian manifold M of two complete parts M_1 and M_2, where occur the following two cases:

(i) Both M_1 and M_2 are irreducible, or

(ii) M_1 is irreducible and M_2 is a straight line or a circle.

Let the dimension of M_1 be m. We denote a separate coordinate system in M by (x^α, x^σ), (x^α) belonging to M_1 and (x^σ) to M_2[18]. The metric tensor of M is given in the form

$$(g_{\mu\lambda}) = \begin{bmatrix} g_{\gamma\beta}(x^\alpha) & 0 \\ 0 & g_{\tau\sigma}(x^\rho) \end{bmatrix}.$$

In such a system, we have the equations

$$(6.1) \qquad \nabla_\gamma v_\beta + \nabla_\beta v_\gamma = 2\rho g_{\gamma\beta}, \quad \nabla_\tau v_\sigma + \nabla_\sigma v_\tau = 2\rho g_{\tau\sigma}$$

and

$$(6.2) \qquad \nabla_\gamma \nabla_\beta \rho = \phi g_{\gamma\beta}, \quad \nabla_\tau \nabla_\sigma \rho = -\phi g_{\tau\sigma},$$

see [16]. These equations mean that the restrictions $v_{(1)} = (v^\alpha)$ and $v_{(2)} = (v^\sigma)$ define concircular vector fields in $M_1(P)$ and $M_2(P)$ for any point P, respectively. For the proof of Theorem 5, it is sufficient to show that we have

$$(6.3) \qquad \phi = -k\rho$$

in the whole manifold M, k being a nonzero constant, and to apply Theorem 2.

If ϕ vanishes identically, then the restriction ρ_1 of ρ on $M_1(P)$ through any point P is constant, because M_1 is irreducible. Hence $\rho_\beta = \partial_\beta \rho$ vanish in $M_1(P)$ for any point P and we have $\nabla_\tau \rho_\beta = \partial_\tau \rho_\beta = 0$ in M. Hence ρ is parallel in M and the vector field v is reduced to an isometric one. By the same reason, the case where either ρ_1 is constant on $M_1(P)$ or ρ_2 is constant on $M_2(P)$ for any point P can be excluded from our consideration. Hence ϕ does not identically vanish, and if (6.3) holds then k is not equal to zero. Therefore there exists a point P such that the restrictions ρ_1 and ρ_2 of ρ on $M_1(P)$ and $M_2(P)$ are properly concircular scalar fields and $v_{(1)}$ and $v_{(2)}$ are concircular vector fields in $M_1(P)$ and $M_2(P)$ respectively. By the assumptions of the theorem, we may suppose $m \geqq 3$.

(18) Greek indices α, β, γ run from 1 to m, and ρ, σ, τ from $m+1$ to n. The first Latin indices a, b, c belong to ρ_1-hypersurfaces V_1 in $M_1(P)$ and run from 1 to $m-1$.

The case where ρ_1-hypersurfaces V_1 in M_1 are not locally euclidean. By Lemma 4.3, ρ_1 is a special concircular field in $M_1(P)$, and we have

$$\nabla_\gamma \nabla_\beta \rho_1 = -k\rho_1 g_{\gamma\beta}, \tag{6.4}$$

k being a nonzero constant. By Lemma 1.2, there exists an adapted coordinate system (u^a, y) in $M_1(P)$ except stationary points of ρ_1, and the metric form of $M_1(P)$ is given by

$$ds_1^2 = (\rho_1')^2 f_{cb} du^c du^b + dy^2$$

in which we denote the mth coordinate by y in place of u^m and prime indicates ordinary derivatives in y. The coordinate system $(u^a, y; x^\sigma)$ is a separate one of M in a neighborhood containing $M_2(P)$. Then the Christoffel symbol

$$\left\{ {\alpha \atop \gamma\beta} \right\}$$

belonging to M_1 is given by (1.4) in which the indices are replaced by those belonging to M_1 and ρ by the restriction ρ_1 on $M_1(P)$. By the same method as we have derived (4.4) from (4.1), the equation $(6.1)_1$ is decomposed into

$$\begin{aligned} \bar{\nabla}_c v_b + \bar{\nabla}_b v_c &= 2(\rho_1'^2 \rho - \rho_1' \rho_1'' v_m) f_{cb}, \\ \partial_y v_b + \partial_b v_m &= 2(\rho_1''/\rho_1') v_b, \\ \partial_y v_m &= \rho, \end{aligned} \tag{6.5}$$

