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Completely bounded multipliers of the Fourier algebra of a simple Lie group of real rank one

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The Fourier algebra $A(G)$ of a locally compact group G is the space of matrix coefficients of the regular representation, and is the predual of the von Neumann algebra $VN(G)$ generated by the regular representation of G on $L^2(G)$. A multiplier m of $A(G)$ is a bounded operator on $A(G)$ given by pointwise multiplication by a function on G, also denoted m. We say m is a completely bounded multiplier of $A(G)$ if the transposed operator on $VN(G)$ is completely bounded (definition below). It may be possible to find a net of $A(G)$-functions, $(m_i : i \in I)$ say, such that m_i tends to 1 uniformly on compacta, and, for some L in $\mathbb{R}^+$, $\|m_i\|_{M_0} \leqq L$ ($\| \quad \|_{M_0}$ being the completely bounded operator norm). We define Λ_G to be the infimum of all values of L, as we consider all possible nets of this type; in particular Λ_G is set equal to $+\infty$ if there is no such net. In this paper, we calculate Λ_G for all non-compact real-rank-one simple Lie groups with finite center: If G is locally isomorphic to $SO(1,n)$ or $SU(1,n)$ (where $n \geqq 2$), then $\Lambda_G = 1$; if G is locally isomorphic to $\mathrm{Sp}(1,n)$ (with $n \geqq 2$), then $\Lambda_G = 2n-1$, and if G is locally isomorphic to the exceptional Lie group $F_{4(-20)}$, then $\Lambda_G = 21$. The second-named author [16] has shown that if G is simple and of real rank greater than one, then $\Lambda_G = +\infty$; he has also shown, that if Γ is a lattice in G, then $\Lambda_G = \Lambda_\Gamma$, and that the von Neumann algebras of lattices Γ and Γ' contained in the Lie groups G and G' cannot be isomorphic unless $\Lambda_G = \Lambda_{G'}$. Consequently, if Γ and Γ' are lattices in $\mathrm{Sp}(1,n)$ and $\mathrm{Sp}(1,n')$ respectively and $n \neq n'$, then the von Neumann algebras of the two lattices are not isomorphic.

0. Notation and definition

For a locally compact group, G, we let $B(G)$ be the space of all coefficients of continuous unitary representations of G; $u \in B(G)$ iff there exists a unitary representation (from now on, representation means continuous representation) π of G acting on a Hilbert space $\mathfrak{H}_\pi$, and vectors ξ and η in $\mathfrak{H}_\pi$ so that

$$u(x) = \langle \pi(x)\xi, \eta \rangle \quad \forall x \in G. \tag{0.1}$$

Because the sum and tensor product of unitary representations is again a unitary representation, $B(G)$ is closed under sums and products (with pointwise operations); it is easy to see that, equipped with the norm $\| \ \|_B$

$$\|u\|_B = \min\{\|\xi\| \ \|\eta\| : (0.1) \text{ holds}\}$$

$B(G)$ is a Banach algebra. The closed ideal $A(G)$ in $B(G)$ generated by compactly supported $B(G)$-functions turns out to be just the space of coefficients of the left regular representation λ of G on $L^2(G)$ (the usual Lebesgue space constructed using a left-invariant Haar measure), i.e. $u \in A(G)$ iff there are functions h and k in $L^2(G)$ so that

$$u(x) = \langle \lambda(x)h, k\rangle \quad \forall x \in G, \tag{0.2}$$

and

$$\|u\|_B = \min\{\|h\|_2 \|k\|_2 : (0.2) \text{ holds}\}.$$

Often we write $\|u\|_A$ instead of $\|u\|_B$ to emphasize that $u \in A(G)$. We remind the reader that $A(G)$ is the predual of $VN(G)$, the algebra of bounded linear operators on $L^2(G)$ commuting with right translations.

If G is compact, then $A(G) = B(G)$, but otherwise $A(G) \subset B(G)$, as elements of $A(G)$ vanish at infinity while $1 \in B(G)$. The group G is amenable exactly when it is possible to find a net of $A(G)$-functions $(u_i : i \in I)$ so that

$$\|u_i\|_A \leqq 1$$

and

$$\|u_i v - v\|_A \to 0 \quad \text{as} \quad i \to \infty \quad \forall v \in A(G).$$

We refer the reader to P. Eymard [12], F. Greenleaf [15] and H. Leptin [25] for details of these assertions on $A(G)$, $B(G)$ and amenability.

Various authors have considered some related spaces, starting with C.S. Herz [19]. We denote by $M(G)$ the space of multipliers of $A(G)$, i.e. $M(G)$ is the space of functions on G so that the pointwise product $mu \in A(G)$ whenever $u \in A(G)$, and we equip $M(G)$ with the operator norm, denoted $\| \ \|_M$. An important subspace of $M(G)$ is the set $M_0(G)$ of completely bounded multipliers of $A(G)$, which can be defined in various ways, viz:

(i) $u \in M_0(G)$ if $u \in M(G)$ and the induced operator on $VN(G)$ is completely bounded;

(ii) $u \in M_0(G)$ if the function $u \otimes 1 \in M(G \times H)$, where H is the group $SU(2)$;

(iii) $u \in M_0(G)$ if the function $u \otimes 1 \in M(G \times H)$ for any locally compact group H;

(iv) $u \in M_0(G)$ if there exist bounded continuous mappings $P, Q: G \to \mathfrak{H}$ ($\mathfrak{H}$ a Hilbert space) so that

$$u(y^{-1}x) = \langle P(x), Q(y)\rangle \quad \forall x, y \in G, \tag{0.3}$$

(v) $u \in M_0(G)$ if the function $\tilde{u}: G \times G \to C$ given by $\tilde{u}(x, y) = u(y^{-1}x)$ multiplies pointwise the projective tensor product $L^2(G) \otimes_\gamma L^2(G)$.

The natural norms associated to each of these definitions coincide, so we may use the completely bounded norm, the maximum of the norms $\|u \otimes 1\|_M$ as H varies, the minimum of the expressions

$$\sup\{\|P(x)\| \ \|Q(y)\| : x, y \in G\}$$

as different representations of the sort (0.3) are considered, or the operator norm on $L^2(G)\otimes_\gamma L^2(G)$; we denote the norm in $M_0(G)$ by $\| \ \|_{M_0}$. It is known that

$$B(G)\subseteq M_0(G)\subseteq M(G),$$

and that the inclusion maps are norm-non-increasing. It is also easy to see that these spaces coincide (isometrically) when G is amenable. It is very likely, that these inclusions are all strict, when G is not amenable (see M. Bozejko [4], C. Nebbia [28] and V. Losert [26]). The main results on $M_0(G)$ can be found in Herz (op. cit.), in unpublished works of J. E. Gilbert [14], and in papers of J. De Canniere and U. Haagerup [11], M. Bozejko and G. Fendler [5] and U. Haagerup [16].

We shall say that G is weakly amenable if there exists a net in $A(G)$, $(u_i : i\in I)$ say, such that

$$\begin{aligned} &\|u_i\|_{M_0}\leqq L \\ &u_i\to 1 \quad \text{uniformly on compacta}. \end{aligned} \tag{0.4}$$

We let Λ_G be the infimum of all such values L, as we consider all possible such nets. We shall prove the following surprising result.

Main theorem. *Let G be a connected real Lie group with finite centre. If G is locally isomorphic to* $SO(1,n)$ *or* $SU(1,n)$ *then* $\Lambda_G=1$. *If G is locally isomorphic to* $\mathrm{Sp}(1,n)$, *then* $\Lambda_G=2n-1$, *while if G is the exceptional* rank-*one group* $F_{4(-20)}$, *then* $\Lambda_G=21$.

This was known for $SO(1,n)$ and $SU(1,n)$, by results of De Cannière and Haagerup [11] and Cowling [9]. It is curious and perhaps significant that, for connected non-compact real semisimple Lie groups G with finite centre, $\Lambda_G>1$ exactly when the group has D.A. Kazhdan's Property T [22], so that Λ_G provides a measure of the degree of isolation of the identity representation in the dual space $\hat{G}$.

This paper contains another six sections. In Sect. 1, we discuss briefly some properties of the index Λ_G and we consider $K-bi$-invariant approximate identities on a semisimple Lie group G. In Sect. 2, we look at the structure of the real rank-one simple Lie groups, and describe some of their representations, and in Sect. 3, the calculations begin. Section 3 contains some Fourier transform computations for the Iwasawa nilpotent group N, while Sect. 4 involves working with the maximal solvable subgroup AN; in these two sections we prove that $\Lambda_G\leqq 2n-1$ (respectively 21) for the case when $G=\mathrm{Sp}(1,n)$ (respectively $F_{4(-20)}$). In Sect. 5, by working on N, we obtain the lower bounds for Λ_G for $\mathrm{Sp}(1,n)$ and $F_{4(-20)}$. Section 6 is dedicated to applications in the theory of von Neumann algebras; various non-isomorphic II_1-factors are constructed. Since the $SO(1,n)$ case has already been published, we shall not consider this case here. Further, the result that $\Lambda_G=1$ for $SO(1,n)$ follows readily by restriction from the $SU(1,n)$ case.

1. Completely bounded multipliers of $A(G)$ and $K-bi$-invariant functions

In this section we set down some basic results about completely bounded multipliers of $A(G)$, first for arbitrary groups and then for groups with compact subgroups.

