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Complete Solution For The Time Fractional Diffusion Problem With Mixed Boundary Conditions by Operational Method

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Abstract

In this study, we present some new results for the time fractional mixed boundary value problems. We consider a generalization of the Heat - conduction problem in two dimensions that arises during the manufacturing of p - n junctions. Constructive examples are also provided throughout the paper. The main purpose of this article is to present mathematical results that are useful to researchers in a variety of fields.

Keywords: Integral transform method; Laplace transform; Caputo fractional derivative; Newmann function; Modified Bessel's functions; Hankel transform; Fourier series; P- N junctions.
AMS 2010 codes: 26A33; 44A10; 44A15; 44A35.

1 Introduction

Fractional partial differential equations provide an excellent model for the description of memory and hereditary properties of various processes and materials. This is the main advantage of fractional partial differential equations in comparison with classical integer - order models, in which such effects are in fact neglected. It is well - known that the mixed boundary value problems occur in the theory of elasticity in connection with punching and crack problems. The main objective of present study is to justify, in a clear fasion, the interesting possibility that fractional methods represent for modelling the dynamics certain phenomena which ordinary models cannot. The solution of the mixed boundary value problems requires considerable mathematical skills. Most mixed boundary value problems are solved using integral transform method or separation of variables [7,13,14]. Transform method are usually led to the problem of solving dual or triple Fourier or Bessel integral equations. For a discussion of such equation see [4,5]. The main goal of this study is to give an updated treatment of this subject. An alternative method of solving mixed boundary value problem involves Green's function.

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Conformal mapping is a mathematical technique to solve certain types of mixed boundary value problems [4]. It should be emphasized that the focus of this paper is only on integral transform method for solving fractional partial differential equations. However, some papers have recently presented numerical techniques for this class of problems. In[9], the authors used a q- homotopy analysis transform method to find the solution for fractional Drinfeld - Sokolov - Wilson equation, where fractional derivative defined with Atangana - Baleanu (AB) operator. In[16], P.Veeresha used a numerical technique called q - homotopy analysis transform method to solve a non - linear Fisher's equation of fractional order. Finally, we list a number of research articles where the background and many applications of numerical methods of solution could be found (see [8,9,12,15-17]) and focus mostly on the solution of non - linear equations.

1.1 Definitions and Notations

In the last three decades or so, fractional derivatives and notably fractional calculus have played a very important role in the various fields such as chemistry, biology, engineering, economics and signal processing. At this point, it should be pointed out that several definitions have been proposed of a fractional derivative, among those the Riemann - Liouville and Caputo fractional derivatives are the most popular. The differential equations defined in terms of Riemann - Liouville derivatives require fractional initial conditions, whereas the differential equations defined in terms of Caputo derivatives require regular boundary conditions. For this reason, the Caputo fractional derivatives are popular among engineers and scientists.

Definition 1.1. *The left Caputo fractional derivative of order α $(0<\alpha<1)$ of $\phi(t)$ is as follows*

$$D_a^{c,\alpha}\phi(t)=\tfrac{1}{\Gamma(1-\alpha)}\int_a^t\tfrac{1}{(t-\xi)^{\alpha}}\phi'(\xi)d\xi, \qquad (1.1)$$

Definition 1.2. *The Laplace transform of the function $f(t)$ is as follows[6]*

$$\mathscr{L}\{f(t)\}=\int_0^{\infty}e^{-st}f(t)dt:=F(s). \qquad (1.2)$$

If $\mathscr{L}\{f(t)\}=F(s)$, then $\mathscr{L}^{-1}\{F(s)\}$ is given by

$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{st}F(s)ds, \qquad (1.3)$$

where $F(s)$ is analytic in the region $\mathrm{Re}(s)>c$. The parameter s is generally complex, but for the present time it is more convenient to consider it as real. The existence of the Laplace transform will depend upon the parameter s and the function $f(t)$. The above complex integration along vertical line $Res=c$ is known as Bromwich's integral[6,10]. The real merit of the Laplace transform is revealed by its effect on fractional derivatives. Here we derive a relation between the Laplace transform of the Caputo fractional derivative of a function and the Laplace transform of the function itself [11].

$$\mathscr{L}[D_{0,t}^{c,\alpha}f(t);s]=s^{\alpha}\mathscr{L}[f(t);s]-s^{\alpha-1}f(0^+).0<\alpha<1. \qquad (1)$$

Definition 1.3. *A two - parameter Mittag - Leffler function is defined by the series expansion [11]*

$$E_{\alpha,\beta}(\xi)=\sum_{k=0}^{+\infty}\frac{\xi^k}{\Gamma(\alpha k+1)}. \qquad (2)$$

