J. Geom. (2020) 111:35

0047-2468/20/020001-6
published online July 4, 2020
https://doi.org/10.1007/s00022-020-00547-2

Journal of Geometry

Complete spherical convex bodies

Marek Lassak

Abstract. Similarly to the classic notion in Euclidean space, we call a set on the sphere S^d complete, provided adding any extra point increases its diameter. Complete sets are convex bodies on S^d. Our main theorem says that on S^d complete bodies of diameter δ coincide with bodies of constant width δ.

Mathematics Subject Classification. 52A55.

Keywords. Sphere, lune, convex body, complete body, constant width, constant diameter.

1. On spherical geometry

Let S^d be the unit sphere in the $(d+1)$-dimensional Euclidean space E^{d+1}, where $d \geq 2$. By a *great circle* of S^d we mean the intersection of S^d with any two-dimensional subspace of E^{d+1}. The common part of the sphere S^d with any hyper-subspace of E^{d+1} is called a $(d-1)$*-dimensional great sphere* of S^d. By a pair of *antipodes* of S^d we mean any pair of points of intersection of S^d with a straight line through the origin of E^{d+1}.

Clearly, if two different points $a, b \in S^d$ are not antipodes, there is exactly one great circle containing them. By the *arc* ab connecting a with b we mean the shorter part of the great circle containing a and b. By the *spherical distance* $|ab|$, or shortly *distance*, of these points we mean the length of the arc connecting them. The *diameter* $\mathrm{diam}(A)$ of a set $A \subset S^d$ is the number $\sup_{a,b\in A}|ab|$. By a *spherical ball* $B_\rho(r)$ *of radius* $\rho \in (0, \frac{\pi}{2}]$, or shorter *a ball*, we mean the set of points of S^d having distance at most ρ from a fixed point, called the *center* of this ball. Spherical balls of radius $\frac{\pi}{2}$ are called *hemispheres*. Two hemispheres whose centers are antipodes are called *opposite hemispheres*.

We say that a subset of S^d is *convex* if it does not contain any pair of antipodes and if together with every two points a, b it contains the arc ab. By a *convex body*, or shortly *body*, on S^d we mean any closed convex set with non-empty interior.

Recall a few notions from [6]. If for a hemisphere H containing a convex body $C \subset S^d$ we have $\mathrm{bd}(H) \cap C \neq \emptyset$, then we say that H *supports* C. If hemispheres G and H of S^d are different and not opposite, then $L = G \cap H$ is called *a lune* of S^d. The $(d-1)$-dimensional hemispheres bounding the lune L and contained in G and H, respectively, are denoted by G/H and H/G. We define the *thickness of a lune* $L = G \cap H$ as the spherical distance of the centers of G/H and H/G. For a hemisphere H supporting a convex body $C \subset S^d$ we define the *width* $\mathrm{width}_H(C)$ *of* C *determined by* H as the minimum thickness of a lune of the form $H \cap H'$, where H' is a hemisphere, containing C. If for all hemispheres H supporting C we have $\mathrm{width}_H(C) = w$, we say that C is *of constant width* w.

2. Spherical complete bodies

Similarly to the traditional notion of a complete set in the Euclidean space E^d (for instance, see [1–3] and [10]) we say that a set $K \subset S^d$ of diameter $\delta \in (0, \pi)$ is *complete* provided $\mathrm{diam}(K \cup \{x\}) > \delta$ for every $x \notin K$.

Theorem 1. *An arbitrary set of diameter* $\delta \in (0, \pi)$ *on the sphere* S^d *is a subset of a complete set of diameter* δ *on* S^d.

We omit the proof since it is similar to the proof by Lebesgue [9] in E^d (it is recalled in Part 64 of [1]). Let us add that earlier Pál [12] proved this for E^2 by a different method.

The following fact permits to use the term a *complete convex body* for a complete set.

Lemma 1. *Let* $K \subset S^d$ *be a complete set of diameter* δ. *Then* K *coincides with the intersection of all balls of radius* δ *centered at points of* K. *Moreover,* K *is a convex body.*

Proof. Denote by I the intersection of all balls of radius δ with centers in K.

Since $\mathrm{diam}(K) = \delta$, then K is contained in every ball of radius δ whose center is a point of K. Consequently, $K \subset I$.

Let us show that $I \subset K$; so let us show that $x \notin K$ implies $x \notin I$. Really, from $x \notin K$ we get $|xy| > \delta$ for a point $y \in K$, which means that x is not in the ball of radius δ with center y, and thus $x \notin I$.

The first thesis implies that K is a convex body. □

Lemma 2. *If* $K \subset S^d$ *is a complete body of diameter* δ, *then for every* $p \in \mathrm{bd}(K)$ *there exists* $p' \in K$ *such that* $|pp'| = \delta$.