Proposition 1.1. *Suppose that G is a locally compact group, and that $(u_i : i \in I)$ is a net of $A(G)$-functions satisfying conditions* (0.4) *above. Then there exists a net $(v_j : j \in J)$ of $A_c(G)$ functions satisfying the conditions*

$$\begin{aligned} &\|v_j\|_{M_0} \leqq L \qquad \forall j \in J \\ &\|uv_j - u\|_A \to 0 \qquad \forall u \in A(G) \\ &v_j \to 1 \ \text{uniformly on compacta.} \end{aligned} \tag{1.1}$$

Further, if K is any compact subset of G and $\varepsilon \in \mathbb{R}^+$, then there exists w in $A_c(G)$ so that

$$\begin{aligned} &\|w\|_{M_0} \leqq L + \varepsilon \\ &w(x) = 1 \quad \forall x \in K. \end{aligned} \tag{1.2}$$

Finally, if G is a Lie group, then the functions v_j and w may be chosen to have the extra property that $v_j \in C_c^\infty(G)$ and $w \in C_c^\infty(G)$.

Proof. We first show how to construct the net $(v_j : j \in J)$. Take a nonnegative $C_c(G)$-function f on G of integral 1, and define

$$u_i' = f * u_i \quad \forall i \in I.$$

Because $M_0(G)$ is translation-invariant, and translations act isometrically

$$\|u_i'\|_{M_0} \leqq \|f\|_1 \, \|u_i\|_{M_0} \leqq L.$$

We shall now show that

$$\|uu_i' - u\|_A \to 0 \quad \forall u \in A_c(G):$$

by the boundedness of $\|u_i'\|_{M_0}$ and the density of $A_c(G)$ in $A(G)$, this will then hold for any u in $A(G)$. Fix u in $A_c(G)$, and write S and $\mathbb{1}_S$ for the compact set $\operatorname{supp}(f)^{-1} \operatorname{supp}(u)$ and its characteristic function. For x in $\operatorname{supp}(u)$,

$$\begin{aligned} u_i'(x) &= \int_G f(y) u_i(y^{-1}x)\, dy \\ &= \int_G f(y)(1_S u_i)(y^{-1}x)\, dy. \end{aligned}$$

because only y's in $\operatorname{supp}(f)$ contribute to the integral. Now

$$(uu_i')(x) = (u[f * 1_S u_i])(x) \quad \forall x \in G.$$

since if $x \notin \operatorname{supp}(u)$, both sides are zero. Similarly

$$u(x) = (u[f * 1_S])(x) \quad \forall x \in G.$$

As $1_S u_i \to 1_S$ uniformly and S is compact, $f * 1_S u_i \to f * 1_S$ in $A(G)$: because $A(G)$ is a Banach algebra,

$$uu_i' = u[f * 1_S u_i] \to u[f * 1_S] = u$$

in $A(G)$, as claimed.

We observe that from this property it follows that $u_i' \to 1$ locally uniformly. For given any compact set K in G, there exists u in $A_c(G)$ which takes the value 1 on K. Since $u_i' u \to u$ in $A(G)$ and a fortiori uniformly, $u_i' 1_K \to 1_K$ uniformly.

The net (u_i') has the required properties (1.1), except that u_i may not have compact support. So now for each n in $\mathbb{N}$ ($\mathbb{N}=\{1,2,3,\dots\}$), we choose an element $u_{i,n}$ of $A_c(G)$ so that

$$\|u_i-u_{i,n}\|_A<n^{-1}.$$

Since $A(G)\subseteq M_0(G)$ and $\|u\|_{M_0}\leqq\|u\|_A$ for every u in $A(G)$,

$$\|u_{i,n}\|_{M_0}\leqq L+n^{-1}.$$

We define $v_{i,n}$ by the formula

$$v_{i,n}=[L/(L+n^{-1})]u_{i,n}\quad \forall i\in I,\ \forall n\in\mathbb{N}.$$

It is now easy to check that, if the net $I\times\mathbb{N}$ is given the product ordering, then $(v_{i,n}: i\in I,\ n\in\mathbb{N})$ is a net of $A_c(G)$-functions with properties (1.1).

Next, we take a compact subset K of G and ε in $\mathbb{R}^+$. Then there is an $A_c(G)$-function u which takes the value 1 on K. Take a net of $A_c(G)$-functions $(v_j: j\in I)$ satisfying (1.1), and choose j so that

$$\|uv_j-u\|_A<\varepsilon.$$

Write w for $v_j-(uv_j-u)$. Then $w\in A_c(G)$ and has properties (1.2).

Finally, if G is a Lie group, we may ensure that the functions v_j and w are C^∞ by convolving then with a $C_c^\infty(G)$-function of small compact support, and integral 1. □

Corollary 1.2. *In the definition of Λ_G, it is equivalent to consider nets satisfying properties* (0.4) *or* (1.1), *and if G is Lie we may in addition require the functions to be smooth. Further, if U is open and relatively compact in G, then, setting*

$$\Lambda_U=\inf\{\|w\|_{M_0}: w\in A_c(G),\ w(x)=1\ \ \forall x\in U\}$$

we have that

$$\Lambda_G=\sup\{\Lambda_U: U \textit{ open and relatively compact}\},$$

where the net of such subsets of G is ordered by inclusion.

Proof. It is clear from Proposition 1.1 that, in the definition of Λ_G, it makes no difference whether we consider nets satisfying conditions (0.4) or only nets satisfying the stronger conditions (1.1). We shall, rather loosely, refer to both types of nets as approximate identities, although the terminology might be more properly used for the latter type only.

It is easy to see that $\Lambda_U<\infty$ for any relatively compact open set U, and that $\Lambda_U\leqq\Lambda_V$ if $U\subseteq V$. From (1.2), $\Lambda_U\leqq\Lambda_G$ for any U, so that $\lim_U\Lambda_U\leqq\Lambda_G$. On the other hand, if $\sup\{\Lambda_U\}<\Lambda_G$, then we can construct an approximate identity $(v_j: j\in J)$ of $A_c(G)$-functions with $\|v_j\|_{M_0}$ bounded by a constant less than Λ_G, which is absurd. Thus $\Lambda_G=\lim_U\Lambda_U$, as required. □

Our next results concern the computation of Λ_G, for general locally compact groups G.

Proposition 1.3. (a) *If G is a locally compact group, and H is a closed subgroup of G, then the restriction $u|_H$ of any $M_0(G)$ function u to H belongs to $M_0(G)$, and $\|u|_H\|_{M_0} \leqq \|u\|_{M_0}$: consequently $\Lambda_H \leqq \Lambda_G$.*

(b) *If $(G_i : i \in I)$ is the net of compactly generated open subgroups of the locally compact group G, ordered by inclusion, then $\Lambda_G = \lim_i \Lambda_{G_i}$.*

(c) *If H is a compact normal subgroup of the locally compact group G, then the space $M_0(G/H)$ may be canonically and isometrically identified with the subspace of $M_0(G)$ of functions constant on the cosets of H in G; further $\Lambda_G = \Lambda_{G/H}$.*

Proof. From condition (iv) that a function belong to $M_0(G)$ (0.3), (a) follows immediately; (b) follows from (a) and Corollary 1.2. It is also clear that any $M_0(G/H)$-function gives rise to an $M_0(G)$-function which is constant on cosets of H in (c). The rest follows by averaging over H. □

We are now going to consider direct products of groups, and shall prove that $\Lambda_{G \times H} = \Lambda_G \Lambda_H$ for arbitrary locally compact groups G and H. We shall need the following definitions and a preliminary lemma. First, we define $\bar{A}(G)$ to be the norm closure of $A(G)$ in $M_0(G)$. Next, we define the norms $\|\ \|_P$ and $\|\ \|_Q$ on $L^1(G)$ as follows:

$$\|f\|_Q = \sup\left\{\left|\int_G f(x)u(x)\,dx\right| : u \in M_0(G),\ \|u\|_{M_0} \leqq 1\right\},$$

and

$$\|f\|_P = \sup\left\{\left|\int_G f(x)u(x)\,dx\right| : u \in \bar{A}(G),\ \|u\|_{M_0} \leqq 1\right\},$$

and we denote by $P(G)$ and $Q(G)$ the corresponding completions of $L^1(G)$. We may think of $P(G)$ and $Q(G)$ as analogues of the group C^*-algebras $C_r^*(G)$ and $C^*(G)$ respectively, though the second-named author has shown that, in general, they are not algebras under convolution. It is known that $M_0(G)$ can be identified with the dual space of $Q(G)$ (see Herz [20] or De Cannière and Haagerup [11]). It can also be shown that the dual space of $P(G)$ is the space of locally uniform limits of bounded nets of $\bar{A}(G)$-functions (see Cowling [9]).