Lemma 1.1. The following integral identities hold true[11]

$$1.\int_0^{+\infty}e^{-st}t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm\lambda t^{\alpha})dt=\frac{k!s^{\alpha-\beta}}{(s^{\alpha}\mp\lambda)^{k+1}}. \qquad (3)$$

$$2. \int_0^{+\infty} e^{-st} t^{\beta-1} E_{\alpha,\beta}(\pm\lambda t^{\alpha}) dt = \frac{s^{\alpha-\beta}}{(s^{\alpha} \mp \lambda)}. \tag{4}$$

Note

$$3. \int_0^{+\infty} e^{-st} t^{\alpha} E_{\alpha,\alpha-1}(\pm\lambda t^{\alpha}) dt = \frac{1}{s(s^{\alpha} \mp \lambda)}. \tag{5}$$

Note: With $E^{(k)}_{\alpha,\beta}(\xi) = \frac{d^k}{d\xi^k} E_{\alpha,\beta}(\xi)$.
Many problems of physical interest lead to Laplace transform whose inverses are not readily expressed in terms of tabulated functions. Therefore, it is highly desirable to have methods for inversion. In this section an algorithm to invert the Laplace transform is presented [1,2,3]
Remark. In the next Lemmas, we need the following integral representation for the modified Bessel's function of the second kind

$$1. \quad K_0(a\xi) = \int_0^{+\infty} e^{-a\xi coshz} dz,$$

Lemma 1.2. By using an appropriate integral representation for the modified Bessel's functions of the second kind of order zero, $e^{\lambda s} K_0(\lambda s)$, show that

$$\mathscr{L}^{-1}[e^{\lambda s} K_0(\lambda s)] = \frac{1}{\sqrt{t(t+2\lambda)}}. \tag{6}$$

Proof. In view of the definition1.1 taking the inverse Laplace transform of the given $e^{\lambda s} K_0(\lambda s)$, we obtain

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} (e^{\lambda s} K_0(\lambda s)) ds, \tag{7}$$

at this stage, using the following integral representation for $K_0(\lambda s)$.

$$e^{\lambda s} K_0(\lambda s) = \int_0^{\infty} e^{\lambda s - (\lambda s) coshz} dz. \tag{8}$$

By setting relation (8) in (7), we obtain

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} (\int_0^{\infty} e^{\lambda s - (\lambda s) coshz} dz) ds, \tag{9}$$

let us change the order of integration in relation (9), we arrive at

$$f(t) = \int_0^{\infty} (\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{s(t-(\lambda coshz - \lambda))} ds) dz, \tag{10}$$

the value of the inner integral is $\delta(t - (\lambda coshz - \lambda))$, therefore

$$f(t) = \int_0^{\infty} \delta(t - (\lambda coshz - \lambda)) dz, \tag{11}$$

after making a change of variable $t - (\lambda coshz - \lambda) = \psi$, and considerable algebra, we obtain

$$f(t) = \int_{-\infty}^{0} \frac{\delta(\psi)}{\lambda \sqrt{(\frac{t+\lambda-\psi}{\lambda})^2 - 1)}} d\psi = \frac{1}{\sqrt{t(t+2\lambda)}}. \tag{12}$$

In view of the relation (1.3), we obtain the following integral relation

$$\int_0^\infty \frac{e^{-st}}{\sqrt{t(t+2\lambda)}}dt = e^{\lambda s}K_0(\lambda s). \tag{13}$$

Thus, we get the following integral representation for the modified Bessel's function of the second kind of order zero,

$$K_0(\lambda s) = \int_0^{+\infty} \frac{e^{-(\lambda+t)s}}{\sqrt{t(t+2\lambda)}}.$$

Definition 1.4. *The Hankel transform of order ν of a function $f(t)$ is given by*

$$\mathscr{H}_\nu[f(t);\rho] = \int_0^{+\infty} f(t)tJ_\nu(\rho t)dt = F(\rho). \tag{14}$$

In order for a transformation to be useful in solving boundary value problems, it must have an inverse. The inverse Hankel transform of a function $F(\rho)$ is given by [6]

$$\mathscr{H}_\nu^{-1}[F(\rho);t] = \int_0^{+\infty} F(\rho)\rho J_\nu(t\rho)d\rho = f(t). \tag{15}$$

Lemma 1.3. We have the following relation

$$\mathscr{H}_\nu[1;\rho] = \int_0^{+\infty} tJ_\nu(\rho t)dt = \frac{\nu}{\rho^2}. \tag{16}$$