and $(6.2)_1$ into

$$\begin{aligned} \bar{\nabla}_c \bar{\nabla}_b \rho &= [\rho_1'^2 \phi - \rho_1' \rho_1'' (\partial_y \rho)] f_{cb}, \\ \partial_y \partial_b \rho &= (\rho_1''/\rho_1') \partial_b \rho, \\ \partial_y \partial_y \rho &= \phi, \end{aligned} \tag{6.6}$$

where $\bar{\nabla}$ indicates covariant differentiation with respect to f_{cb} in V_1. The equation (6.4) is now reduced to

$$\rho_1'' = -k\rho_1 \tag{6.7}$$

and we have

$$\rho_1'^2 = -k\rho_1^2 - A, \tag{6.8}$$

A being a constant. From $(6.6)_2$ we obtain

$$\rho_1' \partial_y \rho = -k\rho_1 \rho - \alpha, \tag{6.9}$$

where α is a function independent of u^a. From $(6.5)_1$ and its derivative in y, we eliminate v_m by the use of (6.5)–(6.9) and obtain

$$\bar{\nabla}_c p_b + \bar{\nabla}_b p_c = 2\rho_1'^2 (A\rho - \rho_1 \alpha) f_{cb}, \tag{6.10}$$

where we have put

$$p_b = (k\rho_1^2 - \rho_1'^2)v_b + \rho_1\rho_1'\partial_y v_b.$$

Moreover, from (6.10) and its derivative in y, we eliminate ρ and obtain

$$\bar{\nabla}_c q_b + \bar{\nabla}_b q_c = -2\rho_1\rho_1'^3(\partial_y\alpha)f_{cb}, \tag{6.11}$$

where we have put

$$q_b = 3k\rho_1 p_b + \rho_1'\partial_y p_b.$$

Since the coefficient of the right hand side of (6.11) is independent of the variables u^a belonging to V_1, the equation (6.11) means that q_b defines a homothetic vector field in ρ_1-hypersurfaces in $M_1(P)$ and their homeomorphes in the part through any point of M. However, since ρ_1-hypersurfaces are complete and not locally euclidean, the vector field q_b is isometric by means of Theorem 4.1, and thus we have $\partial_y\alpha = 0$, that is, α is independent of u^a and y belonging to M_1. Differentiating (6.9) in y and substituting (6.7), we have

$$\partial_y\partial_y\rho = -k\rho. \tag{6.12}$$

Comparing (6.12) with $(6.6)_3$, we obtain the equation (6.3). Since a stationary point O of ρ_1 in M_1 is isolated if there is any, and the set of points corresponding to O in $M = M_1 \times M_2$ is the submanifold $M_2(O)$, the equation (6.3) holds in the whole manifold M.

The case where the ρ_1-hypersurfaces in M_1 are locally euclidean. In this case it follows from the discussions in §5 that the part M_1 is conformal to the direct product $\bar{M}_1 = V_1 \times J$ of a locally euclidean complete manifold V_1 with an open interval $J = (r_1, r_2)$.

If ρ_1 is of the first kind in the sense of §5, then the part M_1 is locally euclidean and ρ itself is parallel. Hence, by means of Lemma 4.2, the vector field v is isometric unless M itself is locally euclidean. If ρ_1 is of the second kind, then the part M_1 is locally hyperbolic of curvature $\kappa_1 = -c^2$. By (4.3), the restriction of ρ on the part M_1 through any point of M is a special concircular field with characteristic constant $k = \kappa_1$, and we have hence the equation (6.3) in the whole manifold M.

If ρ_1 is of the third kind, then ρ_1 is a solution of (6.15) rewritten now in the form

$$(Ar + B)\rho_1' = \rho_1^2 - A\rho_1 + C, \qquad C \neq 0, \tag{6.13}$$

with respect to the parameter r. In the remainder of this paragraph, prime indicates ordinary derivatives in r. The restriction ρ_1 satisfies the relations

$$\begin{aligned} \rho_1''/2\rho_1' &= (\rho_1 - A)/(Ar + B), \\ (\rho_1''/2\rho_1')' - (\rho_1''/2\rho_1'^2) &= C/(Ar + B)^2. \end{aligned} \tag{6.14}$$

Using the solution ρ_1, the metric form of M is given by

$$ds^2 = \rho_1'(f_{cb}du^c du^b + dr^2) + (ds_2)^2$$

in a separate coordinate system $(u^a, r; x^\sigma)$, ds_2 being the metric of M_2. The Christoffel symbol belonging to M_1 is there given by (5.2) with indices suitably replaced and ρ replaced by ρ_1.