Lemma 1.4. *Suppose that G and H are locally compact groups.*

(a) *If $u \in M_0(G)$ (respectively $\bar{A}(G)$) and $v \in M_0(H)$ (resp. $\bar{A}(H)$), then $u \otimes v \in M_0(G \times H)$ (resp. $\bar{A}(G \times H)$) and $\|u \otimes v\|_{M_0} = \|u\|_{M_0} \|u\|_{M_0}$.*

(b) *If $g \in P(H)$ and $h \in P(H)$, then $g \otimes h \in P(G \times H)$ and*

$$\|g \otimes h\|_P = \|g\|_P \|h\|_P.$$

Proof. By duality, it will suffice to prove only the inequalities $\|u \otimes v\|_{M_0} \leqq \|u\|_{M_0} \|v\|_{M_0}$, $\|g \otimes h\|_Q \leqq \|g\|_Q \|h\|_Q$, and $\|g \otimes h\|_P \leqq \|g\|_P \|h\|_P$, for the converse inequalities then follow quickly. From the characterisation (iv) of $M_0(G)$, it is clear that if $u \in M_0(G)$ and $v \in M_0(H)$, then $u \otimes v \in M_0(G \times H)$ and $\|u \otimes v\|_{M_0} \leqq \|u\|_{M_0} \|v\|_{M_0}$. Since further $u \otimes v \in A(G \times H)$ if $u \in A(G)$ and $v \in A(H)$, $u \otimes v \in \bar{A}(G \times H)$ if $u \in \bar{A}(G)$ and $v \in \bar{A}(H)$.

The main ingredient of the proof of (b) is Herz' result [20] that $Q(G)$ is the image of the tensor product space $T(G)$ –

$$T(G)=(L^2(G)\otimes_\gamma L^2(G))\otimes_\gamma(L^2(G)\otimes_\lambda L^2(G)),$$

where γ and λ denote the greatest and least cross norms respectively – under the linear mapping π (Herz' "contraction linéaire bizarre") which is defined on simple tensors by the formula

$$\pi(f\otimes g\otimes h\otimes k)=(fh)*(gk)^*,$$

$*$ denoting the usual involution of $L^1(G)$. In fact, $g\in Q(G)$ if and only if there exists s in $T(G)$ so that $\pi(s)=g$, and

$$\|g\|_Q=\inf\{\|s\|_T : s\in T(G),\ \pi(s)=g\}.$$

Now we claim that if $s\in T(G)$ and $t\in T(H)$, then $s\otimes t\in T(G\times H)$ and $\|s\otimes t\|_T\leqq\|s\|_T\|t\|_T$. Indeed, it suffices to consider tensors s and t of the form

$$s=[f\otimes g]\otimes\left[\sum_{i=1}^m h_i\otimes k_i\right]$$

$$t=[o\otimes p]\otimes\left[\sum_{j=1}^n q_j\otimes r_j\right];$$

then $s\otimes t$ may be identified with

$$[(f\otimes o)\otimes(g\otimes p)]\otimes\left[\sum_{i=1}^m\sum_{j=1}^n (h_i\otimes q_j)\otimes(k_i\otimes r_j)\right]$$

in

$$[L^2(G\times H)\otimes_\gamma L^2(G\times H)]\otimes_\gamma[L^2(G\times H)\otimes_\lambda L^2(G\times H)],$$

and

$$\|s\otimes t\|\leqq\|f\|_2\|o\|_2\|g\|_2\|p\|_2\left\|\sum_{i=1}^m\sum_{j=1}^n (h_i\otimes q_j)\otimes(k_i\otimes r_j)\right\|_\lambda.$$

Now the least cross norm $\left\|\sum_{i=1}^m h_i\otimes k_i\right\|_\lambda$ is exactly the operator norm $\|L\|_{\mathrm{op}}$ of the linear map L on $L^2(G)$ sending f to $\sum_{i=1}^m (f,h_i)k_i$, and $\|L\otimes M\|_{\mathrm{op}}=\|L\|_{\mathrm{op}}\|M\|_{\mathrm{op}}$ for operators L and M on $L^2(G)$ and $L^2(H)$; it follows that, as required,

$$\|s\otimes t\|_T\leqq\|s\|_T\|t\|_T.$$

Since $\pi(s\otimes t)=\pi(s)\otimes\pi(t)$, it follows immediately from Herz' result that, if $g\in Q(G)$ and $h\in Q(H)$, then $g\otimes h\in Q(G\times H)$ and

$$\|g\otimes h\|_Q\leqq\|g\|_Q\|h\|_Q.$$

To prove (c), we argue from (b). Given g in $P(G)$, which we may assume by density lies in $L^1(G)$, we have a linear functional, L say, on $\bar{A}(G)$ –

$$L(u)=\int_G g(x)u(x)\,dx\quad\forall u\in\bar{A}(G) –$$

which, by the Hahn-Banach theorem, extends to a linear functional on $M_0(G)$ of the same norm, still denoted L. Since the unit ball of $Q(G)$, the predual of $M_0(G)$, is weak-star dense in the unit ball of $M_0(G)^*$, there exists a net $(g_i : i \in I)$ of $L^1(G)$-functions such that

$$\|g_i\|_Q \leqq \|g\|_P \quad \forall i \in I$$

and

$$L(u) = \lim_i \int_G g_i(x)u(x)dx \quad \forall u \in M_0(G).$$

In particular, we have that

$$\int_G g(x)u(x)dx = \lim_i \int_G g_i(x)u(x)dx \quad \forall u \in \bar{A}(G).$$

Similarly, we can find a net $(h_j : j \in J)$ of $L^1(H)$-functions such that

$$\|h_j\|_Q \leqq \|h\|_P \quad \forall j \in J$$

and

$$\int_H h(y)v(y)dy = \lim_j \int_H h_j(y)v(y)dy \quad \forall v \in \bar{A}(H).$$

Consequently, by (b),

$$\|g_i \otimes h_j\|_Q \leqq \|g\|_P \|h\|_P \quad \forall i \in I, \forall j \in J,$$

and

$$\lim_{ij} \int_H \int_G g_i(x)h_j(y)u(x)v(y)dxdy = \int_H \int_G g(x)h(y)u(x)v(y)dxdy$$
$$\forall u \in \bar{A}(G), \forall v \in \bar{A}(H),$$

where the net $(g \otimes h_j : i \in I, j \in J)$ has the product order on $I \times J$. Since $M_0(G \times H)^*$ is a dual space, there is a subnet $(g_{i_k} \otimes h_{j_k} : k \in K)$ of the product net with a weak-star limit point in $M_0(G \times H)^*$, L say, i.e. there exists L in $M_0(G \times H)^*$ of norm at most $\|g\|_P \|h\|_P$ such that

$$L(w) = \lim_k \int_H \int_G g_{i_k}(x)h_{j_k}(y)w(x,y)dxdy \quad \forall w \in M_0(G \times H)$$

If w lies in the algebraic tensor product $A(G) \otimes A(H)$, then clearly

$$L(w) = \int_H \int_G g(x)h(y)w(x,y)dxdy.$$

However, $A(G) \otimes A(H)$ is dense in $A(G \times H)$ and hence in $\bar{A}(G \times H)$; it follows that $g \otimes h \in P(G \times H)$ and

$$\|g \otimes h\|_P \leqq \|g\|_P \|h\|_P,$$

as required. $\square$

We can now prove our main general result about Λ_G.

Corollary 1.5. *Let G and H be locally compact groups. Then $\Lambda_{G \times H} = \Lambda_G \Lambda_H$.*

Proof. It follows from Lemma 1.4(a) that $\Lambda_{G \times H} \leqq \Lambda_G \Lambda_H$; we must prove the converse inequality.

Take relatively compact open sets U and V in G and H respectively. By applying the Hahn-Banach theorem to $\bar{A}(G)/I_U$, where I_U is the subspace of $\bar{A}(G)$ of functions which vanish on U, it can be seen that there is L in $\bar{A}(G)^*$ with the properties that $\|L\| \leqq 1$ and $L(u) = \Lambda_U$ if u takes the value one on U. As $A(G)$ is dense in $\bar{A}(G)$ and $A(G)^* = VN(G)$, L is implemented by an element T of $VN(G)$. By convolving T with an $A_c(G)$-function of compact support K and integral 1, it can be seen that there exists a $C_c(G)$-function g, support in $K\bar{U}$, with the properties that

$$\|g\|_P \leqq 1$$

and

$$\int_G g(x)u(x)dx = \Lambda_U$$

if $u \in \bar{A}(G)$ and u takes the value 1 on $K\bar{U}$. Equivalently, $\operatorname{supp}(g) \subseteq K\bar{U}$,

$$\|g\|_P \leqq 1$$

and

$$\int_G g(x)dx = \Lambda_U .$$

Analogously, there is a $C_c(H)$-function h supported in $L\bar{V}$, where L is a compact set, with the properties that $\|h\|_P \leqq 1$ and

$$\int_H h(y)dy = \Lambda_V .$$

Now if $w \in \bar{A}(G \times H)$ and w takes the value 1 on $K\bar{U} \times L\bar{V}$, then

$$\int_H \int_G g(x)h(y)w(x,y)dxdy = \Lambda_U \Lambda_V .$$

As $\|g \otimes h\|_P \leqq 1$ by Lemma 1.4(c), we deduce that $\Lambda_{K\bar{U} \times L\bar{V}} \geqq \Lambda_U \Lambda_V$, and by letting U and V grow, it follows that $\Lambda_{G \times H} \geqq \Lambda_G \Lambda_H$, as required. □

Our next result concerns groups with large compact subgroups, and alternative definitions of Λ_G. Before we state it, let us denote by Λ_G' the infimum of all positive real numbers L for which there exists a net $(u_i : i \in I)$ of $A(G)$-functions such that

$$\|u_i\|_M \leqq L \quad \forall i \in I$$

$$u_i \to 1 \quad \text{uniformly on compacta},$$

where $\|u_i\|_M$ denotes the norm of the multiplier u_i of $A(G)$. Since $\|u_i\|_M \leqq \|u_i\|_{M_0}$, $\Lambda_G' \leqq \Lambda_G$.