Proof: Let us start with the following Laplace transform relation

$$\mathscr{L}[J_\nu(\lambda t)] = \int_0^{+\infty} e^{-st}J_\nu(\lambda t)dt = \frac{(\sqrt{s^2+\lambda^2}-s)^\nu}{\lambda^\nu\sqrt{s^2+\lambda^2}}, \tag{17}$$

taking derivative with respect to the parameter s from both sides of the above relation, we obtain

$$\int_0^{+\infty} -te^{-st}J_\nu(\lambda t)dt = -\lambda^\nu[\frac{-1}{2}(s^2+\lambda^2)^{\frac{-3}{2}}(\sqrt{s^2+\lambda^2}-s)^\nu]+......$$

$$..+\lambda^\nu[(s^2+\lambda^2)^{\frac{-1}{2}}(\sqrt{s^2+\lambda^2}-s)^{\nu-1}(\frac{s}{\sqrt{s^2+\lambda^2}}-1)], \tag{18}$$

in the above relation, if we set $s=0$ and $\lambda=\rho$ after simplifying, we obtain

$$\int_0^{+\infty} tJ_\nu(\rho t)dt = \frac{\nu}{\rho^2}. \tag{19}$$

Note. In special case $\nu = n+\frac{1}{2}$, we have

$$\int_0^{+\infty} tJ_{n+\frac{1}{2}}(\rho t)dt = \frac{n+\frac{1}{2}}{\rho^2}. \tag{20}$$

Corollary 1.1. The following integral identities hold true

1. $$\mathscr{H}_\nu[\delta(t-\lambda);\rho] = \int_0^{+\infty} tJ_\nu(\rho t)\delta(t-\lambda)dt = \lambda J_\nu(\lambda\rho). \tag{21}$$

$$2. \qquad \mathscr{H}_\nu[t^{\nu-1}e^{-at};\rho] = \int_0^{+\infty} t^\nu e^{-at} J_\nu(\rho t)dt = \frac{(2\rho)^\nu \Gamma(\nu+0.5)}{\sqrt{\pi}(\rho^2+a^2)^{\nu+0.5}} \tag{22}$$

Corollary 1.2. Parseval's relation for Hankel transform
If $F(\rho)$ and $G(\rho)$are Hankel transforms of $f(t)$ and $g(t)$, respectively, then we have the following relation

$$\int_0^{+\infty} tf(t)g(t)dt = \int_0^{+\infty} \rho F(\rho)G(\rho)d\rho. \tag{23}$$

Like the Laplace transform, the Hankel transforms used in a variety of applications. Perhaps the most common usage of the Hankel transform is in the solution of boundary value problems. However, there are other situations for which the properties of the Hankel transform are also very useful, such as in the evaluation of certain integrals.
Example 1.1. By using Parseval's relation show that

$$\int_0^{+\infty} \frac{\rho^{2\nu+1}d\rho}{[(\rho^2+a^2)(\rho^2+b^2)]^{\nu+0.5}} = \frac{\pi\Gamma(2\nu)}{4^\nu(a+b)^{2\nu}\Gamma^2(\nu+0.5)}. \tag{24}$$

Solution. Let us take $f(t) = t^{\nu-1}e^{-at}$ and $g(t) = t^{\nu-1}e^{-bt}$, with Hankel transforms

$$F(\rho) = \frac{(2\rho)^\nu\Gamma(\nu+0.5)}{\sqrt{\pi}(\rho^2+a^2)^{\nu+0.5}}, \qquad G(\rho) = \frac{(2\rho)^\nu\Gamma(\nu+0.5)}{\sqrt{\pi}(\rho^2+b^2)^{\nu+0.5}}. \tag{25}$$

respectively.
At this point, using Parseval's relation leads to

$$\int_0^{+\infty} e^{-(a+b)t}t^{2\nu-1}dt = \frac{\Gamma(2\nu)}{(a+b)^{2\nu}} = \int_0^{+\infty} \frac{\rho(2\rho)^{2\nu}\Gamma^2(\nu+0.5)}{\pi[(\rho^2+a^2)(\rho^2+b^2)]^{\nu+0.5}}d\rho. \tag{26}$$

After simplifying, we arrive at

$$\int_0^{+\infty} \frac{\rho^{2\nu+1}d\rho}{[(\rho^2+a^2)(\rho^2+b^2)]^{\nu+0.5}} = \frac{\pi\Gamma(2\nu)}{4^\nu(a+b)^{2\nu}\Gamma^2(\nu+0.5)}. \tag{27}$$

Lemma 1.4. The following integral relations hold true.

$$1. \int_0^{+\infty} x^{\nu+1}J_\nu(\rho x)\frac{dx}{(x^2+a^2)^{\lambda+1}} = \frac{a^{\nu-\lambda}\rho^\lambda}{2^\lambda\Gamma(\lambda+1)}K_{\nu-\lambda}(a\rho). \tag{28}$$

$$2. \int_0^{+\infty} \rho^{\lambda+1}J_\nu(x\rho)K_{\nu-\lambda}(a\rho) = \frac{2^\lambda\Gamma(\lambda+1)x^\nu}{a^{\nu-\lambda}(x^2+a^2)^{\lambda+1}}. \tag{29}$$

$$3. \int_0^{+\infty} \rho Y_\nu(x\rho)K_\nu(a\rho)d\rho = \frac{2\pi(a^{-\nu}x^\nu\cos\pi\nu - a^\nu x^{-\nu}}{2(x^2+a^2)\sin\pi\nu}. \tag{30}$$

