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COMPLETE SET OF CUT-AND-JOIN OPERATORS IN THE HURWITZ–KONTSEVICH THEORY

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We define cut-and-join operators in Hurwitz theory for merging two branch points of an arbitrary type. These operators have two alternative descriptions: (1) the GL characters are their eigenfunctions and the symmetric group characters are their eigenvalues; (2) they can be represented as W-type differential operators (in particular, acting on the time variables in the Hurwitz–Kontsevich τ-function). The operators have the simplest form when expressed in terms of the Miwa variables. They form an important commutative associative algebra, a universal Hurwitz algebra, generalizing all group algebra centers of particular symmetric groups used to describe the universal Hurwitz numbers of particular orders. This algebra expresses arbitrary Hurwitz numbers as values of a distinguished linear form on the linear space of Young diagrams evaluated on the product of all diagrams characterizing particular ramification points of the branched covering.

Keywords: matrix model, Hurwitz number, symmetric group character

1. Introduction

1.1. Hurwitz numbers and characters. The Hurwitz numbers $\mathrm{Cov}_q(\Delta_1,\Delta_2,\dots,\Delta_m)$ count the number (weighted in a certain way) of ramified q-fold coverings of a Riemann sphere with fixed positions of m branch points of the given types $\Delta_1,\dots,\Delta_m$. The types are labeled by ordered integer partitions of q, i.e., by the Young diagrams Δ with $|\Delta|=q$ boxes. This seemingly formal problem appears related to numerous directions of research in physics and mathematics and attracts increasing attention in the literature (see [1]–[24] and the references therein). After an accurate definition (see Sec. 2.1 below), the problem becomes a problem in the representation theory of symmetric groups and reduces to the celebrated formula [3]

$$\mathrm{Cov}_q(\Delta_1,\Delta_2,\dots,\Delta_m)=\sum_{|R|=q} d_R^2\varphi_R(\Delta_1)\varphi_R(\Delta_2)\cdots\varphi_R(\Delta_m) \tag{1}$$

(our normalization of $\varphi_R(\Delta)$ differs from that used in textbooks by a factor). The right-hand side is a sum over all representations (Young diagrams) R with $|R|=q$, and the $\varphi_R(\Delta)$ are expansion coefficients (they are in fact proportional to the characters of symmetric groups [25]) of the GL characters (Shur functions) $\chi_R(t)$ [25] in the time variables $p_k=kt_k$:

$$\chi_R(t)=\sum_{|\Delta|=|R|} d_R\varphi_R(\Delta)p(\Delta)=\sum_{\Delta} d_R\varphi_R(\Delta)p(\Delta)\delta_{|\Delta|,|R|}. \tag{2}$$

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For the integer partition $\Delta = [\mu_1, \ldots]$, where $\mu_1 \geq \mu_2 \geq \cdots \geq 0$ with $\sum_j \mu_j = |\Delta|$, a monomial is $p(\Delta) \equiv \prod_i p_{\mu_i} = \prod_j p_j^{m_j}$. In what follows, we also use a differently normalized monomial: if $p(\Delta) = \prod_k p_k^{m_k}$, then

$$\widetilde{p(\Delta)} \equiv \prod_k \frac{1}{m_k!}\left(\frac{p_k}{k}\right)^{m_k} = \left(\prod_k m_k! k^{m_k}\right)^{-1} p(\Delta).$$

In what follows, we use the same definition of $\widetilde{Y(\Delta)}$ to define monomials for arbitrary chains of variables $\{y_k\}$. The definitions of $\chi_R(t)$ and $d_R = \chi_R(t_k = \delta_{k,1})$ are standard (see Sec. 2.6 below).

We can extend the definition of $\varphi_R(\Delta)$ to larger diagrams R with $|R| > |\Delta|$ by

$$\varphi_R([\Delta, \underbrace{1, \ldots, 1}_{k}]) \equiv \begin{cases} 0, & |\Delta| + k > |R|, \\ \varphi_R([\Delta, \underbrace{1, \ldots, 1}_{|R|-|\Delta|}]) C^k_{|R|-|\Delta|}, & |\Delta| + k \leq |R|. \end{cases} \tag{3}$$

Here, $C^a_b = b!/(a!\,(b-a)!)$ are the binomial coefficients, and Δ is a Young diagram that does not contain units, $1 \notin \Delta$. With this extension, we can lift the requirement that all $|\Delta_\alpha| = q$ in (1).

1.2. Hurwitz partition functions. There are two different ways to define generating functions of Hurwitz numbers (also see [26] for other generating functions related to partitions). First, $\varphi_R(\Delta)$ in (1) can be contracted with $p(\Delta)$ and converted into $\chi_R(t)$ using (2). Second, $\varphi_R(\Delta)$ can be exponentiated. This implies the definition of the Hurwitz partition function [23]

$$\mathcal{Z}(t, t', t'', \ldots |\beta) \equiv \sum_R d_R^2 \frac{\chi_R(t)}{d_R} \frac{\chi_R(t')}{d_R} \frac{\chi_R(t'')}{d_R} \cdots \exp\left(\sum_\Delta \beta_\Delta \varphi_R(\Delta)\right), \tag{4}$$

where the sum is now over all representations (Young diagrams) R of an arbitrary size $|R|$ and β is a set of constants depending on the Young diagrams. If only $\beta_{[2]}$ corresponding to the diagrams $\Delta = [2]$ is nonzero, then this reduces to the generating function for the $\mathcal{N}$-Hurwitz numbers. In particular [8], [9],

$$\begin{aligned} \mathcal{N} = 1: &\quad Z(t|\beta) = \sum_R d_R \chi_R(t) e^{\beta_2 \varphi_R([2])} \quad \longrightarrow \quad Z(t|0) = \sum_R d_R \chi_R(t) = e^{t_1}, \\ \mathcal{N} = 2: &\quad Z(t, \bar{t}|\beta) = \sum_R \chi_R(t) \chi_R(\bar{t}) e^{\beta_2 \varphi_R([2])} \quad \longrightarrow \\ &\quad \longrightarrow \quad Z(t, \bar{t}|0) = \sum_R \chi_R(t)\chi_R(\bar{t}) = \exp\left(\sum_k k t_k \bar{t}_k\right) \end{aligned} \tag{5}$$

are the respective KP and Toda-chain τ-functions in t and in t and $\bar{t}$ [27], [9], [19], [22], but integrability is violated for $\mathcal{N} \geq 3$ [23]. It is also violated by including higher β_Δ with $|\Delta| \geq 3$ [16], [20], [23]: to preserve it, we must exponentiate the Casimir eigenvalues $C_R(|\Delta|)$ [27] instead of $\varphi_R(\Delta) \neq C_R(|\Delta|)$. The replacement of φ_R with C_R in the definition of the partition functions was called the transition to complete cycles in [16], [20], and $\mathcal{Z}$ is obtained from the thus-defined τ-function [27] by the action of quite sophisticated operators $\mathcal{B}_\Delta$ (see [23]).

The KP τ-function $Z(t|\beta)$ is in fact related [10], [19], [22] by the equivalent-hierarchies technique [28] to the Kontsevich τ-function [29], and following [22], we call it and its further generalization (4) the Kontsevich–Hurwitz partition function. This remarkable relation allows using the well-developed technique of matrix models [29]–[35] to study the Hurwitz numbers. This paper develops a particular example [24] of such a use.

1.3. General cut-and-join operators. Alternatively, we can introduce the β-deformations in the partition function using the cut-and-join operators $\widehat{\mathcal{W}}$, which are differential operators acting on the time variables $\{t_k\}$ (or, alternatively, on $\{t'_k\}$ or $\{t''_k\}$) and have the characters $\chi_R(t)$ as their eigenfunctions and $\varphi_R(\Delta)$ as the corresponding eigenvalues:

$$\widehat{\mathcal{W}}(\Delta)\chi_R(t) = \varphi_R(\Delta)\chi_R(t). \tag{6}$$

As an immediate corollary of (6) and (4), we then have

$$\mathcal{Z}(t,t',\dots|\beta) = \exp\biggl(\sum_\Delta \beta_\Delta \widehat{\mathcal{W}}(\Delta)\biggr)\mathcal{Z}(t,t',\dots|0). \tag{7}$$

In the simplest case of $\Delta = [2]$, we obtain the standard cut-and-join operator [8]

$$\begin{aligned}\widehat{\mathcal{W}}([2]) &= \frac{1}{2}\sum_{a,b=1}^{\infty}\biggl((a+b)p_a p_b \frac{\partial}{\partial p_{a+b}} + ab p_{a+b}\frac{\partial^2}{\partial p_a \partial p_b}\biggr) = \\ &= \frac{1}{2}\sum_{a,b=1}^{\infty}\biggl(ab t_a t_b \frac{\partial}{\partial t_{a+b}} + (a+b) t_{a+b}\frac{\partial^2}{\partial t_a \partial t_b}\biggr),\end{aligned} \tag{8}$$

which is the zero-mode generator of the W_3-algebra [36]: $\widehat{\mathcal{W}}([2]) = \widehat{W}_0^{(3)}$. The W_3-algebra is a part of the universal enveloping algebra of $GL(\infty)$ and is a symmetry of the universal Grassmannian [37], [38]. Hence, the action of this operator preserves the KP integrability [37], [39], [28] and deforms the Toda integrability [40] in a simple way [27], [9].