Since $\bar{M}_1$ has the same underlying manifold as M_1, we may consider the covariant components of the restriction $v_{(1)}$ on the part $\bar{M}_1$ through any point and denote them by $\xi_\lambda = (\xi_b, \xi_m)$. We denote also by $\bar{\nabla}_\lambda$ the covariant differentiation with respect to the euclidean metric of $\bar{M}_1$, whose part $\bar{\nabla}_c$ defines that with respect to the euclidean metric of V_1. Now, by the same method as we have derived (5.5) from (4.1), we can obtain from $(6.1)_1$ the equations

$$(6.15)\qquad \begin{aligned} \bar{\nabla}_c\xi_b + \bar{\nabla}_b\xi_c &= 2[\rho - (\rho_1''/2\rho_1')\xi_m]f_{cb}, \\ \partial_r\xi_b + \partial_b\xi_m &= 0, \\ \partial_r\xi_m &= \rho - (\rho_1''/2\rho_1')\xi_m. \end{aligned}$$

Putting

$$(6.16)\qquad \alpha = \rho - (\rho_1''/2\rho_1')\xi_m,$$

the equations (6.15) imply that the restriction $v_{(1)}$ of v is a conformal vector field with α as associated scalar field in the part $\bar{M}_1$ through any point. Since $\bar{M}_1$ is locally euclidean and of dimension $m \geqq 3$, we have the equation $\bar{\nabla}_\gamma\bar{\nabla}_\beta\alpha = 0$, which is decomposed into

$$(6.17)\qquad \bar{\nabla}_c\bar{\nabla}_b\alpha = 0, \quad \partial_r\partial_b\alpha = 0, \quad \partial_r\partial_r\alpha = 0.$$

On the other hand, the equation $(6.2)_1$ is separated into

$$(6.18)\qquad \begin{aligned} \bar{\nabla}_c\bar{\nabla}_b\rho &= [\rho_1'\phi - (\rho_1''/2\rho_1')\partial_r\rho]f_{cb}, \\ \partial_r\partial_b\rho &= (\rho_1''/2\rho_1')\partial_b\rho, \\ \partial_r\partial_r\rho &= \rho_1'\phi + (\rho_1''/2\rho_1')\partial_r\rho. \end{aligned}$$

Substituting (6.16) into $(6.17)_2$ and taking account of $(6.15)_{2,3}$, $(6.18)_2$, (6.14), we have $\partial_r\partial_b\alpha = [C/(Ar+B)^2]\partial_b\xi_m = 0$. Since $C \neq 0$, ξ_m is independent of u^a, and it follows from $(6.15)_2$ that ξ_b are independent of r. From $(6.15)_3$ and (6.16), we see that ρ is independent of u^a and so is α. Since the left-hand side of $(6.15)_1$ is independent of r, α is also independent of r. Hence α is constant in the part M_1 through any point, and the restriction $v_{(1)}$ is homothetic with respect to the locally euclidean metric of $\bar{M}_1$.

Since ρ is independent of u^a, we have, from $(6.18)_{1,3}$,

$$(6.19)\qquad \phi = (\rho_1''/2\rho_1'^2)\partial_r\rho$$

and

$$(6.20)\qquad \partial_r\partial_r\rho = (\rho_1''/\rho_1')\partial_r\rho.$$

From $(6.15)_3$, we can put

$$\xi_m = \alpha r + \beta, \tag{6.21}$$

β being a function dependent only of x^σ belonging to M_2, and obtain the equation

$$\rho = (\rho_1''/2\rho_1')(\alpha r + \beta) + \alpha. \tag{6.22}$$

Substituting (6.22) into (6.20), we have the relation

$$A\beta = \alpha B. \tag{6.23}$$

If $A \neq 0$, then we may assume $B = 0$ by a suitable choice of r and we have $\beta = 0$. From (6.21), (6.22) and $(6.14)_1$, we have $\xi_m = \alpha r$ and

$$\rho = \rho_1 \alpha / A. \tag{6.24}$$

Substituting (6.24) into (6.19), we have

$$\phi = (\alpha/A)(\rho_1''/2\rho_1') = \alpha(\rho_1 - A)/A^2 r. \tag{6.25}$$

Substituting (6.24) and (6.25) into $(6.2)_2$, we have the equation

$$\rho_1 \nabla_\tau \nabla_\sigma \alpha = -[(\rho_1 - A)/Ar]\alpha g_{\tau\sigma}.$$