Proof: Let us start with the following elementary integral identity

$$\frac{1}{(x^2+a^2)^{\lambda+1}} = \frac{1}{\Gamma(\lambda+1)}\int_0^{+\infty} e^{-(x^2+a^2)u}u^\lambda du. \tag{31}$$

By substituting this integral on the left hand side of the first integral and interchanging the order of integration, we obtain

$$\int_0^{+\infty} x^{\nu+1} J_\nu(\rho x)[\frac{1}{\Gamma(\lambda+1)}\int_0^{+\infty} e^{-(x^2+a^2)u} u^\lambda du]dx = ...$$

$$= \frac{1}{2\Gamma(\lambda+1)}\int_0^{+\infty} e^{-a^2u} u^\lambda (\int_0^{+\infty} x^{\nu+1} e^{-ux^2} J_\nu(\rho x)dx)du. \tag{32}$$

At this point, let us evaluate the inner integral by making a change of variable $x^2 = \phi$ and $\rho = 2\sqrt{\beta}$, after evaluation of the integral, we arrive at

$$\int_0^{+\infty} x^{\nu+1} J_\nu(\rho x)[\frac{1}{\Gamma(\lambda+1)}\int_0^{+\infty} e^{-(x^2+a^2)u} u^\lambda du]dx = ...$$

$$\frac{\beta^{\frac{\nu}{2}}}{\Gamma(\lambda+1)}\int_0^{+\infty} e^{-a^2u} u^\lambda (\frac{e^{-\frac{\beta}{u}}}{u^{\nu+1}})du = \frac{\rho^\nu}{2^\nu\Gamma(\lambda+1)}\int_0^{+\infty} e^{-(a^2u+\frac{\rho^2}{4u})}\frac{du}{2u^{\nu-\lambda+1}}. \tag{33}$$

Making a change of variable $a^2u = w$, after simplifying, we obtain

$$\int_0^{+\infty} x^{\nu+1} J_\nu(\rho x)\frac{dx}{(x^2+a^2)^{\lambda+1}} = \frac{\rho^\lambda a^{2(\nu-\lambda)}}{2^\lambda\Gamma(\lambda+1)}[(\frac{a\rho}{2})^{\nu-\lambda}\int_0^{+\infty} e^{-w-\frac{a^2\rho^2}{4w}}\frac{dw}{2w^{\nu-\lambda+1}}]. \tag{34}$$

But, the value of the integral in the braces is $K_{\nu-\lambda}(a\rho)$, therefore we get the following

$$\int_0^{+\infty} x^{\nu+1} J_\nu(\rho x)\frac{dx}{(x^2+a^2)^{\lambda+1}} = \frac{\rho^\lambda a^{\nu-\lambda}}{2^\lambda\Gamma(\lambda+1)} K_{\nu-\lambda}(a\rho). \tag{35}$$

At this stage, the above relation can be written as Hankel transform of a function as below

$$\mathscr{H}_\nu[\frac{x^\nu}{(x^2+a^2)^{\lambda+1}};\rho] = \frac{a^{\nu-\lambda}}{2^\lambda\Gamma(\lambda+1)}\rho^\lambda K_{\nu-\lambda}(a\rho). \tag{36}$$

Thus, by taking the inverse Hankel transform, we obtain

$$\int_0^{+\infty} \rho^{\lambda+1} J_\nu(x\rho)K_{\nu-\lambda}(a\rho)d\rho = \frac{2^\lambda\Gamma(\lambda+1)}{a^{\nu-\lambda}}\frac{x^\nu}{(x^2+a^2)^{\lambda+1}}. \tag{37}$$

Let us choose $\lambda = 0$, after simplifying we get

$$\int_0^{+\infty} \rho J_\nu(x\rho)K_\nu(a\rho)d\rho = \frac{2x^\nu}{a^\nu(x^2+a^2)}. \tag{38}$$

Let us consider the following relations,

$$K_\nu(\xi) = K_{-\nu}(\xi), \qquad Y_\nu(\xi) = \frac{\pi}{2}\frac{J_\nu(\xi)\cos(\pi\nu) - J_{-\nu}(\xi)}{\sin(\pi\nu)}. \tag{39}$$

Combination of (38) and (39) leads to the following integral relation

$$\int_0^{+\infty} \rho Y_\nu(x\rho) K_\nu(a\rho) d\rho = \frac{2\pi(a^{-\nu}x^\nu \cos\pi\nu - a^\nu x^{-\nu})}{2(x^2+a^2)\sin\pi\nu}. \tag{40}$$