Proposition 1.6. *Let G be a locally compact group and K a compact subgroup of G. Suppose that S is an amenable closed subgroup of G so that, set-theoretically, $G = SK$. Then*

(a) *if $u \in M_0(G)$ (or $M(G)$) and $\dot{u}$ denotes the function obtained by averaging u over the double cosets $K \times K$ ($x \in G$), then $\dot{u} \in M_0(G)$ and $\|\dot{u}\|_{M_0} \leqq \|u\|_{M_0}$ (or the corresponding result for $M(G)$). Consequently, G is weakly amenable iff there exists a K-bi-invariant approximate identity of completely bounded multipliers, and Λ_G (or Λ_G') is the infimum of the numbers L where* (0.4) *(or* (1.1)*) holds and the approximate identity is K-bi-invariant;*

(b) *If u is K-bi-invariant, then* $u \in M_0(G)$ iff $u \in M(G)$ iff $u|_S \in B(S)$, *and*

$$\|u\|_M = \|u\|_{M_0} = \|u|_S\|_B .$$

Consequently, the existence of an approximate identity in $M_0(G)$ is equivalent to the existence of an approximate identity in $M(G)$, and $\Lambda_G = \Lambda'_G$.

Proof. We observe that $M_0(G)$ ($M(G)$) is closed under left and right translations, and that these act isometrically; (a) then follows. To see (b), we note that $u \in M_0(G)$ implies $u \in M(G)$ trivially, and that $u \in M(G)$ implies $u|_S \in M(S)$ because $A(G)|_S = A(S)$ (see C.S. Herz [19]), whence $u|_S \in B(S)$, S being amenable. There are also norm inequalities corresponding to these implications:

$$\|u\|_M \leqq \|u\|_{M_0}, \quad \text{and} \quad \|u|_S\|_B \leqq \|u\|_M .$$

It therefore suffices to show that, if $u|_S \in B(S)$, then $u \in M_0(G)$, and prove an appropriate norm inequality. Now if $u|_S \in B(S)$, there are a unitary representation π and vectors ξ and η in $\mathfrak{H}_\pi$ so that

$$\begin{aligned} u(y^{-1}x) &= \langle \pi(y^{-1}x)\xi, \eta \rangle \\ &= \langle \pi(x)\xi, \pi(y)\eta \rangle \quad \forall x, y \in S, \end{aligned}$$

and

$$\|u|_S\|_B = \|\xi\| \, \|\eta\| .$$

We assume, without loss of generality, that ξ and η are cyclic vectors, and then, if $z \in S \cap K$, we see that, for all x, y in S,

$$\begin{aligned} \langle \pi(xz)\xi, \pi(y)\eta \rangle &= u(y^{-1}xz) \\ &= u(y^{-1}x) \\ &= \langle \pi(x)\xi, \pi(y)\eta \rangle , \end{aligned}$$

so that $\pi(xz)\xi = \pi(x)\xi$ for all x in S. Consequently, we may define $P: G \to \mathfrak{H}_\pi$ by requiring that

$$P(xk) = \pi(x)\xi \quad \forall x \in S, \ \forall k \in K ;$$

similarly, we may define $Q: G \to \mathfrak{H}_\pi$ by requiring that

$$Q(yk') = \pi(y^{-1})\eta \quad \forall y \in S, \ \forall k' \in K .$$

Now it is straightforward to check that

$$u(y^{-1}x) = \langle P(x), Q(y) \rangle$$

for any choice of x, y in G; the proposition follows. □

It is probably worth remarking explicitly that, for connected semisimple Lie groups G with finite centres, Λ_G depends only on the local isomorphism class of G, by Proposition 1.3(c); also, if S is a Borel or minimal parabolic subgroup of G, then the convolution algebra of K-*bi*-invariant functions on G is isomorphic to the convolution algebra on S of restrictions of K-*bi*-invariant functions to S. Our strategy is going to be to work with such restrictions.

2. Some real-rank-one simple Lie groups

In this section, we describe the class-one principal series of the simple Lie groups $SU(1,n)$, $\mathrm{Sp}(1,n)$, and $F_{4(-20)}$. General references for this are S. Helgason's texts [17, 18].

Let G be a connected real-rank-one simple Lie group with finite centre, not locally isomorphic to $SO_0(1,n)$. We denote by K a maximal compact subgroup, by θ the corresponding Cartan involution of G and its Lie algebra $\mathfrak{g}$, and by B the Killing form on $\mathfrak{g}\times\mathfrak{g}$. Given a connected subgroup H of G, we normally denote its Lie algebra by $\mathfrak{h}$, and vice versa. We let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$, the complement of $\mathfrak{k}$ in $\mathfrak{g}$, and decompose $\mathfrak{g}$ into root spaces:

$$\mathfrak{g}=\mathfrak{m}+\mathfrak{a}+\sum_{\alpha\in\Sigma}\mathfrak{g}_\alpha$$

where $\mathfrak{m}$ is the centraliser of $\mathfrak{a}$ in $\mathfrak{k}$, and Σ is the sets of roots. Then $\mathfrak{a}$ is one dimensional and $\Sigma=\{-2\alpha,\ -\alpha,\alpha,2\alpha\}$, since G is real-rank-one but not locally isomorphic to $SO_0(1,n)$. We write $\mathfrak{n}$ for $\mathfrak{g}_\alpha+\mathfrak{g}_{2\alpha}$, $\bar{\mathfrak{n}}$ for $\theta\mathfrak{n}$, $2p$ for $\dim\mathfrak{g}_\alpha$ and q for $\dim\mathfrak{g}_{2\alpha}$. Then we have the following direct sum decompositions of the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$$

and

$$\mathfrak{g}=\bar{\mathfrak{n}}+\mathfrak{m}+\mathfrak{a}+\mathfrak{n}\,;$$

at the group level, we have the Iwasawa decomposition $G=KAN$ and the Bruhat big cell decomposition $\dot{G}=\bar{N}MAN$, where $\dot{G}$ is a dense open submanifold of G whose complement is a lower-dimensional submanifold. There is a unique element H_α of $\mathfrak{a}$ with the property that

$$\mathrm{ad}(H_\alpha)|_{\mathfrak{g}_\alpha}=I\,;$$

we write $\bar{A}^+$ for $\{\exp(tH_\alpha): t\geqq 0\}$, where exp denotes the exponential map. Then

$$\begin{aligned}\mathrm{tr}(\mathrm{ad}(H_\alpha)|_{\mathfrak{n}})&=\dim\mathfrak{g}_\alpha+2\dim\mathfrak{g}_{2\alpha}\\&=2p+2q\\&=2r,\end{aligned}$$

say. The Cartan decomposition of $G-G=K\bar{A}^+K-$holds, though not uniquely, and any $K-bi$-invariant function on G is determined by its values on $\bar{A}^+$.

We equip $\bar{\mathfrak{n}}$ with the inner product

$$(X+Y,X'+Y')=-(2p+4q)^{-1}\,B\left(\frac{X}{2}+\frac{Y}{4},\frac{X'}{2}+\frac{Y'}{4}\right)$$

for all X,X' in $\mathfrak{g}_{-\alpha}$ and all Y,Y' in $\mathfrak{g}_{-2\alpha}$, which makes $\bar{N}$ into a H-type group (see Sect. 3). The following formulae relate the Iwasawa, Bruhat, and Cartan decompositions.

Proposition 2.1. *Suppose that $X \in \mathfrak{g}_{-\alpha}$ and $Y \in \mathfrak{g}_{-2\alpha}$. Then there exist k, k' and k'' in K, n in N, s in $\mathbb{R}$ and t in $[0, \infty)$ so that*

$$\exp\left(X+\frac{Y}{4}\right)=k\exp(sH_\alpha)n$$

$$\exp\left(X+\frac{Y}{4}\right)=k'\exp(tH_\alpha)k'' ;$$

k, and n, s, and t are unique, and

$$e^s=((1+|X|^2)^2+|Y|^2)^{1/2}$$

and

$$4\sinh^2 t=4|X|^2+|X|^4+|Y|^2.$$

Proof. This follows from Helgason's [17] Theorem IX.3.8, once the different normalisations are taken into account. Indeed, if we write $(\cdot|\cdot)_0$ for the inner product $-B(\cdot,\theta\cdot)$, then Helgason proves that

$$e^s=\left((1+c|X|_0^2)^2+\frac{c}{4}|Y|_0^2\right)^{1/2}$$

and

$$2\cosh(2t)=2+4c|X|_0^2+c^2|X|_0^2+\frac{c}{4}|Y|_0^2$$

where $c^{-1}=4(2p+4q)$ (note that we deal with $\exp\left(X+\frac{Y}{4}\right)$ while Helgason considers $\exp(X+Y)$). We have normalised things so that $c|X|_0^2=|X|^2$ and $c|Y|_0^2=4|Y|^2$, and the desired conclusion follows. □

Corollary 2.2. *The space $C^\infty(K\backslash G/K)|_{\bar{N}}$ of restrictions to $\bar{N}$ of $K-bi$-invariant C_c^∞-functions on G coincides with the space of functions of the form*

$$\exp\left(X+\frac{Y}{4}\right)\to f(4|X|^2+|X|^4+|Y|^2),$$

where f is a $C_c^\infty(\mathbb{R})$-function.