Operator (8) is conveniently rewritten [24] in terms of the matrix Miwa variable X (the matrix variable was called ψ in [24]),

$$p_k = k t_k = \operatorname{tr} X^k, \tag{9}$$

where X is an $N\times N$ matrix,

$$\widehat{\mathcal{W}}([2]) = \frac{1}{2}(\operatorname{tr}\widehat{D}^2 - N\operatorname{tr}\widehat{D}) = \frac{1}{2}:\operatorname{tr}\widehat{D}^2:. \tag{10}$$

Here, we use the matrix operator $\widehat{D} = X(\partial/\partial\widetilde{X})$ involving the transposed matrix $\widetilde{X}$ as usual in matrix model theory [29]–[35] (hereafter, summation over repeated indices is implied):

$$\widehat{D}_{ab} \equiv X_{ac}\frac{\partial}{\partial X_{bc}} \tag{11}$$

i.e., a family of operators $[\widehat{D}_{ab}, \widehat{D}_{cd}] = \widehat{D}_{ad}\delta_{bc} - \widehat{D}_{bc}\delta_{ad}$ acting on the algebra of functions generated by X_{ab}: $\widehat{D}_{ab}X_{cd} \equiv X_{ad}\delta_{bc}$. The normal ordering implies that the X derivatives inside colons do not act on the X between the same pair of colons. This is equivalent to taking a symbol of the operator.

Our goal here is to clarify that (10) has a direct generalization to all other operators $\widehat{\mathcal{W}}(\Delta)$:

$$\widehat{\mathcal{W}}(\Delta) = :\widetilde{D(\Delta)}:, \tag{12}$$

where $\widetilde{D(\Delta)}$ is constructed from the operators $\widehat{D}_k \equiv \operatorname{tr}\widehat{D}^k$ in exactly the same way as $\widetilde{p(\Delta)}$ is constructed from the time variables $p_k = kt_k$: $\widetilde{D(\Delta)} \equiv (\prod_k m_k! k^{m_k})^{-1}\widehat{D}_k^{m_k}$. We note that the operators $\widehat{D}_{ab}$ given by (11) realize the regular representation of the algebra gl. The (commuting) Casimir operators in the universal enveloping algebra of gl can be realized as $\widehat{D}_k$, and the characters $\chi_R(t)$ of the group GL are their eigenfunctions [41], [42]. Because all $\widehat{D}_k$ mutually commute, all the $\widehat{D}(\Delta)$ and $\widehat{\mathcal{W}}(\Delta)$ mutually commute (and their common system of eigenfunctions is still formed by the characters). This allows expressing $\mathcal{Z}(t|\beta)$ in terms of the trivial τ-function $Z(t|0) = e^{t_1}$:

$$\mathcal{Z}(t|\beta) = \exp\biggl(\sum_{\Delta} \beta_\Delta :\widetilde{D(\Delta)}:\biggr) Z(t|0). \tag{13}$$

Moreover, extra sets of time variables can be introduced using the same operators, for example,[1]

$$\mathcal{Z}(t, \bar{t}|\beta) \approx :\exp\biggl(\sum_k \bar{t}_k \operatorname{tr} \widehat{D}^k\biggr): \mathcal{Z}(t|\beta). \tag{14}$$

This opens a way to incorporate Hurwitz partition functions naturally into the M-theory of matrix models [35].

Beyond $\Delta = [2]$, the normal ordering makes even operators $\widehat{\mathcal{W}}([k])$ with a single-row Young diagram $\Delta = [k]$ nonlinear combinations of the Casimir operators; this takes $\widehat{\mathcal{W}}(\Delta)$ out of the universal Grassmannian [37], [38] and leads to a violation of integrability, observed in [16], [20], [23].

Restricting the set $\{\beta_\Delta\}$ of β variables in (13) to a single $\beta_{[2]} = \beta$, we obtain a representation for the Kontsevich–Hurwitz τ-function $Z(t|\beta)$, which was the starting point in [24] for deriving a promising matrix-model representation for this intriguing function. It is actually expressed (in a yet poorly understood but clearly established way [10], [19], [22]) in terms of the standard cubic Kontsevich τ-function [29], [34].

The cut-and-join operators $\widehat{\mathcal{W}}$ form a commutative associative algebra (see Sec. 2.4.3 below):

$$\widehat{\mathcal{W}}(\Delta_1)\widehat{\mathcal{W}}(\Delta_2) = \sum_{\Delta} C^{\Delta}_{\Delta_1\Delta_2} \widehat{\mathcal{W}}(\Delta), \tag{15}$$

where the structure constants are related to the triple Hurwitz numbers $\operatorname{Cov}(\Delta_1, \Delta_2, \Delta_3)$ (see Sec. 1.4). Accordingly, these $\operatorname{Cov}(\Delta_1, \Delta_2, \Delta_3)$ can be alternatively studied in the theory of *dessins d'enfants* and Belyi functions [43]. At $|\Delta_1| = |\Delta_2| = |\Delta|$, these numbers are the structure constants $c^{\Delta}_{\Delta_1\Delta_2}$ of the center of the group algebra of the symmetric group $S_{|\Delta|}$.

Equations (13) and (15) should have an interesting non-Abelian generalization to the case of open Hurwitz numbers [13], [14], [44], counting coverings of Riemann surfaces with boundaries, which should be an open-string counterpart of closed-string formula (13).

1.4. Universal Hurwitz numbers and the universal Hurwitz algebra. The structure constants $C^{\Delta}_{\Delta_1\Delta_2}$ allow introducing the universal Hurwitz numbers defined for arbitrary sets of Young diagrams and not restricted by the condition $|\Delta_1| = \cdots = |\Delta_m|$.

We consider the vector space Y generated by all Young diagrams. The correspondence $\Delta \mapsto \widehat{\mathcal{W}}(\Delta)$ generates the structure of a commutative associative algebra on Y; we let $*$ denote the corresponding multiplication of Young diagrams. We consider the linear form l: $Y \to \mathbb{R}$ such that $l(\Delta) = 1/|\Delta|!$ for $\Delta = [1, 1, \ldots, 1]$ and $l(\Delta) = 0$ for all other Young diagrams. This definition is motivated by Eq. (72) in the theory of characters (see Sec. 2.7 below). We call

$$\operatorname{Cov}(\Delta_1, \Delta_2, \ldots, \Delta_m) = l(\Delta_1 * \Delta_2 * \cdots * \Delta_m) \tag{16}$$

the *Hurwitz number* of $\Delta_1, \Delta_2, \ldots, \Delta_m$. These generalized Hurwitz numbers coincide with the classical ones for $|\Delta_1| = |\Delta_2| = \cdots = |\Delta_m|$, when restricting the $*$-operation reproduces the composition $\circ$ of conjugation classes of permutations, considered in Sec. 2.2.

[1] For a more accurate formulation of what $\approx$ means in this equation, see Sec. 2.8 below.

The symmetric bilinear form $\langle\Delta_1,\Delta_2\rangle = l(\Delta_1 * \Delta_2)$ is nondegenerate and invariant,

$$\langle\Delta_1 * \Delta, \Delta_2\rangle = \langle\Delta_1, \Delta_2 * \Delta\rangle \quad \forall\Delta \tag{17}$$

as a consequence of commutativity and associativity. Moreover,

$$\sum_{\Delta} C^{\Delta}_{\Delta_1\Delta_2}\langle\Delta,\Delta_3\rangle = l(\Delta_1 * \Delta_2 * \Delta_3), \tag{18}$$

i.e.,

$$C^{\Delta}_{\Delta_1\Delta_2} = \sum_{\Delta_3} G^{\Delta\Delta_3} l(\Delta_1 * \Delta_2 * \Delta_3), \tag{19}$$

where $G^{\Delta_2\Delta_3}$ is the inverse matrix of $G_{\Delta_1\Delta_2} = \langle\Delta_1,\Delta_2\rangle$.

Finally, our universal Hurwitz algebra of cut-and-join operators is freely generated by a set of Casimir operators and, as a vector space, in fact coincides with the center of the universal enveloping algebra of $gl(\infty)$ (see Sec. 2.5 below and [23], [42] for more details).

2. Comments

2.1. Hurwitz numbers and counting of coverings. A q-sheet covering Σ of a Riemann surface Σ_0 is a projection $\pi\colon \Sigma \to \Sigma_0$, where almost all the points of Σ_0 have exactly q preimages. The number of preimages decreases at finitely many singular (ramification) points $x_1, \dots, x_m \in \Sigma_0$. In fact, $\pi^{-1}(x_\alpha)$ is a collection of points $y_i^{(\alpha)} \in \Sigma$ such that in the vicinity of each $y_i^{(\alpha)}$, the projection π is

$$\pi\colon (x - x_\alpha) = (y - y_i^{(\alpha)})^{\mu_i^{(\alpha)}}. \tag{20}$$

With each singular point, we then associate an integer partition of q, which can be ordered, $\Delta_\alpha\colon \mu_1^{(\alpha)} \geq \mu_2^{(\alpha)} \geq \cdots \geq 0$, i.e., is in fact a Young diagram. This diagram Δ_α is called the *type* of the ramification point x_α.