Proof. Suppose the contrary, i.e., that $|pq| < \delta$ for a point $p \in \mathrm{bd}(K)$ and for every point $q \in K$. Since K is compact, there is an $\varepsilon > 0$ such that

$|pq| \leq \delta - \varepsilon$ for every $q \in K$. Hence there is a point $s \notin K$ in a positive distance from p which is smaller than ε such that $|sq| \leq \delta$ for every $q \in K$. Thus $\mathrm{diam}(K \cup \{s\}) = \delta$, which contradicts the assumption that K is complete. Consequently, the thesis of our lemma holds true. □

For different points $a, b \in S^d$ at a distance $\delta < \pi$ from a point $c \in S^d$ define the piece of the circle $P_c(a, b)$ as the set of points $v \in S^d$ such that cv has length δ and intersects ab.

We show the next lemma for S^d despite we apply it later only for S^2.

Lemma 3. *Let $K \subset S^d$ be a complete convex body of diameter δ. Take $P_c(a, b)$ with $|ac|$ and $|bc|$ equal to δ such that $a, b \in K$ and $c \in S^d$. Then $P_c(a, b) \subset K$.*

Proof. First let us show the thesis for a ball B of radius δ in place of K. There is unique $S^2 \subset S^d$ with $a, b, c \in S^2$. Consider the disk $D = B \cap S^2$. Take the great circle containing $P_c(a, b)$ and points a^*, b^* of its intersection with the circle bounding D. There is a unique $c^* \in S^2$ such that $P_c(a, b) \subset P_{c^*}(a^*, b^*)$. Clearly, $P_{c^*}(a^*, b^*) \subset D \subset B$. Hence $P_c(a, b) \subset B$.

By the preceding paragraph and Lemma 1 we obtain the thesis of the present lemma. □

3. Complete and constant width bodies on S^d coincide

Here is our main result presenting the spherical version of the classic theorem in E^d proved by Meissner [11] for $d = 2, 3$ and by Jessen [5] for arbitrary d.

Theorem 2. *A body of diameter δ on S^d is complete if and only if it is of constant width δ.*

Proof. ($\Rightarrow$) Let us prove that if a body $K \subset S^d$ of diameter δ is complete, then K is of constant width δ.

Suppose the opposite, i.e., that $\mathrm{width}_I(K) \neq \delta$ for a hemisphere I supporting K. By Theorem 3 and Proposition 1 of [6] $\mathrm{width}_I(K) \leq \delta$. So $\Delta(K) < \delta$. By lines 1-2 of p. 562 of [6] the thickness of K is equal to the minimum thickness of a lune containing K. Take such a lune $L = G \cap H$, where G, H are different and non-opposite hemispheres. Denote by g, h the centers of G/H and H/G, respectively. Of course, $|gh| < \delta$. By Claim 2 of [6] we have $g, h \in K$. By Lemma 2 there exists a point $g' \in K$ in the distance δ from g. Since the triangle ghg' is non-degenerate, there is a unique two-dimensional sphere $S^2 \subset S^d$ containing g, h, g'. Clearly, ghg' is a subset of $M = K \cap S^2$. Hence M is a convex body on S^2. Denote by F this hemisphere of S^2 such that $hg' \subset \mathrm{bd}(F)$ and $g \in F$. There is a unique $c \in F$ such that $|ch| = \delta = |cg'|$. By Lemma 3 for $d = 2$ we have $P_c(h, g') \subset M$.

We intend to show that c is not on the great circle E of S^2 through g and h. In order to see this, for a while suppose the opposite, i.e. that $c \in E$. Then from $|g'g| = \delta$, $|g'c| = \delta$ and $|hc| = \delta$ we conclude that $\angle gg'c = \angle hcg'$. So the spherical triangle $g'gc$ is isosceles, which together with $|gg'| = \delta$ gives $|cg| = \delta$. Since $|gh| = \Delta(L) = \Delta(K) > 0$ and g is a point of ch different from c, we get a contradiction. Hence, really, $c \notin E$.

By the preceding paragraph $P_c(h, g')$ intersects $\text{bd}(M)$ at a point h' different from h and g'. So the non-empty set $P_c(h, g') \setminus \{h, h'\}$ is out of M. This contradicts the result of the paragraph before the last. Consequently, K is a body of constant width δ.

($\Leftarrow$) Let us prove that if K is a spherical body of constant width δ, then K is a complete body of diameter δ. In order to prove this, it is sufficient to take any point $r \notin K$ and to show that $\text{diam}(K \cup \{r\}) > \delta$.