Proof. The space of restrictions to A of $K-bi$-invariant functions on G is exactly the space of even functions on A, and a $K-bi$-invariant function u on G is C^∞ if and only if $u|_A$ is C^∞. The set of functions $\varrho\to f(\varrho^2)$ obtained as f varies over all $C_c^\infty(\mathbb{R})$-functions is exactly the space of even $C_c^\infty(\mathbb{R})$-functions. □

We now describe the class-one principal series of representations of G. If $\lambda\in\mathbb{C}$, then the mapping $\chi_\lambda: MAN\to\mathbb{C}$, given by the rule

$$\chi_\lambda: m\exp(sH_\alpha)n\to\exp(\lambda s)$$

is a character of the parabolic subgroup MAN of G, unitary when λ is purely imaginary. We induce this character to give a representation π_λ of G as follows. Let $\mathfrak{H}^\lambda$ be the completion of the space of all continuous functions $\xi: G\to\mathbb{C}$ which satisfy the condition

$$\xi(xman) = (\varrho\chi_\lambda)(man)^{-1}\xi(x) \quad \forall x \in G, \forall m \in M, \forall a \in A, \forall n \in N$$

in the norm $\|\xi\|$

$$\|\xi\| = \left(\int_K |\xi(k)|^2 dk\right)^{1/2}$$

dk denoting normalised Haar measure on K, and ϱ denoting the character χ_r. We let G act on $\mathfrak{H}^\lambda$ by left translations:

$$[\pi_\lambda(x)\xi](y) = \xi(x^{-1}y) \quad \forall x, y \in G.$$

Then for k in K, $\pi_\lambda(k)$ is unitary, but for general x in G, this is not so unless χ_λ is unitary; nevertheless $\pi_\lambda(x)$ is a bounded operator for each x in G. The spherical function ϕ_λ is defined by the formula

$$\begin{aligned}\phi_\lambda(x) &= \int_K (\pi_\lambda(x)\mathbb{1}_\lambda)(k)dk \\ &= \int_K (\pi_\lambda(x)\mathbb{1}_\lambda)(k)\mathbb{1}_{-\lambda}(k)dk \\ &= \langle \pi_\lambda(x)\mathbb{1}_\lambda, \mathbb{1}_{-\bar\lambda}\rangle,\end{aligned}$$

where $\mathbb{1}_\lambda$ is the unit K-fixed vector in $\mathfrak{H}^\lambda$:

$$\mathbb{1}_\lambda(kan) = (\varrho\chi_\lambda)^{-1}(an) \quad \forall k \in K, \forall a \in A, \forall n \in N.$$

By Proposition 2.1,

$$\mathbb{1}_\lambda\left(\exp\left(X + \frac{Y}{4}\right)\right) = ((1+|X|^2)^2 + |Y|^2)^{-(\lambda+r)/2} \quad \forall X \in \mathfrak{g}_{-\alpha}, \forall Y \in \mathfrak{g}_{-2\alpha}. \tag{2.1}$$

The main facts about the representations π_λ and the spherical functions ϕ_λ are summarised in the following result.

Theorem 2.3. *For ξ in $\mathfrak{H}^\lambda$ and η in $\mathfrak{H}^{-\bar\lambda}$,*

$$\int_K \xi(k)\bar\eta(k)dk = \left[\int_{\bar N} \mathbb{1}_r(\bar n)d\bar n\right]^{-1} \int_{\bar N} \xi(\bar n)\bar\eta(\bar n)d\bar n, \tag{2.2}$$

for any λ in $\mathbb{C}$. Consequently, for all x in G,

$$\langle \pi_\lambda(x)\xi, \pi_{-\bar\lambda}(x)\eta\rangle = \langle \xi, \eta\rangle, \tag{2.3}$$

and in particular π_λ is unitary when λ is purely imaginary. Further,

$$\phi_\lambda(kxk') = \phi_\lambda(x) \quad \forall x \in G, \forall k, k' \in K,$$

and $\lambda \to \phi_\lambda$ is an entire function with values in $C(G)$ with the topology of locally uniform convergence. Finally, if $|\mathrm{Re}(\lambda)| \leq r$, then $\|\phi_\lambda\|_\infty = 1$, and $\phi_{-r}(x) = \phi_r(x) = 1$ for all x in G.

Proof. For simplicity, we assume that Haar measure on $\bar N$ is normalised so that $\int_{\bar N} \mathbb{1}_r(\bar n)d\bar n = 1$, and we take ξ in $\mathfrak{H}^\lambda$ and η in $\mathfrak{H}^{-\bar\lambda}$ which are continuous in G. The formula (2.2) is proved in Theorem I.5.20 of Helgason [18]. Since the inner product $\langle\, ,\rangle$ is expressed in terms of Haar measure on K,

$$\langle \pi_\lambda(k)\xi, \pi_{-\bar\lambda}(k)\eta\rangle = \langle \xi, \eta\rangle \quad \forall k \in K;$$

By (2.2), the inner product can also be expressed as an integral over $\bar N$, so

$$\langle \pi_\lambda(\bar n)\xi, \pi_{-\bar\lambda}(\bar n)\eta\rangle = \langle \xi, \eta\rangle \quad \forall \bar n \in \bar N.$$

Together, K and $\bar N$ generate G, so (2.3) holds. Now we can see that

$$\begin{aligned}\phi_\lambda(kxk') &= \langle \pi_\lambda(kxk')\mathbb{1}_\lambda, \mathbb{1}_{-\bar\lambda}\rangle \\ &= \langle \pi_\lambda(x)\pi_\lambda(k')\mathbb{1}_\lambda, \pi_{-\bar\lambda}(k^{-1})\mathbb{1}_{\bar\lambda}\rangle \\ &= \langle \pi_\lambda(x)\mathbb{1}_\lambda, \mathbb{1}_{-\bar\lambda}\rangle \\ &= \phi_\lambda(x).\end{aligned}$$

That $\lambda \to \phi_\lambda$ is entire is easy. It is obvious that $\phi_{-r}=1$, and, from above, $\phi_r(x) = \phi_{-r}(x^{-1})$ for any x in G, so $\phi_r = 1$ also. It is routine to check that $|\phi_\lambda(x)| \leqq 1$ for any x in G if $\mathrm{Re}(\lambda) = \pm r$; the three lines theorem then implies that $\|\phi_\lambda\|_\infty \leqq 1$ for λ with $\mathrm{Re}(\lambda)$ in $[-r, r]$. As $\phi_\lambda(e) = 1$, we have $\|\phi_\lambda\|_\infty = 1$ for such λ. □

Proposition 2.4. *The spherical function ϕ_λ is given on $A\bar N$ by*

$$\phi_\lambda(a\bar n) = (\varrho\chi_\lambda)(a) \int_{\bar N} \mathbb{1}_\lambda|_{\bar N}(\bar n^{-1}a^{-1}\bar n' a)\mathbb{1}_{-\lambda}|_{\bar N}(\bar n')d\bar n'.$$

Proof. This follows straight from the definitions. □

It is notationally more convenient to work on N rather than on $\bar N$. We equip the Lie algebra $\mathfrak{n}$ with the inner product $(\, ,)$, where

$$(X+Y, X'+Y') = -(2p+4q)^{-1} B\left(\frac{X}{2}+\frac{Y}{4}, \theta\left(\frac{X}{2}+\frac{Y}{4}\right)\right)$$
$$\forall X, X' \in \mathfrak{g}_\alpha,\ \forall Y, Y' \in \mathfrak{g}_{2\alpha}, \tag{2.4}$$

and define $u_\lambda : N \to \mathbb{C}$ by the rule

$$u_\lambda\left(\exp\left(X+\frac{Y}{4}\right)\right) = ((1+|X|^2)^2+|Y|^2)^{-(\lambda+r)/2} \quad \forall X \in \mathfrak{g}_\alpha,\ \forall Y \in \mathfrak{g}_{2\alpha}. \tag{2.5}$$

We denote by a_s the element $\exp(\log(s)H_\alpha/2)$ of A; then

$$a_s \exp(X+Y)a_s^{-1} = \exp(s^{1/2}X+sY) \quad \forall s \in \mathbb{R}^+,\ \forall X \in \mathfrak{g}_\alpha,\ \forall Y \in \mathfrak{g}_{2\alpha}.$$

Theorem 2.5. (a) *On AN, the spherical function ϕ_λ is given by the formula*

$$\phi_\lambda(a_s n) = s^{-(\lambda+r)/2} \int_N u_\lambda(n^{-1}a_s^{-1}n'a_s)u_{-\lambda}(n')dn',$$

provided that Haar measure on N, dn', is normalised so that

$$\int_N u_r(n)dn' = 1.$$

(b) *The space of restrictions to N of K-bi-invariant C_c^∞-functions on G is exactly the space of functions of the form*

$$\exp\left(X+\frac{Y}{4}\right)\to f(4|X|^2+|X|^4+|Y|^2),$$

where f is a $C^\infty(\mathbb{R})$-function.