If we select some nonsingular point $x_* \in \Sigma_0$ and consider a closed path C_* in Σ_0 that begins and ends at x_*, then the q preimages of x_* in Σ are somehow permuted when we travel along C_*. A permutation of q variables is thus associated with a path C_*, i.e., the covering defines a map from the fundamental group $\pi_1(\Sigma_0, x_*)$ into the symmetric (permutation) group S_q. Changing x_* amounts to the common conjugation of all the permutations associated with different contours, and the covering itself is associated with the conjugated classes of maps of $\pi_1(\Sigma_0, x_*)$ into S_q. In fact, the converse is also almost true: given such a map, we can reconstruct the covering. Enumerating ramified coverings thus becomes a pure group theory problem and defines Hurwitz numbers for a Riemann surface of an arbitrary genus g:

$$\mathrm{Cov}_q^g(\Delta_1, \dots, \Delta_m) = \sum \frac{1}{|\operatorname{Aut}(\pi)|} \tag{21}$$

is the number of its coverings π with a fixed set of singular points $x_1, \dots, x_m$ of the types $\Delta_1, \dots, \Delta_m$ divided by the order of the automorphism group. For brevity, we merely set $\mathrm{Cov}_q^0 = \mathrm{Cov}_q$. The sum in (21) is over all possible equivalence classes of coverings, and the equivalence is established by a biholomorphic map $f\colon \Sigma \to \Sigma'$ such that $\pi' = f \circ \pi$.

Because $\mathrm{Cov}_q(\Delta_1, \dots, \Delta_m)$ is simultaneously a group theory quantity, it can also be expressed in terms of symmetric groups, and this approach leads to formula (1). An extension to arbitrary-genus surfaces Σ_0 follows from topological field theory [45], [46] [3], [13]. A certain nontrivial generalization to Σ_0 with boundaries can be found in [13], [14].

2.2. Permutations, cycles, and their compositions. Cut-and-join operators appear when we study the merger of two ramification points x_α and x_β of the types Δ_α and Δ_β. As a result of such a merger, a single singular point arises instead of two, but its type Δ is not defined unambiguously by Δ_α and Δ_β. It depends on the actual distribution of preimages $y_i^{(\alpha)}$ and $y_j^{(\beta)}$ between the sheets of the covering, and this distribution is summed over in the definition of Hurwitz numbers.

The Young diagram Δ labels the monodromy element of a critical value and a conjugation class in the symmetric group S_q. When two critical points merge, the resulting monodromy is a product of the two original monodromies.

Before we consider multiplication of classes, we examine multiplication of permutations. Any permutation can be represented as a product of cycles. For example, S_3 consists of the six elements

$$\{123\} \longrightarrow \{123\},\ \{132\},\ \{213\},\ \{321\},\ \{231\},\ \{312\},$$

which can be respectively expressed in terms of cycles as

$$123,\ 1(23),\ (12)3,\ (13)2,\ (132),\ (123).$$

The notation (132) for a cycle means that $1 \to 3 \to 2 \to 1$. For brevity, we write 123 instead of (1)(2)(3).

The Young diagram Δ describes the conjugation class of group elements. We let the same symbol Δ denote the element of the group algebra equal to the sum of all elements of the conjugation class (with unit coefficients).

For instance, the Young diagrams with three boxes label the three conjugation classes of these permutations as

$$\Delta = [1,1,1] = 123, \qquad \Delta = [2,1] = 1(23),\ (12)3,\ (13)2, \qquad \Delta = [3] = (132),\ (123).$$

The corresponding elements of the group algebra are

$$\Delta = [1,1,1] = 123, \qquad \Delta = [2,1] = 1(23) \oplus (12)3 \oplus (13)2, \qquad \Delta = [3] = (132) \oplus (123).$$

It is convenient to define $\|\Delta\|$ as the number of different permutations in the conjugation class Δ, for example, $\|3\| = 2$, $\|2,1\| = 3$, and $\|1,1,1\| = 1$.

Similarly, there are five conjugation classes for S_4, and the corresponding group algebra elements are

$$\Delta = [4] = (1234) \oplus (1243) \oplus (1324) \oplus (1342) \oplus (1423) \oplus (1432), \quad \|4\| = 3! = 6,$$

$$\Delta = [3,1] = (123)4 \oplus (124)3 \oplus (132)4 \oplus (134)2 \oplus (142)3 \oplus (143)2 \oplus 1(234) \oplus 1(243), \quad \|3,1\| = 8,$$

$$\Delta = [2,2] = (12)(34) \oplus (13)(24) \oplus (14)(23), \quad \|2,2\| = 3,$$

$$\Delta = [2,1,1] = (12)34 \oplus (13)24 \oplus (14)23 \oplus 1(23)4 \oplus 1(24)3 \oplus 12(34), \quad \|2,1,1\| = 6,$$

$$\Delta = [1,1,1,1] = 1234, \quad \|1,1,1,1\| = 1.$$

If we now consider the merger of two ramification points, for example, with $\Delta = [2,1]$ and $\Delta' = [3]$, then we must see what happens when any of the three permutations from the conjugation class $\Delta = [2,1]$ are multiplied by any of the two from $\Delta' = [3]$. This is described by the 3×2 table

$$[2,1] \circ [3] = \begin{array}{|c|c|} \hline 1(23)\circ(132) & 1(23)\circ(123) \\ \hline (12)3\circ(132) & (12)3\circ(123) \\ \hline (13)2\circ(132) & (13)2\circ(123) \\ \hline \end{array} = \begin{array}{|c|c|} \hline (13)2 & (12)3 \\ \hline 1(23) & (13)2 \\ \hline (12)3 & 1(23) \\ \hline \end{array} = 2 \cdot [2,1] \qquad (22)$$

or, simply,

$$[2,1]\circ[3] = \big(1(23)\oplus(12)3\oplus(13)2\big)\circ\big((132)\oplus(123)\big) =$$
$$= 2\cdot\big(1(23)\oplus(12)3\oplus(13)2\big) = 2\cdot[2,1], \tag{23}$$

where $\circ$ denotes the composition of permutations. As usual, the second permutation acts first, for example, $\{123\}\overset{(132)}{\longrightarrow}\{132\}\overset{1(23)}{\longrightarrow}\{321\}$, and the result is the same as $\{123\}\overset{(13)2}{\longrightarrow}\{321\}$. The numbers in the permutation notation refer to *places*, not *elements*: (12) permutes the entries in the first and second places, not the elements 1 and 2.

Having the composition of permutations, we can use the corresponding structure constants $c^{\Delta''}_{\Delta\Delta'}$,

$$\Delta\circ\Delta' = \sum_{\Delta''} c^{\Delta''}_{\Delta\Delta'}\Delta'', \tag{24}$$

to define the cut-and-join operator by the rule

$$\widehat{\mathcal{W}}(\Delta)\widetilde{p(\Delta')} = \sum_{\Delta''} c^{\Delta''}_{\Delta\Delta'}\widetilde{p(\Delta'')}. \tag{25}$$

Equation (22) implies that the operator $\widehat{\mathcal{W}}([2,1])$ thus defined contains an term

$$\widehat{\mathcal{W}}([2,1]) = 2\cdot\widetilde{p([2,1])}\frac{\partial}{\partial\widetilde{p([3])}} + \dots = 3p_1p_2\frac{\partial}{\partial p_3} + \dots, \tag{26}$$

where the dots denote terms that annihilate p_3.

Similarly, the composition table

$[3]\circ[2,1] =$

(132)∘1(23)	(132)∘(12)3	(123)∘(13)2
(123)∘1(23)	(123)∘(12)3	(123)∘(13)2

$=$

(12)3	(13)2	1(23)
(13)2	1(23)	(12)3

$= 2\cdot[2,1]$ (27)

implies that

$$\widehat{\mathcal{W}}([3]) = 2p_1p_2\frac{\partial^2}{\partial p_1\,\partial p_2} + \dots, \tag{28}$$

where the dots now denote terms from the annihilator of p_1p_2.[2]

The elements of the group algebra corresponding to the Young diagrams generate the center of the group algebra. In our example, we can see that the right-hand sides of (27) and (22) are the same, as implied by commutativity of the center.