Take the largest ball $B_\rho(r)$ disjoint with the interior of K. Since K is convex, $B_\rho(r)$ has exactly one point p in common with K. By Theorem 3 of [8] there exists a lune $L \supset K$ of thickness δ with p as the center of one of the two $(d-1)$-dimensional hemispheres bounding this lune. Denote by q the center of the other $(d-1)$-dimensional hemisphere. By Claim 2 of [6] also $q \in K$. Since p and q are the centers of the two $(d-1)$-dimensional hemispheres bounding L, we have $|pq| = \delta$. From the fact that rp and pq are orthogonal to $\text{bd}(H)$ at p, we see that $p \in rq$. Moreover, p is not an endpoint of rq and $|pq| = \delta$, Hence $|rq| > \delta$. Thus $\text{diam}(K \cup \{r\}) > \delta$. Since $r \notin K$ is arbitrary, K is complete. □

We say that a convex body $D \subset S^d$ is of *constant diameter* δ provided $\text{diam}(D) = \delta$ and for every $p \in \text{bd}(D)$ there is a point $p' \in \text{bd}(D)$ with $|pp'| = \delta$ (see [8]). The following fact is analogous to the result in E^d given by Reidemeister [13].

Theorem 3. *Bodies of constant diameter on S^d coincide with complete bodies.*

Proof. Take a complete body $D \subset S^d$ of diameter δ. Let $g \in \text{bd}(D)$ and G be a hemisphere supporting D at g. By Theorem 2 the body D is of constant width δ. So $\text{width}_G(D) = \delta$ and there exists a hemisphere H such that the lune $G \cap H \supset D$ has thickness δ. By Claim 2 of [6] the centers h of H/G and g of G/H belong to D. So $|gh| = \delta$. Thus D is of constant diameter δ.

Consider a body $D \subset S^d$ of constant diameter δ. Let $r \notin D$. Take the largest $B_\rho(r)$ whose interior is disjoint with D. Denote by p the common point of $B_\rho(r)$ and D. A unique hemisphere J supports $B_\rho(r)$ at p. Observe that $D \subset J$ (if not, there is a point $v \in D$ out of J; clearly vp passes through $\text{int}B_\rho(r)$, a contradiction). Since D is of constant diameter δ, there is $p' \in D$ with $|pp'| = \delta$. Observe that $\angle rpp' \geq \frac{\pi}{2}$. If it is $\frac{\pi}{2}$, then $|rp'| > \delta$. If it is larger than $\frac{\pi}{2}$, the triangle rpp' is obtuse and then by the law of cosines $|rp'| > |pp'|$ and hence $|rp'| > \delta$. By $|rp'| > \delta$ in both cases we see that D is complete. □

Theorem 2 permits to change “complete” to “constant width” in Theorem 3. This form is proved earlier as follows. In [8] it is shown that any body of constant width δ on S^d is of constant diameter δ and the inverse is argued for $\delta \geq \frac{\pi}{2}$, and for $\delta < \frac{\pi}{2}$ if $d = 2$. By [4] the inverse holds for any δ. Our short proof of Theorem 3 is quite different from the considerations in [8], [4] and [7].

References

[1] Bonnesen, T., Fenchel, T.W.: Theorie der konvexen Körper. Springer, Berlin (1934). (**English translation: Theory of Convex Bodies, BCS Associates, Moscow, Idaho, 1987**)

[2] Chakerian, G.D., Groemer, H.: Convex bodies of constant width. In: Gruber, P.M., Wills, J.M. (eds.) Convexity and Its Applications, pp. 49–96. Birkhäuser, Basel (1983)

[3] Eggleston, H.G.: Convexity, Cambridge Tracts in Mathematics and Mathematical Physics, vol. 47. Cambridge University Press, New York (1958)

[4] Han, H., Wu, D.: Constant diameter and constant width of spherical convex bodies. arXiv:1905.09098v2

[5] Jessen, B.: *Über konvexe Punktmengen konstanter Breite*. Math. Z. **29**(1), 378–380 (1929)

[6] Lassak, M.: *Width of spherical convex bodies*. Aequat. Math. **89**, 555–567 (2015)

[7] Lassak, M.: *When is a spherical convex body of constant diameter a body of constant width?* Aequat. Math. **94**, 393–400 (2020)

[8] Lassak, M., Musielak, M.: *Spherical bodies of constant width*. Aequat. Math. **92**, 627–640 (2018)

[9] Lebesgue, H.: *Sur quelques questions de minimum, relatives aux courbes orbiformes, et sur leurs rapports avec le calcul des variations*. J. Math. Pures Appl. **8**(4), 67–96 (1921)

[10] Martini, M., Montejano, L., Oliveros, D.: Bodies of Constant Width. An Introduction to Convex Geometry with Applications. Springer Nature Switzerland AG, Cham (2019)

[11] Meissner, E.: *Über Punktmengen konstanter Breite*. Vjschr. Naturforsch. Ges. Zürich **56**, 42–50 (1911)

[12] Pál, J.: *Über ein elementares Variationsproblem (Danish)*. Bull. de l'Acad. de Dan. **3**(2), 35 (1920)

[13] Reidemeister, K.: *Über Körper konstanten Durchmessers*. Math. Z. **10**, 214–216 (1921)

Marek Lassak
University of Technology and Life Sciences
Al. prof. Kaliskiego 7
85-796 Bydgoszcz
Poland
e-mail: `marek.lassak@utp.edu.pl`

Received: April 28, 2020.
Revised: June 12, 2020.