Proof. We could rewrite all this section interchanging α with $-\alpha$, taking $\bar{\mathfrak{n}}$ to be $\mathfrak{g}_\alpha+\mathfrak{g}_{2\alpha}$, $\mathfrak{n}$ to be $\theta\bar{\mathfrak{n}}$, and N to be $\bar{N}$. Mutatis mutandis, Theorem 2.5(a) is Proposition 2.4 and 2.5(b) is Corollary 2.2. $\square$

3. Harmonic analysis on groups of Heisenberg type

In this section, we consider groups of type H, which are a family of two step nilpotent groups which include the nilpotent components of the Iwasawa decompositions of the groups $SU(1,n)$, $\mathrm{Sp}(1,n)$ and $F_{4(-20)}$. We describe briefly their representations, and the Plancherel formula (which are already known). Finally, we calculate some Fourier transforms on H-type groups.

A group of type H is a connected simply connected real Lie group whose Lie algebra is of type H; following A. Kaplan [21], we say that the Lie algebra $\mathfrak{n}$ is of type H if it is the direct sum $\mathfrak{v}\oplus\mathfrak{z}$ of real Euclidean spaces, with a Lie algebra structure such that $\mathfrak{z}$ is the centre of $\mathfrak{n}$ and, for all V in $\mathfrak{v}$ of length one, the map $\mathrm{ad}(V)$ is a surjective isometry of the orthogonal complement $\mathfrak{v}\ominus\ker\mathrm{ad}(V)$ onto $\mathfrak{z}$. For such an algebra, we define a linear map $j:\mathfrak{z}\to\mathrm{End}(\mathfrak{v})$ by the formula

$$\langle j(Z)V,V'\rangle=\langle Z,[V,V']\rangle\quad \forall Z\in\mathfrak{z},\ \forall V,V'\in\mathfrak{v}.$$

It can be readily shown that (see e.g. [21])

$$|j(Z)V|=|Z|\,|V|\quad \forall V\in\mathfrak{v},\ \forall Z\in\mathfrak{z}$$

and

$$j(Z)^2=-|Z|^2I,\quad \forall Z\in\mathfrak{z};$$

in particular, if $|Z|=1$, then $j(Z)$ defines a complex structure on $\mathfrak{v}$. For ω in $\mathfrak{z}$ of length 1, we denote by $\{,\}_\omega$ the corresponding Hermitean inner product, i.e.

$$\begin{aligned}\{V,W\}_\omega&=\langle V,W\rangle+i\langle j(\omega)V,W\rangle\\&=\langle V,W\rangle+i\langle[V,W],\omega\rangle\quad \forall V,W\in\mathfrak{v}.\end{aligned}$$

It will be convenient to denote by $\mathfrak{v}_\omega$ the space $\mathfrak{v}$ equipped with the complex structure $j(\omega)$, by $2p$ and q the (real) dimensions of $\mathfrak{v}$ and $\mathfrak{z}$ respectively, and by r the integer $p+q$. Hereafter, for a group N of type H with Lie algebra $\mathfrak{n}=\mathfrak{v}+\mathfrak{z}$, we write, using lower case rather than upper case letters,

$$(v,z)=\exp(v+z/4)\quad \forall v\in\mathfrak{v},\ \forall z\in\mathfrak{z}.$$

We note that the Iwasawa N-groups from $SU(1,n)$, $\mathrm{Sp}(1,n)$ and $F_{4(-20)}$ are H-type groups, with the Euclidean structure (2.4) used above – see [10] for example.

We define the Haar measure on N by the formula

$$\int_N f(n)\,dn = k(p,q)^{-1} \int_{\mathfrak{v}} \int_{\mathfrak{z}} f(v,z)\,dz\,dv$$

where

$$k(p,q) = \frac{\pi^{(2p+q+1)/2}\, 2^{1-2p-q}}{\Gamma\left(\dfrac{2p+q+1}{2}\right)}.$$

This normalisation is appropriate since, if $u_\lambda : N \to C$ is the function defined by the formula

$$u_\lambda(v,z) = ((1+|v|^2)^2 + |z|^2)^{-(\lambda+r)/2} \quad \forall (v,z) \in N \tag{3.1}$$

(as in (2.5) above), we have the following result.

Lemma 3.1. *With the definitions just made,*

$$k(p,q)^{-1} \int_{\mathfrak{v}} \int_{\mathfrak{z}} u_r(v,z)\,dz\,dv = 1\,.$$

Proof. This is a straighforward integration: by putting $t = (1+s^2)u$,

$$\begin{aligned}
\int_{\mathfrak{v}} \int_{\mathfrak{z}} u_r(v,z)\,dz\,dv &= \frac{2\pi^p}{\Gamma(p)} \frac{2\pi^{q/2}}{\Gamma(q/2)} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} ((1+s^2)^2 + t^2)^{-r} t^{q-1}\,dt\, s^{2p-1}\,ds \\
&= \frac{4\pi^{p+q/2}}{\Gamma(p)\Gamma(q/2)} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} (1+s^2)^{q-2r} (1+u^2)^{-r} u^{q-1}\,du\, s^{2p-1}\,ds \\
&= \frac{\pi^{p+q/2}}{\Gamma(p)\Gamma(q/2)} \int_{\mathbb{R}^+} (1+w)^{q-2r} w^{p-1}\,dw \int_{\mathbb{R}^+} (1+v)^{-r} v^{q/2-1}\,dv,
\end{aligned}$$

where $w = s^2$ and $v = u^2$. Since

$$\int_{\mathbb{R}^+} \frac{w^{a-1}}{(1+w)^b}\,dw = \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)},$$

and

$$2\pi^{1/2}\Gamma(2p+q) = 2^{2p+q}\Gamma\left(\frac{2p+q}{2}\right)\Gamma\left(\frac{2p+q+1}{2}\right),$$

(see E.C. Titchmarsh [31], 1.86), we are done. □

The irreducible unitary representations of a group of type H fall into two classes. Some are trivial on the centre of the group, and factor to characters of $\mathfrak{v}$. These representations do not appear in the Plancherel formula, and we shall not need to discuss them further. The other are parametrised by $\mathbb{R}^+ \times S_{\mathfrak{z}}$, where $S_{\mathfrak{z}}$ is the unit sphere in $\mathfrak{z}$. We define $H(\mathfrak{v}_\omega)$ to be the space of entire functions on $\mathfrak{v}_\omega$, and let $\mathfrak{H}_{\nu,\omega}$ ($\nu \in \mathbb{R}^+$, $\omega \in S_{\mathfrak{z}}$) be the following Hilbert space:

$$\mathfrak{H}_{\nu,\omega} = \left\{\xi \in H(\mathfrak{v}_\omega) : \|\xi\|_\nu^2 = \int_{\mathfrak{v}} |\xi(v)|^2 \exp(-2\nu|v|^2)\,dv < \infty\right\}:$$

here dv denotes Lebesgue measure on $\mathfrak{v}$, and $\|\ \|_\nu$ is, of course, the norm on $\mathfrak{H}_{\nu,\omega}$. The unitary representation $\sigma_{\nu,\omega}$ of N acts on $\mathfrak{H}_{\nu,\omega}$ –

$$[\sigma_{\nu,\omega}(v,z)\xi](w)=\exp(-\nu[|v|^2+2\{w,v\}_\omega+i\langle z,\omega\rangle])\xi(w+v)$$

for any v, w in $\mathfrak{v}$ and z in $\mathfrak{z}$. It is well known that $\sigma_{\nu,\omega}$ is the only irreducible unitary representation (up to unitary equivalence) of N whose central character is $(0,z)\to\exp(-i\nu\langle z,\omega\rangle)$; indeed $\sigma_{\nu,\omega}$ is essentially the Bargmann-Fock model of the Heisenberg group representations.

The representation $\sigma_{\nu,\omega}$ extends to a representation of $L^1(N)$ – for u in $L^1(N)$, one sets

$$\sigma_{\nu,\omega}(u)\xi=k(p,q)^{-1}\int_{\mathfrak{v}}\int_{\mathfrak{z}}\sigma_{\nu,\omega}(v,z)\xi u(v,z)\,dz\,dv\,. \tag{3.2}$$

It is known that, if $u\in C_c^\infty(N)$, then $\sigma_{\nu,\omega}(u)$ is of trace class; the Plancherel formula for N is also known. So that this paper is self-contained, we offer a brief sketch of the proofs of traceability and of the calculation of the Plancherel measure.

First, we choose an orthonormal basis for $\mathfrak{H}_{\nu,\varepsilon}$: we identify $\mathfrak{v}_\varepsilon$ with $\mathbb{C}^p$, and then for m in $\mathbb{N}_0^p$, where $\mathbb{N}_0=\{0,1,2,3,\ldots\}$, we let $e_m=e_{m,\nu,\omega}$ in $\mathfrak{H}_{\nu,\omega}$ be

$$e_{m,\nu,\omega}(w)=e_m(w)=(2\nu/\pi)^{p/2}(2\nu)^{|m|/2}(m!)^{-1/2}w^m\quad\forall w\in\mathbb{C}^p, \tag{3.3}$$

where $|m|=m_1+m_2+\ldots$, $m!=m_1!m_2!\ldots$, and $w^m=w_1^{m_1}w_2^{m_2}\ldots$. If Δ is the usual Laplacean on $\mathfrak{v}$, with sign chosen to be a positive operator, then

$$\sigma_{\nu,\omega}(\Delta)e_m=(4\nu p+8\nu|m|)e_m\,;$$

consequently, if $k\in\mathbb{N}_0$, and $u\in C_c^\infty(N)$,

$$\begin{aligned}\|(4\nu p+8\nu|m|)^k\sigma_{\nu,\omega}(u)e_m\|_\nu&=\|\sigma_{\nu,\omega}(u*\Delta^k)e_m\|_\nu\\&\leqq C(u,k)\,,\end{aligned}$$

so

$$\|\sigma_{\nu,\omega}(u)e_m\|_\nu=0(|m|^{-k})\quad\text{as}\quad|m|\to+\infty\,.$$

More generally, for such k and u

$$|\langle\sigma_{\nu,\omega}(u)e_m,e_n\rangle|=0((|m|+|n|)^{-k})\,,$$

and so $\sigma_{\nu,\omega}(u)$ is indeed of trace class.