Note: In the above relation, $Y_\nu(\xi)$ stands for the Newmann function of order ν.
Theorem 1.1. Let us assume that $\mathscr{L}[f(t)] = F(s)$, then the following relation holds true

$$\exp[-\exp(-\frac{t}{2})] f(\exp\frac{-t}{2}) = \mathscr{L}^{-1}[\int_0^{+\infty} \xi^s J_{2s}(2\sqrt{\xi}) F(\xi) d\xi]. \tag{41}$$

Proof: The right hand side of the above relation can be written as follows

$$\mathscr{L}^{-1}[\int_0^{+\infty} \xi^s J_{2s}(2\sqrt{\xi}) F(\xi) d\xi] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}[\int_0^{+\infty} \xi^s J_{2s}(2\sqrt{\xi}) F(\xi) d\xi] ds. \tag{42}$$

In the above double integral, we set $\mathscr{L}[f(\eta)] = F(\xi) = \int_0^{+\infty} e^{-\xi\eta} f(\eta) d\eta$. to obtain

$$\mathscr{L}^{-1}[\int_0^{+\infty} \xi^s J_{2s}(2\sqrt{\xi}) F(\xi) d\xi] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}[\int_0^{+\infty} \xi^s J_{2s}(2\sqrt{\xi}) \int_0^{+\infty} e^{-\xi\eta} f(\eta) d\eta] d\xi ds, \tag{43}$$

changing the order of integration leads to

$$\mathscr{L}^{-1}[\int_0^{+\infty} \xi^s J_{2s}(2\sqrt{\xi}) F(\xi) d\xi] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}[\int_0^{+\infty} f(\eta)[\int_0^{+\infty} \xi^s J_{2s}(2\sqrt{\xi}) e^{-\xi\eta} d\xi] d\eta] ds, \tag{44}$$

after evaluating the inner integral, we get

$$\mathscr{L}^{-1}[\int_0^{+\infty} \xi^s J_{2s}(2\sqrt{\xi}) F(\xi) d\xi] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}[\int_0^{+\infty} \frac{e^{-\frac{1}{\eta}}}{\eta^{2s+1}} f(\eta) d\eta] ds, \tag{45}$$

at this stage, we change the order of integration and simplifying, we arrive at

$$\mathscr{L}^{-1}[\int_0^{+\infty} \xi^s J_{2s}(2\sqrt{\xi}) F(\xi) d\xi] = \int_0^{+\infty} e^{-\frac{1}{\eta}} f(\eta)[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st-(2s+1)\ln\eta} ds] d\eta, \tag{46}$$

after simplifying the inner integral, we obtain

$$\mathscr{L}^{-1}[\int_0^{+\infty} \xi^s J_{2s}(2\sqrt{\xi}) F(\xi) d\xi] = \int_0^{+\infty} \eta^{-1} e^{-\frac{1}{\eta}} f(\eta)[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{s(t-2\ln\eta)} ds] d\eta, \tag{47}$$

the value of the complex integral is $\delta(t-2\ln\eta)$, therefore

$$\mathscr{L}^{-1}[\int_0^{+\infty} \xi^s J_{2s}(2\sqrt{\xi}) F(\xi) d\xi] = \int_0^{+\infty} \eta^{-1} e^{-\frac{1}{\eta}} f(\eta) \delta(t-2\ln\eta) d\eta, \tag{48}$$

making a change of variable $\phi = t - 2\ln\eta$, then we get $\eta = e^{\frac{t-\phi}{2}}$ and $d\eta = -\frac{1}{2}e^{\frac{t-\phi}{2}}$, after simplifying we have the following

$$\mathscr{L}^{-1}[\int_0^{+\infty} \xi^s J_{2s}(2\sqrt{\xi}) F(\xi) d\xi] = \int_{-\infty}^{+\infty} e^{-e^{\frac{\phi-t}{2}}} f(e^{\frac{\phi-t}{2}}) \delta(\phi) d\phi = e^{-e^{\frac{-t}{2}}} f(e^{\frac{-t}{2}}). \tag{49}$$

Lemman 1.5 Let us assume that $\mathscr{L}[f(t);s]=F(s)$, then the following identities hold true

$$1. \qquad \mathscr{L}[f(t^3);s]=\frac{1}{3\pi}\int_0^{+\infty}\sqrt{\frac{s}{\xi}}K_{\frac{1}{3}}[2(\frac{s}{3\sqrt[3]{\xi}})]F(\xi)d\xi, \tag{50}$$

$$2. \qquad \mathscr{L}[\frac{1}{t}f(\frac{1}{t});s]=\int_0^{+\infty}J_0(2\sqrt{su})F(u)du. \tag{51}$$

$$3. \qquad \mathscr{L}[\frac{1}{t}f(\frac{1}{t^3});s]=\int_0^{+\infty}J_0(2\sqrt{su})[\frac{1}{3\pi}\int_0^{+\infty}\sqrt{\frac{s}{\xi}}K_{\frac{1}{3}}[2(\frac{s}{3\sqrt[3]{\xi}})]F(\xi)d\xi]du. \tag{52}$$

Proof. See [6].