We can similarly analyze the composition of any other pair of conjugation classes and reconstruct all the entries in the operators $\widehat{\mathcal{W}}(\Delta)$. We can thus verify that any continuation of the first column in the Young diagram does not affect the cut-and-join operator,

$$\widehat{\mathcal{W}}([\Delta,1,1,\dots,1]) \cong \widehat{\mathcal{W}}(\Delta), \tag{29}$$

[2] We can compare this formula with the full expression in Eq. (53). The coefficient 2 in (28) arises from the second term in (53) with $abcd = 1212, 1221, 2112, 2121$, and only two of these four terms contribute because of the factor $(1-\delta_{ac}\delta_{bd})$.

if acting on a proper quantity in accordance with (3) (see Sec. 2.4.2 for details). We briefly present just one more example:

$$[2,\underbrace{1,\ldots,1}_{q-2}]\circ[3,\underbrace{1,\ldots,1}_{q-3}] = \begin{array}{|c|c|}\hline (12)3456\ldots q\circ(123)456\ldots q & \ldots \\ \hline (13)2456\ldots q\circ(123)456\ldots q & \\ \hline 1(23)456\ldots q\circ(123)456\ldots q & \\ \hline (14)2356\ldots q\circ(123)456\ldots q & \\ \hline \ldots & \\ \hline 123(45)6\ldots q\circ(123)456\ldots q & \\ \hline \ldots & \\ \hline \end{array} =$$

$$= \begin{array}{|c|c|}\hline (13)2456\ldots q & \ldots \\ \hline 1(23)456\ldots q & \\ \hline (12)3456\ldots q & \\ \hline (1423)56\ldots q & \\ \hline \ldots & \\ \hline (123)(45)6\ldots q & \\ \hline \ldots & \\ \hline \end{array}. \tag{30}$$

There are $\|3,\underbrace{1,\ldots,1}_{q-3}\| = 2C_q^3 = q(q-1)(q-2)/3$ columns and $\|2,\underbrace{1,\ldots,1}_{q-2}\| = C_q^2 = q(q-1)/2$ rows in the tables. Clearly, each column of the second (resultant) table contains three elements from the class $[2,\underbrace{1,\ldots,1}_{q-2}]$, $3(q-3)$ plus $3(q-3)$ elements from the class $[4,\underbrace{1,\ldots,1}_{q-4}]$ plus $(q-3)(q-4)/2$ elements from the class $[3,2,\underbrace{1,\ldots,1}_{q-5}]$. Therefore,

$$[2,1,\ldots,1]\circ\frac{[3,1,\ldots,1]}{\|3,1,\ldots,1\|} = 3\cdot\frac{[2,1,\ldots,1]}{\|2,1,\ldots,1\|} + 3(q-3)\cdot\frac{[4,1,\ldots,1]}{\|4,1,\ldots,1\|} + \\ + \frac{(q-3)(q-4)}{2}\cdot\frac{[3,2,1,\ldots,1]}{\|3,2,1,\ldots,1\|}$$

or

$$[2,1,\ldots,1]\circ[3,1,\ldots,1] = 2(q-2)\cdot[2,1,\ldots,1] + 4\cdot[4,1,\ldots,1] + [3,2,1,\ldots,1]. \tag{31}$$

Because $\widehat{\mathcal{W}}([2,\underbrace{1,\ldots,1}_{q-2}])$ acts on $p_3p_1^{q-3}$ in this example, we have

$$\widehat{\mathcal{W}}([2,\underbrace{1,\ldots,1}_{q-2}])p_3p_1^{q-3} = \frac{1}{2}\left(6p_1p_2\frac{\partial}{\partial p_3} + 6p_4\frac{\partial^2}{\partial p_1\partial p_3} + p_2\frac{\partial^2}{\partial p_1^2} + \cdots\right)p_3p_1^{q-3}. \tag{32}$$

We see that the coefficient in the term $p_1p_2(\partial/\partial p_3)$ is the same as in (26), in full accordance with (29). Both representations in (31) imply the same result for $\widehat{\mathcal{W}}([2,\underbrace{1,\ldots,1}_{q-2}])$ because

$$\widetilde{p(\Delta)} = \frac{\|\Delta\|}{|\Delta|!}p(\Delta) = \frac{p(\Delta)}{\mathrm{Aut}(\Delta)},$$

and both multiplication formulas can be used to extract the cut-and-join operator from (25).

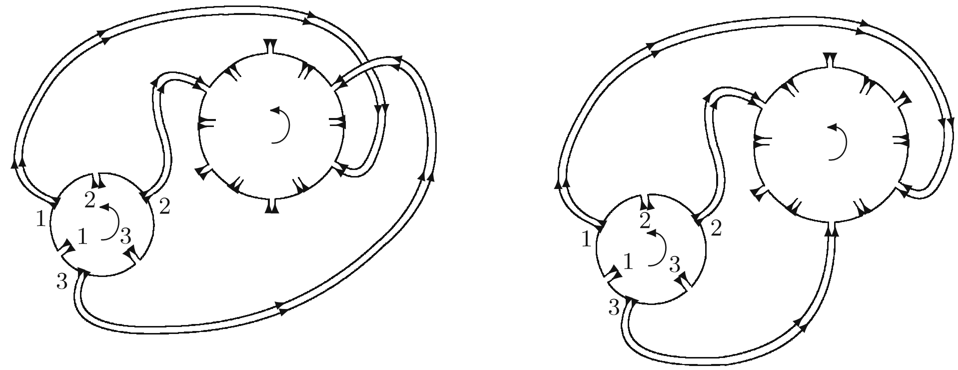

Fig. 1. Composition of two permutations of the cycle (123) with a cycle of length six. At the same time, these Feynman diagrams contribute to multiplication of the normal-ordered matrix differential operators $\widehat{\mathcal{W}}([3])$ and $\widehat{\mathcal{W}}([6])$.

In general, for a composition of conjugation classes, we have

$$\Delta_1 \circ \Delta_2 = \sum_{|\Delta|=|\Delta_1|=|\Delta_2|} c^{\Delta}_{\Delta_1\Delta_2} \cdot \Delta, \tag{33}$$

where we use the lowercase letter c to stress that we deal with the composition of permutations in the group $S_{|\Delta|}$, i.e., $|\Delta| = |\Delta_1| = |\Delta_2|$. The above examples demonstrate that even in this case, the cut-and-join operator is *not exactly* $\sum_{|\Delta_1|=|\Delta_2|} c^{\Delta_1}_{\Delta\Delta_2}\widetilde{p(\Delta_1)}\,\partial/\partial\widetilde{p(\Delta_2)}$. The actual degree of the differential operator satisfying (25) can be much lower than implied by this formula. In fact, the constraint that $|\Delta'| = |\Delta|$ in (25) can be easily lifted: we can extend Δ to a diagram $[\Delta, 1^{|\Delta'|-|\Delta|}]$ by adding a unit-height line of appropriate length and define

$$\widehat{\mathcal{W}}(\Delta)\widetilde{p(\Delta')} = \tilde{p}\big([\Delta, 1^{|\Delta'|-|\Delta|}] \circ \Delta'\big) = \sum_{|\Delta''|=|\Delta'|} c^{\Delta''}_{[\Delta,1^{|\Delta'|-|\Delta|}]\,\Delta'}\widetilde{p(\Delta'')} \quad \text{for } 1 \notin \Delta \tag{34}$$

and

$$\widehat{\mathcal{W}}([\Delta, 1^s])\widetilde{p(\Delta')} = \sum_{|\Delta''|=|\Delta'|} \frac{(|\Delta'| - |\Delta|)!}{s!(|\Delta'| - |\Delta| - s)!}\, c^{\Delta''}_{[\Delta,1^{|\Delta'|-|\Delta|}]\,\Delta'}\widetilde{p(\Delta'')} \quad \text{for } 1 \notin \Delta. \tag{35}$$

Cut-and-join operators can thus be defined as acting on the time variables of an arbitrary level entirely in terms of the structure constants of the universal symmetric group $S(\infty)$. But Eq. (12) provides a much more explicit and transparent alternative representation of these operators, which also allows extending the set of the $S(\infty)$ structure constants by lifting the remaining restriction $|\Delta''| = |\Delta'|$, which is still preserved in (34) and (35). The extended structure constants $C^{\Delta''}_{\Delta\Delta'}$ describe multiplication of the universal operators, which are defined by either (35) or (12).

2.3. Composition of permutations and Feynman diagram technique. A composition of permutations can be conveniently calculated using a simple Feynman diagram technique. On one hand, this literally reflects the geometric definition of the Hurwitz numbers; on the other hand, it is equivalent to the description in terms of differential operators. We represent a cycle (132) of length three by an oriented circle in the left-hand side of the diagram and a cycle of length six by another oriented circle in its right-hand side (see Fig. 1). The composition itself is represented by lines connecting all outgoing lines of the left circle with three arbitrarily chosen incoming lines of the right circle. New cycles are formed as a result: just one of length six for connecting lines as in the left diagram and three of the lengths one, two, and three if one of the connecting lines is changed as in the right diagram.

In Fig. 1, we deal with a situation of the type $(123)\circ(123456)$, where the first cycle is a subset of the second. We should only keep in mind that along with (123456), we should consider all the 5! different cycles formed by the same six elements: only two of these 5! possibilities are shown in the figure. To obtain our operators, we should sum over all these options. We should also add all the other cycles: each Δ is a set of several cycles of given lengths.

An advantage of this diagrammatic representation is that we can further represent such pictures—the Feynman diagrams—by operators. This is the simplest way to obtain (12), which immediately reproduces Eqs. (26) and (28). The normal ordering appears because one connecting line cannot act on another connecting line.