Next, if $u\in C_c^\infty(N)$,

$$\begin{aligned}\langle\sigma_{\nu,\omega}(u)e_m,e_m\rangle&=\int_{\mathfrak{v}}(\sigma_{\nu,\omega}(u)e_m)(v)\overline{e_m(v)}\exp(-2\nu(v)^2)\,dv\\&=\lim_{\delta\to0+}\int_{\mathfrak{v}}(\sigma_{\nu,\omega}(u)e_m)(v)\overline{e_m(v)}\exp(-(\delta+2\nu)|v|^2)\,dv\,.\end{aligned}$$

Since

$$\begin{aligned}&\left|\int_{\mathfrak{v}}(\sigma_{\nu,\omega}(u)e_m)(v)\overline{e_m(v)}\exp(-(\delta+2\nu)|v|^2)\,dv\right|\\&\quad\leqq\|\sigma_{\nu,\omega}(u)e_m\|_\nu\left(\int_{\mathfrak{v}}|e_m(v)|^2\exp(-2(\delta+\nu)|v|^2)\,dv\right)^{1/2}\\&\quad\leqq\|\sigma_{\nu,\omega}(u)e_m\|_\nu=0(|m|^{-k})\,,\end{aligned}$$

for any k in $\mathbb{N}_0$, uniformly for δ in $\mathbb{R}^+$, we have that

$$\begin{aligned}\operatorname{tr}(\sigma_{\nu,\omega}(u)) &= \sum_{m\in\mathbb{N}_0^p} \lim_{\delta\to 0+} \int_{\mathfrak{v}} (\sigma_{\nu,\omega}(u)e_m)(v)\overline{e_m(v)}\exp(-(\delta+2\nu)|v|^2)dv \\ &= \lim_{\delta\to 0+} \sum_{m\in\mathbb{N}_0^p} \int_{\mathfrak{v}} (\sigma_{\nu,\omega}(u)e_m)(v)\overline{e_m(v)}\exp(-(\delta+2\nu)|v|^2)dv.\end{aligned}$$

Now

$$\begin{aligned}&k(p,q) \sum_{m\in\mathbb{N}_0^p} \int_{\mathfrak{v}} (\sigma_{\nu,\omega}(u)e_m)(v)\overline{e_m(v)}\exp(-(\delta+2\nu)|v|^2)dv \\ &= \int_{\mathfrak{v}}\int_{\mathfrak{v}}\int_{\mathfrak{z}} u(w,z) \sum_{m\in\mathbb{N}_0^p} (\sigma_{\nu,\omega}(w,z)e_m)(v)\overline{e_m(v)}\exp(-(\delta+2\nu)|v|^2)dz\,dw\,dv \\ &= \int_{\mathfrak{v}}\int_{\mathfrak{v}}\int_{\mathfrak{z}} \exp(-\delta|v|^2-\nu|w|^2-i\nu\langle z,\omega\rangle+4i\nu\operatorname{Im}\{w,v\}_\omega)(2\nu/\pi)^p u(w,z)\,dz\,dw\,dv \\ &= (2\nu/\pi)^p \int_{\mathfrak{v}} \left[\int_{\mathfrak{z}} u(w,z)\exp(-i\nu\langle z,\omega\rangle)dz\right](\pi/\delta)^p \\ &\quad \exp(-4\nu^2|w|^2/\delta)\exp(-\nu|w|^2)dw.\end{aligned}$$

and therefore

$$\operatorname{tr}(\sigma_{\nu,\omega}(u)) = (\pi/2\nu)^p k(p,q)^{-1} \int_{\mathfrak{z}} u(0,z)\exp(-i\nu\langle z,\omega\rangle)dz.$$

Now, by Fourier inversion, we find that the Plancherel measure of N is given by the following result:

$$\begin{aligned}u(0,0) &= 2^{p-q}\pi^{-q-p}k(p,q)\int_{\mathfrak{z}} \nu^p \operatorname{tr}(\sigma_{\nu,\omega}(u))d(\nu\omega) \\ &= \frac{2^{2-2q-p}}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{2p+q+1}{2}\right)} \int_{S_{\mathfrak{z}}}\int_{\mathbb{R}^+} \nu^{r-1}\operatorname{tr}(\sigma_{\nu,\omega}(u))\,d\nu\,d\omega,\end{aligned}$$

where $d\omega$ denotes normalised surface measure on the sphere $S_{\mathfrak{z}}$. We define

$$c(p,q) = \frac{2^{2-2q-p}\pi^{1/2}}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{2p+q+1}{2}\right)}. \tag{3.4}$$

We now come to the new results on harmonic analysis on groups of type H of this paper. These involve some Fourier transform calculations, on radial functions on a nilpotent group N of type H; more precisely, we call a function u on N $\mathfrak{v}$-radial if $u(v,z)=u(v',z)$ whenever $|v|=|v'|$, $\mathfrak{z}$-radial if $u(v,z)=u(v,z')$ whenever $|z|=|z'|$, and bi-radial if it is both $\mathfrak{v}$-radial and $\mathfrak{z}$-radial. We shall compute explicitly the Fourier transforms of a certain family of $\mathfrak{v}$-radial measures, namely, Φ_R, $R>0$, where

$$\Phi_R(u) = \int_{\mathfrak{v}} u(v,0)\exp(-R|v|^2)dv \quad \forall u\in C_0(N), \tag{3.5}$$

and then of the family of bi-radial functions u_λ, defined by (3.1) above, where $\lambda\in\mathbb{C}$, with $\operatorname{Re}(\alpha)$ sufficiently positive. The techniques we use could readily be applied to compute Fourier transforms of other $\mathfrak{v}$-radial functions. The first step, calculating the Fourier transform of Φ_R, relies on a simple calculation of an integral in $\mathbb{R}^2$, Lemma 3.2 below. In Proposition 3.3, we compute $\sigma_{v,\omega}(\Phi_R)$. Radon and Laplace transform methods are then used to compute $\sigma_{v,\omega}(u_\lambda)$.

Lemma 3.2. *Given v, R in $\mathbb{R}^+$, r, s in $\mathbb{R}$, and k in $\mathbb{N}_0$,*

$$\iint_{\mathbb{R}^2} (x+iy)^k \exp(-v(x^2+y^2))\exp(-R((x-r)^2+(y-s)^2))\exp(2iv(yr-xs))\,dxdy$$
$$=\pi(r+is)^k(R-v)^k(R+v)^{-k-1}\exp(-v(r^2+s^2)). \tag{3.6}$$

Proof. We first show that, if $\mu\in\mathbb{R}^+$, $u, v\in\mathbb{R}$, and $k\in\mathbb{N}_0$, then

$$\iint_{\mathbb{R}^2} (x+iy)^k \exp(-\mu(x^2+y^2))\exp(2i\mu(yu-xv))\,dxdy$$
$$=\frac{\pi}{\mu}(-\mu(u+iv))^k\exp(-\mu(u^2+v^2)). \tag{3.7}$$

To see this, take g in $S(\mathbb{R}^2)$ and define $\tilde{g}$ in $S(\mathbb{R}^2)$ by the rule

$$\tilde{g}(u,v)=\iint_{\mathbb{R}^2} g(x,y)\exp(2i\mu(yu-xv))\,dxdy.$$

Then

$$\frac{1}{2}(\partial/\partial u+i\partial/\partial v)\tilde{g}(u,v)=\mu\iint_{\mathbb{R}^2}(x+iy)g(x,y)\exp(2i\mu(yu-xv))\,dxdy;$$

by induction,

$$\left[\frac{1}{2}(\partial/\partial u+i\partial/\partial v)\right]^k\tilde{g}(u,v)=\mu^k\iint_{\mathbb{R}^2}(x+iy)^k g(x,y)\exp(2i\mu(yu-xv))\,dxdy.$$

If $g(x,y)=\exp(-\mu(x^2+y^2))$, then

$$\tilde{g}(u,v)=(\pi/\mu)\exp(-\mu(u^2+v^2))=(\pi/\mu)\exp(-\mu(u+iv)(u-iv)),$$

and (3.7) follows. By analytic continuation (3.7) is also valid for complex μ, with $\mathrm{Re}(\mu)>0$, and complex u and v.