Since ρ_1 is a function of r and α is independent of r, $(\rho_1 - A)/Ar\rho_1$ should be a constant, say k, and hence ρ_1 is given by $\rho_1 = A/(1 - kAr)$. However, in order that this expression is consistent with (6.13), C should vanish and it leads to a contradiction. If $A = 0$, then we see that $\alpha = 0$, $\xi_m = \beta$, $\rho = \rho_1 \beta / B$ and $\phi = \rho/B$. Putting $k = -1/B$, we obtain the equation (6.3). Q.E.D.

7. Decomposition of a conformal vector field.

THEOREM 7.1. *Let M be a complete product Riemannian manifold $M_1 \times M_2$. A conformal vector field v with associated scalar field ρ in M is decomposed into*

$$v = (1/k)\rho^* + w_1 + w_2, \tag{7.1}$$

where ρ^* is a conformal vector field in M defined by

$$\rho^{*\alpha} = -\rho^\alpha = -g^{\alpha\beta}\rho_\beta, \quad \rho^{*\sigma} = \rho^\sigma = g^{\sigma\tau}\rho_\tau, \tag{7.2}$$

in a separate coordinate system, w_1 an isometric vector field along M_1 and w_2 an isometric one along M_2. In case (ii), *w_2 is in particular a parallel vector field.*

Proof. For a conformal vector field v, we know the equation

$$\nabla_\mu \nabla_\lambda v_\kappa + K_{\nu\mu\lambda\kappa} v^\nu = g_{\mu\kappa}\rho_\lambda + g_{\lambda\kappa}\rho_\mu - g_{\mu\lambda}\rho_\kappa. \tag{7.3}$$

Putting $\kappa = \beta$, $\lambda = \gamma$, $\mu = \tau$, we have $\nabla_\tau \nabla_\gamma v_\beta = \rho_\tau g_{\gamma\beta}$ and, by $(6.2)_1$ with (6.3),

$$\nabla_\gamma[\partial_\tau v_\beta + (1/k)\partial_\tau \rho_\beta] = 0.$$

Since the expressions in brackets give a parallel vector field in M_1 for each value of τ and M_1 is irreducible, the expressions should vanish, and hence the components v_β are expressed as

$$v_\beta = -(1/k)\rho_\beta + w_\beta, \tag{7.4}$$

where w_β are dependent only on x^α belonging to M_1. Substituting (7.4) into $(6.1)_1$, we see that w_β satisfy $\nabla_\gamma w_\beta + \nabla_\beta w_\gamma = 0$. Therefore the vector field w_1 defined by $(w^\alpha, 0)$ in the separate coordinate system is an isometric one along M_1 through any point.

In case (i), by the same argument, we can put

$$v_\sigma = (1/k)\rho_\sigma + w_\sigma \tag{7.5}$$

and the vector field w_2 defined by $(0, w^\sigma)$ is an isometric one along M_2. In case (ii), putting $\lambda = \beta$ and $\mu = n$ in (4.1) and substituting (7.4), we have $\partial_\beta v_n = (1/k)\partial_\beta \rho_n$ and hence may put

$$v_n = (1/k)\rho_n + w_n, \tag{7.6}$$

where w_n is a function of the nth coordinate x^n. Moreover, from (4.1) for $\lambda = \mu = n$, it follows that w_n is a constant and the vector field w_2 defined by $(0, w_n)$ in the separate coordinate system is a parallel vector field along M_2. It is easily seen that the vector field ρ^* defined by (7.2) satisfies

$$\pounds_{\rho^*} g_{\mu\lambda} = \nabla_\mu \rho^*_\lambda + \nabla_\lambda \rho^*_\mu = 2k\rho g_{\mu\lambda},$$

and hence it is conformal. Q.E.D.