Example 1.2. The following integral identity holds true

$$\frac{\lambda^{\frac{7}{6}}}{3}e^{-\frac{s}{\sqrt[3]{\lambda}}}=\int_0^{+\infty}J_0(2\sqrt{su})[\frac{1}{3\pi}\int_0^{+\infty}\sqrt{\frac{s}{\xi}}K_{\frac{1}{3}}(\frac{2s}{3\sqrt[3]{\xi}})e^{-\lambda\xi}d\xi]du. \tag{53}$$

Solution. Let us take $f(t)=\delta(t-\lambda)$, then $F(s)=e^{-\lambda s}$, so we may evaluate $\mathscr{L}[\frac{1}{t}\delta(\frac{1}{t^3}-\lambda)]$ in two different ways as below, first, directly by the definition of the Laplace transform, we get

$$\mathscr{L}[\frac{1}{t}\delta(\frac{1}{t^3}-\lambda)]=\int_0^{+\infty}e^{-st}\frac{1}{t}\delta(\frac{1}{t^3}-\lambda)dt=\frac{\lambda^{\frac{7}{6}}}{3}e^{-\frac{s}{\sqrt[3]{\lambda}}}. \tag{54}$$

On the other hand, by using part three of the Lemma 1.5, we arrive at

$$\frac{\lambda^{\frac{7}{6}}}{3}e^{-\frac{s}{\sqrt[3]{\lambda}}}=\int_0^{+\infty}J_0(2\sqrt{su})[\frac{1}{3\pi}\int_0^{+\infty}\sqrt{\frac{s}{\xi}}K_{\frac{1}{3}}(\frac{2s}{3\sqrt[3]{\xi}})e^{-\lambda\xi}d\xi]du. \tag{55}$$

2 Main Results

In the past three decades, considerable research efforts have been expended to study anomalous diffusion using the time fractional equation. Anomalous diffusion transport appears to be a universal experimental phenomenon. A number of works have been published dealing with anomalous transport in fractals and disordered media, glass - forming liquids and colloidal structures. Let us consider the following two-dimensional heat conduction problem that arises during the manufacture of p-n junctions. To the best of the author's knowledge this kind of fractional mixed boundary value problem is not considered in the literature.

Problem 2.1. Let us consider the following time fractional diffusion problem with mixed boundary conditions

$$\frac{\partial^{c,\alpha}u(x,y,t)}{\partial t^\alpha}=a^2\Delta u(x,y,t), \qquad 0<\alpha<1. \tag{56}$$

where $-\infty<x<\infty$, $0<y,t<+\infty$ and subject to the mixed conditions

$$u_y(x,0,t)=0, \tag{57}$$

$$u(x,y,0)=0, \tag{58}$$

$$u(x,\pi,t)=\frac{U_0 t^{\alpha}}{\Gamma(\alpha+1)}. \tag{59}$$

$$lim_{|x|->+\infty}u(x,y,t)=lim_{y->+\infty}u(x,y,t)=0. \tag{60}$$

Note: In this study, it is assumed that the time fractional derivatives have been defined in the sense of the Caputo fractional derivatives.
Solution: In order to solve the above mixed boundary value problem, we reformulating it in cylindrical coordinates, to obtain

$$\frac{\partial^{c,\alpha}u(r,\phi,t)}{\partial t^{\alpha}}=a^2(u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\phi\phi}), \qquad 0<\alpha<1. \tag{61}$$

where $0<\phi<\pi$, $0<r,t<+\infty$ and subject to the mixed conditions

$$u_{\phi}(r,0,t)=0, \tag{62}$$

$$u(r,\pi,0)=0, \tag{63}$$

$$u(r,\pi,t)=\frac{U_0 t^{\alpha}}{\Gamma(\alpha+1)}. \tag{64}$$

$$lim_{r->0}u(r,\phi,t)=lim_{r->+\infty}u(r,\phi,t)=0. \tag{65}$$

The above mixed boundary value problem can be solved via the Laplace transform.
Let us define

$$U(r,\phi,s)=\int_0^{+\infty}e^{-st}u(r,\phi,t)dt. \tag{66}$$

Then the transformed equation becomes

$$s^{\alpha}U(r,\phi,s)=a^2(U_{rr}+\frac{1}{r}U_r+\frac{1}{r^2}U_{\phi\phi}), \qquad 0<\alpha<1. \tag{67}$$