This Feynman diagram technique ties together the geometric interpretation of the Hurwitz numbers, their combinatorial expressions, and the normal-ordered differential matrix operators.

2.4. Algebra of cut-and-join operators.

2.4.1. Examples of normal ordering. We begin with a few examples illustrating the role of normal ordering:

$$:\operatorname{tr}\widehat{D}^2: = \operatorname{tr}\widehat{D}^2 - N\operatorname{tr}\widehat{D} = \operatorname{tr}(\widehat{D}-N)\widehat{D}$$

or

$$\begin{aligned}
\operatorname{tr}\widehat{D}^2 &= :\operatorname{tr}\widehat{D}^2: + N\operatorname{tr}\widehat{D},\\
\operatorname{tr}\widehat{D}^3 &= :\operatorname{tr}\widehat{D}^3: + 2N:\operatorname{tr}\widehat{D}^2: + :(\operatorname{tr}\widehat{D})^2: + N^2\operatorname{tr}\widehat{D},\\
\operatorname{tr}\widehat{D}^4 &= :\operatorname{tr}\widehat{D}^4: + 3N:\operatorname{tr}\widehat{D}^3: + 3:\operatorname{tr}\widehat{D}\operatorname{tr}\widehat{D}^2: + (3N^2+1):\operatorname{tr}\widehat{D}^2: +\\
&\quad + 3N:(\operatorname{tr}\widehat{D})^2: + N^3:\operatorname{tr}\widehat{D}:,\\
&\ \vdots
\end{aligned}$$

and similarly

$$(\operatorname{tr}\widehat{D})^2 = :(\operatorname{tr}\widehat{D})^2: + \operatorname{tr}\widehat{D}, \tag{36}$$

$$\begin{aligned}
(\operatorname{tr}\widehat{D}^2)^2 &= :(\operatorname{tr}\widehat{D}^2)^2: + 2N:\operatorname{tr}\widehat{D}\operatorname{tr}\widehat{D}^2: + 4:\operatorname{tr}\widehat{D}^3: +\\
&\quad + 4N:\operatorname{tr}\widehat{D}^2: + (N^2+2):(\operatorname{tr}\widehat{D})^2: + N^2\operatorname{tr}\widehat{D}
\end{aligned} \tag{37}$$

and so on.

2.4.2. Comment on formula (29): Inserting an extra $\widehat{\boldsymbol{D}}_{\mathbf{1}}$ in the cut-and-join operator. We completely describe normal ordering for a small but important class of operators containing powers of $\widehat{D}_1 = \operatorname{tr}\widehat{D} = \sum_a a p_a(\partial/\partial p_a)$,

$$\widehat{\mathcal{W}}([\Delta,\underbrace{1,\dots,1}_{k}]) = \frac{1}{k!}:\widetilde{D(\Delta)\widehat{D}_1^k}:, \tag{38}$$

where we assume that Δ contains only k units (see [42] for a more systematic description).

The relations

$$
\begin{aligned}
&:\widetilde{D(\Delta)}\widehat{D}_1: = :\widetilde{D(\Delta)}: \widehat{D}_1 - |\Delta| :\widetilde{D(\Delta)}: = :\widetilde{D(\Delta)}: (\widehat{D}_1 - |\Delta|),\\
&:\widetilde{D(\Delta)}(\widehat{D}_1)^2: = :\widetilde{D(\Delta)}\widehat{D}_1: \widehat{D}_1 - (|\Delta|+1) :\widetilde{D(\Delta)}\widehat{D}_1: = :\widetilde{D(\Delta)}\widehat{D}_1: (\widehat{D}_1 - |\Delta| - 1) =\\
&\qquad\qquad = :\widetilde{D(\Delta)}: (\widehat{D}_1 - |\Delta|)(\widehat{D}_1 - |\Delta| - 1),\\
&\qquad\qquad \vdots\\
&:\widetilde{D(\Delta)}(\widehat{D}_1)^k: = :\widetilde{D(\Delta)}: \prod_{i=0}^{k-1}(\widehat{D}_1 - |\Delta| - i)
\end{aligned}
\tag{39}
$$

follow directly from the definition of normal ordering. This implies that when we act with $:\widetilde{D(\Delta)}:$ on some quantity of weight $|R|$, for example, on $:\widetilde{D(R)}:$, then $\widehat{D}_1$ acts as multiplication by $|R|$, and we can always replace $:\widetilde{D(\Delta)}:$ with

$$
\frac{1}{(|R|-|\Delta|)!} :\widetilde{D(\Delta)}(\widehat{D}_1)^{|R|-|\Delta|}: = :\widetilde{D([\Delta, \underbrace{1,\ldots,1}_{|R|-|\Delta|}])}: , \quad 1 \notin \Delta,
\tag{40}
$$

without changing the result, in accordance with rule (3) and formula (29).

If $\widehat{\mathcal{W}}(\Delta)$ contains $\widehat{D}_1$ factors, then this rule should be modified by a numerical factor. For example, $:\widetilde{D([1])}:$ is replaced with

$$
\begin{aligned}
\frac{1}{(|R|-1)!} :\widetilde{D([1])}(\widehat{D}_1)^{|R|-1}: = \frac{1}{(|R|-1)!} :(\widehat{D}_1)^{|R|}: = |R| :\widetilde{(\widehat{D}_1)^{|R|}}: =\\
= |R| :\widetilde{D([\underbrace{1,\ldots,1}_{|R|}])}: ,
\end{aligned}
\tag{41}
$$

which contains an extra factor of $|R|$, again in accordance with (3).

2.4.3. Multiplication algebra of $\widehat{\mathcal{W}}$ operators. Using the relations in Sec. 2.4.1, we can now multiply different cut-and-join operators:

$$
\widehat{\mathcal{W}}(\Delta_1)\widehat{\mathcal{W}}(\Delta_2) = \sum_{\Delta} C^{\Delta}_{\Delta_1\Delta_2}\widehat{\mathcal{W}}(\Delta).
\tag{42}
$$

We note that in contrast to (33), there is no restriction on the sizes of the Young diagrams Δ_1, Δ_2, and Δ and there is only the selection rule

$$
\max(|\Delta_1|, |\Delta_2|) \le |\Delta| \le |\Delta_1| + |\Delta_2|.
\tag{43}
$$

These new structure constants with $|\Delta_1| = |\Delta_2| = |\Delta|$ nevertheless coincide with the structure constants of conjugation-class algebra (33). The Feynman diagram technique in Sec. 2.3 can be considered a pictorial representation of (42), and expression (25) for the $\widehat{\mathcal{W}}$ operator in terms of the time variables can be considered a corollary of (42) projected to the $|\Delta_1| = |\Delta_2| = |\Delta|$ subset. In this case, it implies that

$$
\widehat{\mathcal{W}}(\Delta_1)\widetilde{p(\Delta_2)} = \sum_{\Delta} C^{\Delta}_{\Delta_1,\Delta_2}\widetilde{p(\Delta)} = \widehat{\mathcal{W}}(\Delta_2)\widetilde{p(\Delta_1)}, \qquad |\Delta_1| = |\Delta_2|.
\tag{44}
$$

Table 1

$R \backslash \Delta$	1	2	11	3	21	111	4	31	22	211	1111	5	41	32	311	221	2111	11111
1	1																	
2	2	1	1															
11	2	−1	1															
3	3	3	3	2	3	1												
21	3	0	3	−1	0	1												
111	3	−3	3	2	−3	1												
4	4	6	6	8	12	4	6	8	3	6	1							
31	4	2	6	0	4	4	−2	0	−1	2	1							
22	4	0	6	−4	0	4	0	−4	3	0	1							
211	4	−2	6	0	−4	4	2	0	−1	−2	1							
1111	4	−6	6	8	−12	4	−6	8	3	−6	1							
5	5	10	10	20	30	10	30	40	15	30	5	24	30	20	20	15	10	1
41	5	5	10	5	15	10	0	10	0	15	5	−6	0	−5	5	0	5	1
32	5	2	10	−4	6	10	−6	−8	3	6	5	0	−6	4	−4	3	2	1
311	5	0	10	0	0	10	0	0	−5	0	5	4	0	0	0	−5	0	1
221	5	−2	10	−4	−6	10	6	−8	3	−6	5	0	6	−4	−4	3	−2	1
2111	5	−5	10	5	−15	10	0	10	0	−15	5	−6	0	5	5	0	−5	1
11111	5	−10	10	20	−30	10	−30	40	15	−30	5	24	−30	−20	20	15	−10	1

Furthermore, in accordance with (6), the eigenvalues $\varphi_R(\Delta)$ satisfy the same algebra (42):

$$\varphi_R(\Delta_1)\varphi_R(\Delta_2) = \sum_{\Delta} C^{\Delta}_{\Delta_1\Delta_2}\varphi_R(\Delta). \tag{45}$$

The structure constants in this relation are independent of R, which is not so obvious if we extract $\varphi_R(\Delta)$ from character expansion (2). We can verify this explicitly for the first few $\varphi_R(\Delta)$ using Table 1 for them ($\varphi_R(\Delta)$ differs by a factor from symmetric group character [25]).