Since an isometric vector field in a complete Riemannian manifold is complete, we can state

LEMMA 7.2. *If a conformal vector field v is complete in $M = M_1 \times M_2$, then so is the conformal vector field ρ^*.*

THEOREM 7.3(19). *Let M be a complete, reducible Riemannian manifold and v a complete conformal vector field with associated scalar field ρ. If the gradient vector field ρ_λ has bounded length in M, then v is isometric, unless M is locally euclidean.*

Proof. In the case where M is not simply connected, the universal covering space $\tilde{M}$ of M is a complete product Riemannian manifold [1] and the natural extensions of v and ρ into $\tilde{M}$ satisfy the assumptions of the theorem. So we may suppose that M is a product Riemannian manifold. The length $\|\rho\|$ of ρ_λ is equal to the length $\|\rho^*\|$ of ρ^*. A trajectory of ρ^* is an integral curve of the equation $dx^\kappa/dt = \rho^{*\kappa}$ with respect to the canonical parameter t. The length $\|\rho^*\|$ satisfies

(19) Theorem 5 of S. Tachibana [13] can be obtained from this theorem.

the equation $d\|\rho^*\|/dt = k$ and it is given by $\|\rho^*\| = C\exp kt$ along any trajectory of ρ^*, C being a constant. Therefore, if v or ρ^* is complete, then $\|\rho^*\|$ is unbounded in M, unless $C = 0$. If $C = 0$, ρ is constant and should be equal to zero, unless M is locally euclidean. Q.E.D.

8. **Projective vector field in a product Riemannian manifold.** A *projective vector field*, or an infinitesimal projective transformation, v is characterized by

$$(8.1)\qquad £_v\begin{Bmatrix}\kappa\\ \mu\lambda\end{Bmatrix} = \nabla_\mu\nabla_\lambda v^\kappa + K_{\nu\mu\lambda}{}^\kappa v^\nu = \rho_\mu\delta^\kappa_\lambda + \rho_\lambda\delta^\kappa_\mu,$$

where ρ_λ is a vector field, called the *associated* vector field with v. As is known, ρ_λ is locally the gradient vector field of a scalar field, say ρ. If ρ_λ vanishes identically, the vector field v is affine. For a projective vector field v, we know the equation

$$(8.2)\qquad \begin{aligned} £_v K_{\nu\mu\lambda}{}^\kappa &= v^\omega\nabla_\omega K_{\nu\mu\lambda}{}^\kappa + K_{\omega\mu\lambda}{}^\kappa\nabla_\nu v^\omega + K^\kappa_{\nu\omega\lambda}\nabla_\mu v^\omega + K_{\nu\mu\omega}{}^\kappa\nabla_\lambda v^\omega - K_{\nu\mu\lambda}{}^\omega\nabla_\omega v^\kappa \\ &= -\delta^\kappa_\nu\nabla_\mu\rho_\lambda + \delta^\kappa_\mu\nabla_\nu\rho_\lambda. \end{aligned}$$

LEMMA 8.1. *Let M be a locally reducible Riemannian manifold of dimension $n \geqq 2$. If v is a projective vector field in M, then the associated vector field ρ_λ is parallel*:

$$(8.3)\qquad \nabla_\mu\rho_\lambda = 0.$$

Proof. If M is locally euclidean, then (8.3) follows immediately from (8.2). A two-dimensional reducible Riemannian manifold is locally euclidean. We may therefore assume that $M = M_1 \times M_2$ and M_1 is irreducible, locally. Using the conventions in §6, and putting $\kappa = \sigma$, $\lambda = \beta$, $\mu = \gamma$, $\nu = \tau$ in (8.2), we have $\nabla_\gamma\rho_\beta = 0$ and similarly $\nabla_\tau\rho_\sigma = 0$. Since M_1 is irreducible, we have $\rho_\beta = 0$ and hence $\nabla_\tau\rho_\beta = 0$. Q.E.D.

Proof of Theorem 6. Since M is supposed to be simply connected, there exists globally a scalar field ρ whose gradient vector field is ρ_λ. By virtue of Lemma 8.1 and Theorem 2 (I,A), M is globally the product $V \times I$ of a complete manifold V with a straight line I. In an adapted coordinate system (u^h, u^n), ρ is given by $\rho = au^n + b$, a and b being constant, and $a \neq 0$, otherwise v would be affine. Putting $\kappa = h$, $\lambda = n$, $\mu = j$ in (8.1), we have $\nabla_j\nabla_n v^i = a\delta^i_j$, which means that the vector field consisting of components $\partial_n v^i$ in $V(P)$ is concurrent. Hence, by virtue of Theorem 2 (I,B), V is a euclidean space and consequently so is M itself. Q.E.D.

THEOREM 8.2. *A complete projective vector field in a complete, reducible Riemannian manifold is always affine.*

Proof. A euclidean space cannot admit a complete nonaffine projective vector field. Q.E.D.

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