where $0<\phi<\pi$, $0<r,t<+\infty$ and subject to the mixed conditions

$$U_{\phi}(r,0,s)=0, \tag{68}$$

$$U(r,\pi,0)=0, \tag{69}$$

$$U(r,\pi,s)=\frac{U_0}{s^{\alpha+1}}. \tag{70}$$

$$lim_{r->0}U(r,\phi,s)=lim_{r->+\infty}U(r,\phi,s)=0. \tag{71}$$

At this stage, let us choose

$$U(r,\phi,s)=\frac{U_0}{s^{\alpha+1}}+V(r,\phi,s). \tag{72}$$

Then we have the following relations

$$\frac{s^{\alpha}}{a^2}V(r,\phi,s) = V_{rr} + \frac{1}{r}V_r + \frac{1}{r^2}V_{\phi\phi} - \frac{U_0}{a^2 s}, \qquad 0 < \alpha < 1. \tag{73}$$

where $0 < \phi < \pi$, $0 < r,t < +\infty$ and subject to the mixed conditions

$$V_{\phi}(r,0,s) = 0, \tag{74}$$

$$V(r,\pi,0) = 0, \tag{75}$$

$$lim_{r->0}V(r,\phi,s) = lim_{r->+\infty}V(r,\phi,s) = 0. \tag{76}$$

Now, we express the Fourier series solution to the above equation as follows

$$V(r,\phi,s) = \sum_{n=0}^{+\infty} V_n(r,s)cos(n+\frac{1}{2})\phi. \tag{77}$$

Note that the Equation (77) satisfies the boundary conditions (74),(75) and each Fourier coefficient $V_n(r,\phi,s)$ is governed by the following non - homogenous second order Bessel's differential equation.

$$V_n'' + \frac{1}{r}V_n' - (\frac{(n+\frac{1}{2})^2}{r^2} + \frac{s^{\alpha}}{a^2})V_n = \frac{2(-1)^n U_0}{\pi a^2 (n+\frac{1}{2})s} \tag{78}$$

Equation (78) is known as non - homogeneous modified Bessel equation of order $(n+\frac{1}{2})$ with complete solution as below

$$V_n(r,s) = c_1 K_{n+\frac{1}{2}}(\frac{r}{a}\sqrt{s^{\alpha}}) + c_2 I_{n+\frac{1}{2}}(\frac{r}{a}\sqrt{s^{\alpha}}) + \Psi_c(r,s), \tag{79}$$

where, $\Psi_c(r,s)$ is the complementary solution and not known. Since the modified Bessel's functions $I_{\nu}(r), K_{\nu}(r)$ are unbounded at origin and infinity respectively, so that in view of the boundary conditions (60), we should have $c_1 = c_2 = 0$, therefore, we get the following formal solution

$$V_n(r,s) = \Psi_c(r,s). \tag{80}$$

One of the principal uses of the Hankel transform is in the solution of boundary value problems involving cylindrical coordinates. At this stage, we apply the Hankel transform of order $(n+\frac{1}{2})$ to the variable r in (78), this action leads to

$$\mathscr{H}_{n+\frac{1}{2}}[V_n'' + \frac{1}{r}V_n' - \frac{(n+\frac{1}{2})^2}{r^2}V_n] - \mathscr{H}_{n+\frac{1}{2}}[\frac{s^{\alpha}}{a^2}V_n] = \mathscr{H}_{n+\frac{1}{2}}[\frac{2(-1)^n U_0}{\pi a^2 (n+\frac{1}{2})s}], \tag{81}$$

after using some properties of the Hankel transform and in view of the Lemma 1.3, we arrive at

$$-\rho^2 A(\rho,s) - \frac{s^{\alpha}}{a^2}A(\rho,s) = \frac{2U_0(-1)^n}{\pi a^2 (n+\frac{1}{2})s}(\frac{n+\frac{1}{2}}{\rho^2}), \tag{82}$$

from which we deduce that

$$A(\rho,s) = -\frac{2(-1)^n U_0}{\pi \rho^2 s(s^{\alpha} + a^2\rho^2)}. \tag{83}$$

Inverting this result by means of the Hankel inversion formula, we have

$$V_n(r,s) = \Psi_c(r,s) = -\int_0^{\infty} \frac{2(-1)^n U_0}{\pi s(s^{\alpha}+a^2\rho^2)} J_{n+\frac{1}{2}}(r\rho)\frac{d\rho}{\rho}, \tag{84}$$

and

$$V(r,\phi,s) = -\sum_{n=0}^{+\infty}\left[\int_0^{\infty} \frac{2(-1)^n U_0}{\pi s(s^{\alpha}+a^2\rho^2)} J_{n+\frac{1}{2}}(r\rho)\frac{d\rho}{\rho}\right] cos(n+\frac{1}{2})\phi. \tag{85}$$