2.4.4. Examples of structure constants. We now give some explicit examples of (42): a multiplication table restricted to the case where $|\Delta| \leq 4$. Many of the examples are direct consequences of relations in Sec. 2.4.2. We note that the explicit dependence on N that appeared in the normal-ordered products in Sec. 2.4.1 disappears when we consider products of the normal-ordered operators $\widehat{\mathcal{W}}$. The components satisfying $|\Delta_1| = |\Delta_2| = |\Delta|$, dictated by permutation composition (33), are underlined:

$$\underline{\widehat{\mathcal{W}}([1])\widehat{\mathcal{W}}([1])} = \underline{\widehat{\mathcal{W}}([1])} + 2\widehat{\mathcal{W}}([1,1]),$$

$$\widehat{\mathcal{W}}([1])\widehat{\mathcal{W}}([2]) = 2\widehat{\mathcal{W}}([2]) + \widehat{\mathcal{W}}([2,1]),$$

$$\widehat{\mathcal{W}}([1])\widehat{\mathcal{W}}([1,1]) = 2\widehat{\mathcal{W}}([1,1]) + 3\widehat{\mathcal{W}}([1,1,1]),$$

$$\widehat{\mathcal{W}}([1])\widehat{\mathcal{W}}([3]) = 3\widehat{\mathcal{W}}([3]) + \widehat{\mathcal{W}}([3,1]),$$

$$\widehat{\mathcal{W}}([1])\widehat{\mathcal{W}}([2,1]) = 3\widehat{\mathcal{W}}([2,1]) + 2\widehat{\mathcal{W}}([2,1,1]),$$

$$\widehat{\mathcal{W}}([1])\widehat{\mathcal{W}}([1,1,1]) = 3\widehat{\mathcal{W}}([1,1,1]) + 4\widehat{\mathcal{W}}([1,1,1,1]),$$

$$\widehat{\mathcal{W}}([1])\widehat{\mathcal{W}}([4]) = 4\widehat{\mathcal{W}}([4]) + \widehat{\mathcal{W}}([4,1]),$$

$$\widehat{\mathcal{W}}([1])\widehat{\mathcal{W}}([3,1]) = 4\widehat{\mathcal{W}}([3,1]) + 2\widehat{\mathcal{W}}([3,1,1]),$$

$$\widehat{\mathcal{W}}([1])\widehat{\mathcal{W}}([2,2]) = 4\widehat{\mathcal{W}}([2,2]) + \widehat{\mathcal{W}}([2,2,1]),$$

$$\widehat{\mathcal{W}}([1])\widehat{\mathcal{W}}([2,1,1]) = 4\widehat{\mathcal{W}}([2,1,1]) + 3\widehat{\mathcal{W}}([2,1,1,1]),$$

$$\widehat{\mathcal{W}}([1])\widehat{\mathcal{W}}([1,1,1,1]) = 4\widehat{\mathcal{W}}([1,1,1,1]) + 5\widehat{\mathcal{W}}([1,1,1,1,1]),$$

$$\underline{\widehat{\mathcal{W}}([1,1])\widehat{\mathcal{W}}([2])} = \underline{\widehat{\mathcal{W}}([2])} + 2\widehat{\mathcal{W}}([2,1]) + \widehat{\mathcal{W}}([2,1,1]),$$

$$\underline{\widehat{\mathcal{W}}([1,1])\widehat{\mathcal{W}}([1,1])} = \underline{\widehat{\mathcal{W}}([1,1])} + 6\widehat{\mathcal{W}}([1,1,1]) + 6\widehat{\mathcal{W}}([1,1,1,1]),$$

$$\underline{\widehat{\mathcal{W}}([2])\widehat{\mathcal{W}}([2])} = \underline{\widehat{\mathcal{W}}([1,1])} + 3\widehat{\mathcal{W}}([3]) + 2\widehat{\mathcal{W}}([2,2]),$$

$$\widehat{\mathcal{W}}([1,1])\widehat{\mathcal{W}}([3]) = 3\widehat{\mathcal{W}}([3]) + 3\widehat{\mathcal{W}}([3,1]) + \widehat{\mathcal{W}}([3,1,1]),$$

$$\widehat{\mathcal{W}}([1,1])\widehat{\mathcal{W}}([2,1]) = 3\widehat{\mathcal{W}}([2,1]) + 6\widehat{\mathcal{W}}([2,1,1]) + \widehat{\mathcal{W}}([2,1,1,1]),$$

$$\widehat{\mathcal{W}}([1,1])\widehat{\mathcal{W}}([1,1,1]) = 3\widehat{\mathcal{W}}([1,1,1]) + 12\widehat{\mathcal{W}}([1,1,1,1]) + 10\widehat{\mathcal{W}}([1,1,1,1,1]),$$

$$\widehat{\mathcal{W}}([2])\widehat{\mathcal{W}}([3]) = \widehat{\mathcal{W}}([3,2]) + 4\widehat{\mathcal{W}}([4]) + 2\widehat{\mathcal{W}}([2,1]),$$

$$\widehat{\mathcal{W}}([2])\widehat{\mathcal{W}}([2,1]) = 2\widehat{\mathcal{W}}([2,2,1]) + 3\widehat{\mathcal{W}}([3,1]) + 4\widehat{\mathcal{W}}([2,2]) + 3\widehat{\mathcal{W}}([3]) + 3\widehat{\mathcal{W}}([1,1,1]),$$

$$\widehat{\mathcal{W}}([2])\widehat{\mathcal{W}}([1,1,1]) = \widehat{\mathcal{W}}([2,1]) + 2\widehat{\mathcal{W}}([2,1,1]) + \widehat{\mathcal{W}}([2,1,1,1]).$$

2.5. From $\widehat{D}$ to differential operators in time variables. One way to express the operators $\widehat{\mathcal{W}}$ in terms of the time variables is already given in (12). But it is much simpler to extract such expressions directly from (25), i.e., by making a Miwa transformation back from the matrix variable X to the times $p_k = \operatorname{tr} X^k$. This is done by a simple rule: when acting on a function of time variables, the X derivatives give

$$\widehat{D}_{ab}F(p) = X_{ac}\,\frac{\partial}{\partial X_{bc}}F(p) = \sum_{k=1}^{\infty} k(X^k)_{ab}\frac{\partial F(p)}{\partial p_k}. \tag{46}$$

Further, the operators $\widehat{D}$ act both on the X that appeared in the first stage and on the remaining function of time variables:

$$\widehat{D}_{a'b'}\widehat{D}_{ab}F(p) = \sum_{k,l=1}^{\infty} kl(X^l)_{a'b'}(X^k)_{ab}\frac{\partial^2 F(p)}{\partial p_k\,\partial p_l} + \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} k(X^j)_{ab'}(X^{k-j})_{a'b}\frac{\partial F(p)}{\partial p_k}, \tag{47}$$

where we use

$$\widehat{D}_{a'b'}(X^k)_{ab} = X_{a'c'}\frac{\partial}{\partial X_{b'c'}}(X^k)_{ab} = \sum_{j=0}^{k-1} X_{a'c'}(X^j)_{ab'}(X^{k-j-1})_{c'b} = \\ = \sum_{j=0}^{k-1}(X^j)_{ab'}(X^{k-j})_{a'b}. \tag{48}$$

We note that the power of X in the second factor in the right-hand side of (48) is always nonzero, while it can vanish in the first factor. If we considered a normal-ordered product of operators instead of (47), then this power would also be nonzero:

$$:\widehat{D}_{a'b'}\widehat{D}_{ab}\colon F(p) = \sum_k \left(k\sum_{j=1}^{k-1}(X^j)_{ab'}(X^{k-j})_{a'b}\right)\frac{\partial F(p)}{\partial p_k} + \\ + \sum_{k,l} kl(X^k)_{ab}(X^l)_{a'b'}\frac{\partial^2 F(p)}{\partial p_k\,\partial p_l}. \tag{49}$$

This is the property that guarantees that the potential dependence on N is eliminated from the formulas, as it should be for operators expressible in terms of time variables, and hence independent of the details of the Miwa transformation (of which N is an additional parameter).