Finally, we get the solution to transformed Equation (67) as follows

$$U(r,\phi,s) = \frac{U_0}{s^{\alpha+1}} - \frac{2U_0}{\pi}\sum_{n=0}^{+\infty}(-1)^n\left[\int_0^{\infty} \frac{1}{s(s^{\alpha}+a^2\rho^2)\rho} J_{n+\frac{1}{2}}(r\rho)\frac{d\rho}{\rho}\right] cos(n+\frac{1}{2})\phi, \tag{86}$$

and thus by taking the inverse Laplace transform, we obtain

$$\frac{u(r,\phi,t)}{U_0} = \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2}{\pi}\sum_{n=0}^{+\infty}(-1)^n\left[\int_0^{\infty} \mathscr{L}^{-1}\left[\frac{1}{s(s^{\alpha}+a^2\rho^2)}\right] J_{n+\frac{1}{2}}(r\rho)\frac{d\rho}{\rho}\right] cos(n+\frac{1}{2})\phi, \tag{87}$$

at this point, by virtue of the part three of the Lemma 1.1, and the inversion of Equation (87) yields

$$\frac{u(r,\phi,t)}{U_0} = \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2t^{\alpha}}{\pi}\sum_{n=0}^{+\infty}(-1)^n\left[\int_0^{\infty} E_{\alpha,\alpha-1}(\pm\lambda t^{\alpha}) J_{n+\frac{1}{2}}(r\rho)\frac{d\rho}{\rho}\right] cos(n+\frac{1}{2})\phi, \tag{88}$$

Note: It is easy to check that $u(r,\phi,0) = u_{\phi}(r,0,t) = 0, u(r,\pi,t) = \frac{U_0 t^{\alpha}}{\Gamma(\alpha+1)}$

3 Conclusions

In this work, the author presents analytical techniques to solve time fractional diffusion problem with mixed bounadry conditions.We consider a generalization of the fractional heat conduction problem in two dimensions that arises during analysis of the impurity atom distribution near the diffusion mask for a planar p - n junction. The article is intended for scientists and researchers of different disciplines of engineering and science dealing with the solutions of fractional mixed boundary value problems. The results reveal that the integral transforms method is very convenient and effective. It is hoped that this study will lead to further investigations in the field and more elegant solutions would be found.

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References

[1] A.Aghili, Solution time fractional non - homogeneous first order PDE with non - constant coefficients. Tbilisi Mathematical Journal 12(4) (2019),pp. 149 - 155.
[2] A. Aghili, Special functions, integral transforms with applications, Tbilisi Mathematical Journal 12 (1) (2019), 33-44.
[3] A.Aghili, Fractional Black - Scholes equation. International Journal of Financial Engineering, Vol. 4, No. 1 (2017) 1750004 (15 pages)© World Scientific Publishing Company. DOI: 10.1142/S2424786317500049
[4] A.Aghili, Mixed boundary value problem for a quarter- plane with a robin condition. Journal of Sciences, Islamic Republic of Iran 13(1): 65-69 (2002) National Center For Scientific Research, ISSN 1016-1104

[5] A.Aghili, A.Parsania, Mixed boundary value problems in semi infinite strip. Int. J. Contemp. Math. Sci., Vol. 1, 2006, no. 7, 305 - 311
[6] A.Apelblat, Laplace Transforms and Their Applications, Nova science publishers, Inc, New York, 2012.
[7] V.I.Fabrikant, Mixed Boundary Value Problems of Potential Theory and Their Applications in Engineering. Kluwer Academic,1991, 451 pp.
[8] J.P. Feng, and S. Weinbaum,The general motion of a circular disk in a Brinkman medium. Phys. Fluids,1998, 10, 2137-2146.
[9] W.Gao, P.Veeresha, D.G.Prakasha, Haci Mehmet Baskonus, Gulnur. Yel, A powerful approach for fractional Drinfeld - Sokolov- Wilson equation with Mittag - Leffler law. Alexandria Engineering Journal.(2019) 58, 1301-1311.
[10] H. J. Glaeske, A.P.Prudnikov, K. A. Skornik, Operational Calculus And Related Topics.Chapman and Hall / CRC 2006.
[11] I. Podlubny, Fractional differential equations, Academic Press, San Diego, CA,1999.
[12] D. G. Prakasha, P. Veeresha, H. M. Baskonus, Two novel computational techniques for fractional Gardner and Cahn-Hilliard equations, Comp. and Math. Methods,1(2019), 1-19.
[13] Y.Shindo, The linear magnetoelastic problemof a uniform current flow distributed by a penny - shaped crack in a constant axial magnetic field. Eng. Fract.Mech.1986,23,977-982.
[14] I.N.Sneddon, Mixed Boundary Value Problems in Potential Theory. North Holland, 1966,283 pp.
[15] P. Veeresha, D. G. Prakasha, Solution for fractional Zakharov-Kuznetsov equations by using two reliable techniques, Chinese J. Phys., 60(2019), 313-330.
[16] P. Veeresha, D. G. Prakasha, Haci Mehmet Baskonus, Novel simulations to the time-fractional Fisher's equation. Mathematical Sciences (2019) 13:33-42.
[17] P. Veeresha, D. G. Prakasha, Jagdev.Singh, Solution for fractional forced KdV equation using fractional natural decomposition method.AIMS Mathematics, 2019, 5(2):798-810.