The first few examples of the cut-and-join operators in terms of the time variables are

$$\widehat{\mathcal{W}}([1]) = \operatorname{tr}\widehat{D} = \sum_{k=1} kp_k\frac{\partial}{\partial p_k}, \tag{50}$$

$$\widehat{\mathcal{W}}([2]) = \frac{1}{2}:\operatorname{tr}\widehat{D}^2\colon = \frac{1}{2}\sum_{a,b=1}^{\infty}\left((a+b)p_a p_b\frac{\partial}{\partial p_{a+b}} + abp_{a+b}\frac{\partial^2}{\partial p_a\,\partial p_b}\right), \tag{51}$$

$$\widehat{\mathcal{W}}([1,1]) = \frac{1}{2!}:(\operatorname{tr}\widehat{D})^2\colon = \frac{1}{2}\left(\sum_{a=1}^{\infty} a(a-1)p_a\frac{\partial}{\partial p_a} + \sum_{a,b=1}^{\infty} abp_a p_b\frac{\partial^2}{\partial p_a\,\partial p_b}\right), \tag{52}$$

$$\widehat{\mathcal{W}}([3]) = \frac{1}{3}:\operatorname{tr}\widehat{D}^3\colon = \frac{1}{3}\sum_{a,b,c\geq 1}^{\infty} abcp_{a+b+c}\frac{\partial^3}{\partial p_a\,\partial p_b\,\partial p_c} + \\ + \frac{1}{2}\sum_{a+b=c+d} cd(1-\delta_{ac}\delta_{bd})p_a p_b\frac{\partial^2}{\partial p_c\,\partial p_d} + \\ + \frac{1}{3}\sum_{a,b,c\geq 1}(a+b+c)(p_a p_b p_c + p_{a+b+c})\frac{\partial}{\partial p_{a+b+c}}, \tag{53}$$

$$\widehat{\mathcal{W}}([2,1]) = \frac{1}{2} :\mathrm{tr}\, \widehat{D}^2\, \mathrm{tr}\, \widehat{D}: = \frac{1}{2}\sum_{a,b\geq 1}(a+b)(a+b-2)p_a p_b \frac{\partial}{\partial p_{a+b}} +$$
$$+ \frac{1}{2}\sum_{a,b\geq 1} ab(a+b-2)p_{a+b}\frac{\partial^2}{\partial p_a\, \partial p_b} +$$
$$+ \frac{1}{2}\sum_{a,b,c\geq 1}(a+b)cp_a p_b p_c \frac{\partial^2}{\partial p_{a+b}\, \partial p_c} +$$
$$+ \frac{1}{2}\sum_{a,b,c\geq 1} abcp_a p_{b+c}\frac{\partial^3}{\partial p_a\, \partial p_b\, \partial p_c}, \tag{54}$$
$$\widehat{\mathcal{W}}([1,1,1]) = \frac{1}{3!} :(\mathrm{tr}\, \widehat{D})^3: = \frac{1}{6}\sum_{a\geq 1} a(a-1)(a-2)p_a\frac{\partial}{\partial p_a} +$$
$$+ \frac{1}{4}\sum_{a,b} ab(a+b-2)p_a p_b \frac{\partial^2}{\partial p_a\, \partial p_b} +$$
$$+ \frac{1}{6}\sum_{a,b,c\geq 1} abcp_a p_b p_c \frac{\partial^3}{\partial p_a\, \partial p_b\, \partial p_c}. \tag{55}$$

As should be expected from (39) and (41), it follows from these formulas that

$$\begin{aligned}
&\widehat{\mathcal{W}}([1,1]) = \frac{1}{2}\widehat{\mathcal{W}}([1])(\widehat{\mathcal{W}}([1]) - 1),\\
&\widehat{\mathcal{W}}([2,1]) = \widehat{\mathcal{W}}([2])(\widehat{\mathcal{W}}([1]) - 2),\\
&\widehat{\mathcal{W}}([1,1,1]) = \frac{1}{6}\widehat{\mathcal{W}}([1])(\widehat{\mathcal{W}}([1]) - 2)(\widehat{\mathcal{W}}([1]) - 1).
\end{aligned} \tag{56}$$

The manifest expressions for higher operators rapidly become much more involved. But there is a much more compact representation for the operators: when expressed in terms of the time variables, operators are in fact constructed from the scalar field current

$$\partial\Phi(z) = \sum_k \left(kt_k z^k + \frac{1}{z^k}\frac{\partial}{\partial t_k}\right) = \sum_k \left(p_k z^k + \frac{k}{z^k}\frac{\partial}{\partial p_k}\right) \tag{57}$$

and from an additional dilatation operator

$$\widehat{R} = \left(z\frac{\partial}{\partial z}\right)^2 \tag{58}$$

(see [23], [42] for more details; we give only the first few examples here).

The normal ordering in these formulas means that all factors with p are to the left of factors with $\partial/\partial p$ and we do not take p derivatives of the p when constructing an operator from $\partial\Phi(z)$. The subscript 0 means that the coefficient of z^0 in the z series should be taken. Because adding units to the Young diagram, as we have seen, is a trivial procedure, we here list only the operators corresponding to Young diagrams without units [42]:

$$
\begin{aligned}
\widehat{\mathcal{W}}([1]) &= \widehat{C}_1, \\
\widehat{\mathcal{W}}([2]) &= \frac{1}{2}\widehat{C}_2, \\
\widehat{\mathcal{W}}([3]) &= \frac{1}{3}\widehat{C}_3 - \frac{1}{2}\widehat{C}_1^2 + \frac{1}{3}\widehat{C}_1, \\
\widehat{\mathcal{W}}([2,2]) &= \frac{1}{8}\widehat{C}_2^2 - \frac{1}{2}\widehat{C}_3 + \frac{1}{2}\widehat{C}_1^2 - \frac{1}{4}\widehat{C}_1, \\
\widehat{\mathcal{W}}([4]) &= \frac{1}{4}\widehat{C}_4 - \widehat{C}_1\widehat{C}_2 + \frac{5}{4}\widehat{C}_1,
\end{aligned} \tag{59}
$$

where the Casimir operators are [42]

$$
\begin{aligned}
\widehat{C}_1 &= \frac{1}{2}:[(\partial\Phi)^2]_0:, \\
\widehat{C}_2 &= \frac{1}{3}:[(\partial\Phi)^3]_0:, \\
\widehat{C}_3 &= \frac{1}{4}:[(\partial\Phi)^4 + \partial\Phi(\widehat{R}\partial\Phi)]_0:, \\
\widehat{C}_4 &= \frac{1}{5}:\left[(\partial\Phi)^5 + \frac{5}{2}(\partial\Phi)^2(\widehat{R}\partial\Phi)\right]_0:, \\
\widehat{C}_k &= \frac{1}{k+1}:\left[(\partial\Phi)^{k+1} + \frac{(k+1)!}{4!\,(k-2)!}(\partial\Phi)^{k-1}(\widehat{R}\partial\Phi) + \cdots\right]_0:.
\end{aligned}
$$

2.6. The $GL(\infty)$ characters. The GL characters $\chi_R(t)$ are defined using the first Weyl determinant formula

$$\chi_R(t) = \det_{ij} s_{\mu_i+j-i}(t), \tag{60}$$

where $s_k(t)$ are the Shur polynomials,

$$\exp\left(\sum_k t_k z^k\right) = \sum_k s_k(t) z^k. \tag{61}$$

As a result of the Miwa transformation $p_k = kt_k = \operatorname{tr} X^k$, the same characters are expressed in terms of the eigenvalues of X by the second Weyl formula

$$\chi_R[X] = \chi_R\left(t_k = \frac{1}{k}\operatorname{tr} X^k\right) = \frac{\det_{ij} x_i^{\mu_j - j}}{\det_{ij} x_i^{-j}}. \tag{62}$$

The expansion of $\chi_R(t)$ in powers of the p defines the coefficients $\varphi_R(\Delta)$ by Eq. (2) for $|R| = |\Delta|$ and by Eq. (3) for all other Δ. In (2), the parameter d_R is the value of the character at the point $t_k = \delta_{k,1}$,

$$d_R = \chi_R(\delta_{k1}), \tag{63}$$

and is given by the hook formula

$$d_R = \prod_{\text{all boxes of } R} \frac{1}{\text{hook length}} = \frac{\prod_{i<j=1}^{|R|}(\mu_i - \mu_j - i + j)}{\prod_{i=1}^{|R|}(\mu_i + |R| - i)!}. \tag{64}$$

We can also introduce a natural scalar product on the characters,

$$\langle \chi_R(t), \chi_{R'}(t) \rangle = \delta_{RR'}, \tag{65}$$

given by the explicit formula

$$\langle A(t), B(t) \rangle \equiv A\left(\frac{\partial}{\partial p}\right) B(t) \Big|_{t_k=0}. \tag{66}$$

In particular,

$$\langle p(\Delta), \widetilde{p(\Delta')} \rangle = \delta_{\Delta\Delta'}. \tag{67}$$

These formulas together with (2) immediately lead to the inverse expansion

$$\widetilde{p(\Delta)} = \sum_R d_R \varphi_R(\Delta) \chi_R(t) \delta_{|\Delta|,|R|}. \tag{68}$$

Hence, (6) is actually an exhaustive alternative definition of the operators $\widehat{\mathcal{W}}(\Delta)$, and it can be verified that (12) does satisfy this definition (see Sec. 2.7 and [23], [42]). The equivalence of the two definitions, (25) and (6), follows from formula (1).

2.7. Deriving (6) from (12). The idea of a straightforward derivation of (6) from (12) is as follows. For simplicity, we consider a Δ that does not contain units (the generalization is straightforward). Obviously,

$$:\widetilde{D(\Delta)}: e^{t_1} = :\widetilde{D(\Delta)}: e^{\operatorname{tr} X} = \widetilde{p(\Delta)} e^{t_1}. \tag{69}$$

Because

$$e^{t_1} = \sum_R d_R \chi_R(t), \tag{70}$$

it hence follows that

$$\sum_R d_R :\widetilde{D(\Delta)}: \chi_R(t) = \widetilde{p(\Delta)} e^{t_1}. \tag{71}$$

The right-hand side of this formula can be rewritten using (68) and (3) as

$$\sum_k \widetilde{p(\Delta)} \frac{t_1^k}{k!} = \sum_k \widetilde{p([\Delta, \underbrace{1, \dots, 1}_k])} = \sum_k \sum_R d_R \varphi_R([\Delta, \underbrace{1, \dots, 1}_k]) \chi_R =$$
$$= \sum_k \sum_{R: |R| = |\Delta| + k} d_R \varphi_R(\Delta) \chi_R = \sum_R d_R \varphi_R(\Delta) \chi_R, \tag{72}$$

which together with (71) and the fact that $\chi_R(t)$ are the eigenfunctions of $:\widetilde{D(\Delta)}:$ ultimately leads to (6).

2.8. Details of (14). Deviation from the naive formula (14) arises because we should carefully impose the condition $|\Delta| = R$ in (2) when passing from $\varphi_R(\Delta)$ in (1) to the characters $\chi_R(t')$ in (4):

$$Z(t, t', \dots) = \sum_q \left\{ \sum_{\Delta, \Delta'} p(\Delta) p'(\Delta') \delta_{|\Delta|,q} \delta_{|\Delta'|,q} \operatorname{Cov}_q(\Delta, \Delta', \dots) \right\} =$$
$$= \sum_R \chi_R(t) \chi_R(t') \dots \tag{73}$$

or, alternatively,

$$Z(t,t',\dots)=\sum_{\Delta',R} d_R\chi_R(t)\varphi_R(\Delta)p'(\Delta)\cdots\delta_{|\Delta|,|R|}=$$

$$=\oint\frac{dz}{z}\sum_{\Delta,R} d_R\chi_R(t)\varphi_R(\Delta)p'(\Delta)z^{|\Delta|-|R|}\dots=$$

$$=\oint\frac{dz}{z}\sum_{\Delta} z^{|\Delta|}p'(\Delta)\,{:}\widetilde{D(\Delta)}{:}\sum_R d_R\chi_R(t)z^{|R|}\dots=$$

$$=\oint\frac{dz}{z}\,{:}\exp\biggl(\sum_{k=1}^{\infty}z^k t'_k\widehat{D}_k\biggr){:}\sum_R d_R\chi_R(t)z^{|R|}\dots.$$

This is the full (correct) version of Eq. (14). If we consider $Z(t,t'|0)$, then the sum over R is equal to $e^{t_1/z}$, and as a simplest example, we obtain

$$Z(t,t'|0)=\oint\frac{dz}{z}\,{:}\exp\biggl(\sum_{k=1}^{\infty}z^k t'_k\widehat{D}_k\biggr){:}e^{t_1/z}=$$

$$=\exp\biggl(\sum_k kt_kt'_k\biggr)\oint\frac{dz}{z}e^{t_1/z}=\exp\biggl(\sum_k kt_kt'_k\biggr).\qquad(74)$$

Generalizations are easily derived. If we want to consider multiple Hurwitz numbers with more sets of t variables, then we must consider additional δ-functions and integrals over z.

3. Summary and conclusion

The general cut-and-join operator $\widehat{\mathcal{W}}(\Delta)$ is associated with an arbitrary Young diagram Δ and can be defined in two alternative ways.

First, we can define $\widehat{\mathcal{W}}(\Delta)$ in terms of the characters, requiring that

$$\widehat{\mathcal{W}}(\Delta)\chi_R(t)=\varphi_R(\Delta)\chi_R(t)\qquad(75)$$

for any Young diagram R. Then the Hurwitz partition function for two sets of time variables, for example, is equal to

$$\mathcal{Z}(t,\bar{t}|\beta)\equiv\sum_R\chi_R(t)\chi_R(\bar{t})\exp\biggl(\sum_\Delta\beta_\Delta\varphi_R(\Delta)\biggr)=\exp\biggl(\sum_\Delta\beta_\Delta\widehat{\mathcal{W}}(\Delta)\biggr)Z(t,\bar{t}|0),\qquad(76)$$

where

$$Z(t,\bar{t}|0)=\sum_R\chi_R(t)\chi_R(\bar{t})=\exp\biggl(\sum_k kt_k\bar{t}_k\biggr)=\exp\biggl(\sum_k\frac{1}{k}p_k\bar{p}_k\biggr).$$

The Hurwitz Toda τ-function [27], [9] for the *double Hurwitz numbers*,

$$Z(t,\bar{t}|\beta)=\sum_R\chi_R(t)\chi_R(\bar{t})e^{\beta_2\varphi_R([2])}=e^{\beta_2\widehat{\mathcal{W}}([2])}Z(t,\bar{t},0)$$

is a particular case, and the Kontsevich–Hurwitz KP τ-function for the *single* (or *ordinary*) *Hurwitz numbers* is a further restriction to $\bar{t}_k=\delta_{k,1}$. The integrability in these two examples is preserved because the simplest cut-and-join operator $\widehat{\mathcal{W}}([2])$ coincides with the (second) Casimir operator, and integrability is present when any combination of Casimir operators acts on the τ-function [27]. The integrability is violated by the general cut-and-join operators with $|\Delta| \geq 3$. Of course, the β variables can be regarded as associated with some *new* integrable structure, reflected in the commutativity of the operators $\widehat{\mathcal{W}}(\Delta)$,

$$[\widehat{\mathcal{W}}(\Delta_1), \widehat{\mathcal{W}}(\Delta_2)] = 0 \quad \forall \Delta_1, \Delta_2. \tag{77}$$

But there is no obvious way to relate this integrability to the group-theory-induced Hirota-like bilinear relations [47], and there are even fewer chances that it is somehow induced by the free-fermion representation of $\widehat{U(1)}$ (these are the two features built into the definition of integrable hierarchies of the KP/Toda type).

Second, we can define $\widehat{\mathcal{W}}(\Delta)$ in terms of permutations and their cyclic decompositions. This is the problem directly related to the merger of ramification points of the Hurwitz covering. The central formula is (25),

$$\Delta \circ \mathcal{P}(p) = \widehat{\mathcal{W}}(\Delta)\mathcal{P}(p), \tag{78}$$

and in Sec. 2.2, we explained how $\widehat{\mathcal{W}}(\Delta)$ is explicitly reconstructed from the knowledge of permutation compositions.

Both these definitions, conditions (75) and (78), are explicitly resolved by (12), which is a direct generalization of the representation of the simplest cut-and-join operator $\widehat{\mathcal{W}}([2])$ in [24]. The first few operators in this set are listed in Sec. 2.5. These operators form a commutative associative algebra

$$\widehat{\mathcal{W}}(\Delta_1)\widehat{\mathcal{W}}(\Delta_2) = \sum_{\Delta} C^{\Delta}_{\Delta_1\Delta_2} \widehat{\mathcal{W}}(\Delta). \tag{79}$$

We call it the universal Hurwitz algebra because it allows defining the universal Hurwitz numbers for an arbitrary collection of Young diagrams, not necessarily of the same size. That is, if complemented by (43), formula (1) allows defining the Hurwitz number as

$$\mathrm{Cov}(\Delta_1, \ldots, \Delta_m) = \sum_{\Delta} C^{\Delta}_{\Delta_1\cdots\Delta_m} \left(\sum_R d_R^2 \varphi(\Delta) \right), \tag{80}$$

where C is a combination of structure constants, for example,

$$C^{\Delta}_{\Delta_1\ldots\Delta_m} = \sum_{\Delta_a, \Delta_b, \ldots, \Delta_c} C^{\Delta_a}_{\Delta_1\Delta_2} C^{\Delta_b}_{\Delta_a\Delta_3} \cdots C^{\Delta}_{\Delta_c\Delta_m} \tag{81}$$

(the order of pairing is in fact inessential because the algebra is associative and commutative). Evaluating the Hurwitz numbers thus reduces to evaluating the single form on the linear algebra of Young diagrams

$$\sum_R d_R^2 \varphi_R(\Delta) = \begin{cases} 0, & \Delta \text{ contains more than one column}, \\ \dfrac{1}{n!}, & \Delta = [\underbrace{1, \ldots, 1}_{n}]. \end{cases} \tag{82}$$

That the form is zero except on the single-column diagrams is a direct consequence of fundamental sum rule (70).

More details about the character-related description and the integrability aspects of the problem can be found in [23], [42]. All these relations here and in [23], [42] deserve better understanding. Studying the matrix-model representation [24] for the Hurwitz KP τ-function, its application (using [35]) to understanding the mysterious relation of twisting of the Hurwitz KP τ-function to the Kontsevich τ-function [10], [19], [22], and the further non-Abelian generalization of this entire formalism to the Hurwitz numbers for coverings of Riemann surfaces with boundaries [14] (e.g., a disk instead of the Riemann sphere) is especially interesting.

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