Now the left hand side of (3.6) is equal to

$$\exp(-R(r^2+s^2))\iint_{\mathbb{R}^2}\exp(-(v+R)(x^2+y^2))\exp(2R(xr+ys)+2iv(yr-xs))\,dxdy$$
$$=\exp(-R(r^2+s^2))\iint_{\mathbb{R}^2}(x+iy)^k\exp(-\mu(x^2+y^2))\exp(2i\mu(yu-xv))\,dxdy,$$

where $\mu=v+R$, $u=(vr-iRs)/(v+R)$ and $v=(vs+iRr)/(v+R)$. By applying the analytically continued version of (3.7), we obtain the desired result. $\square$

Proposition 3.3. *Let Φ_R be as in* (3.5) *above, and suppose $v\in\mathbb{R}^+$ and $\omega\in S_3$. If ξ is a homogeneous polynomial of degree d in $\mathfrak{H}_{v,\omega}$, then*

$$\sigma_{v,\omega}(\Phi_R)\xi=\pi^p(R-v)^d(R+v)^{-d-p}\xi.$$

Proof. By definition, for any w in $\mathfrak{v}$, and ξ in $\mathfrak{H}_{v,\omega}$,

$$\sigma_{v,\omega}(\Phi_R)\xi(w) = \int_{\mathfrak{v}} \exp(-R|v|^2)\exp(-v[|v|^2 + 2\{w,v\}_\omega])\xi(v+w)\,dv.$$

By changing the variable of integration to $v+w$, we obtain that

$$\begin{aligned}
&\sigma_{v,\omega}(\Phi_R)\xi(w)\\
&\quad= \int_{\mathfrak{v}} \xi(v)\exp(-R(v-w)^2)\exp(-v\{v+w, v-w\}_\omega)\,dv\\
&\quad= \exp(v|w|^2) \int_{\mathfrak{v}} \xi(v)\exp(-v|v|^2)\exp(-R(v-w)^2)\exp(2iv\,\mathrm{Im}\{v,w\}_\omega)\,dv.
\end{aligned}$$

Now $\mathfrak{v}_\omega$ may be identified with $\mathbb{C}^p$, and ξ is a sum of homogeneous monomials of degree d. For each monomial, the integral splits into a product of p integrals, each of which is of the type dealt with in Lemma 3.2. The proposition follows immediately. □

One can use this result to calculate the Fourier transforms of many $\mathfrak{v}$-radial functions on N. For instance, Laplace transform methods enable one to calculate the Fourier transforms of other $\mathfrak{v}$-radial measures supported in $\mathfrak{v}$. We first calculate a Radon transform.

Lemma 3.4. *Let u_λ be as above* (3.1), *with* $\mathrm{Re}(\lambda) > -p-1$. *Then, given v in $\mathfrak{v}$, ω in $S_\mathfrak{z}$ and t in $\mathbb{R}$,*

$$\int_{\omega^\perp} u_\lambda(v, t\omega + z')\,dz' = \pi^{q/2-1/2}\left[\Gamma\left(\frac{\lambda+p+1}{2}\right)\Big/\Gamma\left(\frac{\lambda+r}{2}\right)\right]u_{\lambda-q+1}(v,t\omega),$$

where dz' denotes Lebesgue measure on the orthogonal complement to ω in $\mathfrak{z}$.

Proof. This is routine:

$$\begin{aligned}
&\int_{\omega^\perp} ((1+|v|^2)^2 + |t\omega + z'|^2)^{-(\lambda+r)/2}\,dz'\\
&\quad= \int_{\omega^\perp} ((1+|v|^2)^2 + t^2 + |z'|^2)^{-(\lambda+r)/2}\,dz'\\
&\quad= \int_{\omega^\perp} ((1+|v|^2)^2 + t^2)^{-(\lambda+p+1)/2}(1+|z''|^2)^{-(\lambda+r)/2}\,dz''\\
&\quad= u_{\lambda-q+1}(v,t\omega)\left[2\pi^{q/2-1/2}/\Gamma\left(\frac{q-1}{2}\right)\right]\int_0^\infty (1+\varrho^2)^{-(\lambda+r)/2}\varrho^{q-2}\,d\varrho\\
&\quad= u_{\lambda-q+1}(v,t\omega)\pi^{q/2-1/2}\Gamma\left(\frac{\lambda+p+1}{2}\right)\Big/\Gamma\left(\frac{\lambda+r}{2}\right). \quad \square
\end{aligned}$$

We recall a Laplace transform formula: if $f, g \in C(\mathbb{R}^+)$ are such that

$$|f(x)| + |g(x)| = 0(x^{\varepsilon-1/2}) \quad \text{as} \quad x \to 0+$$

and

$$|f(x)| + |g(x)| = 0(\exp(ax)) \quad \text{as} \quad x \to +\infty$$

for some ε in $\mathbb{R}^+$ and all a in $\mathbb{R}^+$, then, setting

$$F(a+ib) = \int_{\mathbb{R}^+} \exp(-(a+ib)x) f(x)\,dx$$

and

$$G(a+ib) = \int_{\mathbb{R}^+} \exp(-(a+ib)x) g(x)\,dx$$

for a in $\mathbb{R}^+$ and b in $\mathbb{R}$, we have, for all a and v in $\mathbb{R}^+$,

$$\int_{\mathbb{R}} F(a+ib)\overline{G(a+ib)}\, e^{-ivb}\,db = 2\pi \int_{\mathbb{R}^+} f(x)\overline{g(x+v)} \exp(-a(2x+v))\,dx, \tag{3.8}$$

where both integrals converge absolutely. To check this formula, it suffices to use the Plancherel formula –

$$\int_{\mathbb{R}} \hat{h}(b)\,[\hat{k}(b)]^{-}\,db = 2\pi \int_{\mathbb{R}} h(x)\bar{k}(x)\,dx \quad \forall h, k \in L^2(\mathbb{R})$$

where the Fourier transform $\hat{h}$ of h is given by

$$\hat{h}(b) = \int_{\mathbb{R}} h(x) e^{-ixb}\,dx.$$

For fix a in $\mathbb{R}^+$; if

$$h(x) = \begin{cases} e^{-ax} f(x) & \forall x > 0 \\ 0 & \forall x \leqq 0 \end{cases}$$

and

$$k(x) = \begin{cases} e^{-a(x+v)} g(x+v) & \forall x > -v \\ 0 & \forall x \leqq -v \end{cases},$$

then it is immediate that $\hat{h}(b) = F(a+ib)$ and $\hat{k}(b) = G(a+ib)e^{ivb}$ for all b in $\mathbb{R}$, and (3.8) follows.

We now define an integral expression which we shall need: for a, b in $\mathbb{R}^+$ and c in $\mathbb{R}$,

$$L(a,b,c) = \int_{\mathbb{R}^+} \exp(-a(2x+1)) x^{b-1} (x+1)^{-c}\,dx. \tag{3.9}$$

Theorem 3.5. *Suppose that* $\mathrm{Re}(\lambda) > 0$. *Then* $u_\lambda \in L^1(N)$, *and for* v *in* $\mathbb{R}^+$, ω *in* S_3, *and for any homogeneous* ξ *of degree* d *in* $\mathfrak{H}_{v,\omega}$,

$$\sigma_{v,\omega}(u_\lambda)\xi = T(v,\lambda,d)\xi, \tag{3.10}$$

where

$$T(v,\lambda,d) = \frac{2^r \Gamma\left(\frac{2p+q+1}{2}\right)}{\Gamma\left(\frac{\lambda+r}{2}\right)\Gamma\left(\frac{\lambda+p+1}{2}\right)} v^\lambda L\left(v, \frac{2d+\lambda+p+1}{2}, \frac{2d-\lambda+p+1}{2}\right).$$

This formula continues to hold when $\mathrm{Re}(\lambda) > -(r/2)$.

Proof. The function $\lambda \to u_\lambda$ is an analytic $L^1(N)$-valued function in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$. Consequently the Fourier transform is analytic there; it suffices to prove

(3.10) for λ in $\mathbb{R}^+$, and then it will follow that (3.10) holds for λ with $\mathrm{Re}(\lambda)>0$ by analytic continuation. Now in $\{\lambda\in\mathbb{C}:\mathrm{Re}(\lambda)>-(r/2)\}$, the function $\lambda\to u_\lambda$ is also analytic in $L^2(N)$, and its Fourier transform will be analytic in $L^2(\hat{N})$. Certainly then, if (3.10) holds for λ in $\mathbb{R}^+$, it holds for all λ with $\mathrm{Re}(\lambda)>-(r/2)$.

Now we take

$$\beta=(\lambda+p+1)/2, \quad \gamma=(\lambda+r)/2$$

and set

$$f(x)=g(x)=\begin{cases}\Gamma(\beta)^{-1}x^{\beta-1}\exp(-x) & \forall x\in\mathbb{R}^+\\ 0 & \forall x\in\mathbb{R}\backslash\mathbb{R}^+\end{cases}$$

so that the Laplace transforms F and G of f and g are given by

$$F(a+ib)=G(a+ib)=(1+a+ib)^{-\beta} \quad \forall a\in\mathbb{R}^+,\ \forall b\in\mathbb{R}.$$

Now, by Lemma 3.4, Fubini's theorem, and Proposition 3.3, for ξ in $\mathfrak{H}_{\nu,\omega}$ of degree d,

$$\begin{aligned}
&\int_{\mathfrak{v}}\int_{\mathfrak{z}} u_\lambda(v,z)\sigma_{\nu,\omega}(v,z)\xi\,dz\,dv\\
&=\pi^{q/2-1/2}\Gamma(\beta)/\Gamma(\gamma)\int_{\mathfrak{v}}\int_{\mathbb{R}} u_{\lambda-q+1}(v,t\omega)\sigma_{\nu,\omega}(v,t\omega)\xi\,dt\,dv\\
&=\pi^{q/2-1/2}\Gamma(\beta)/\Gamma(\gamma)\int_{\mathfrak{v}}\int_{\mathbb{R}} F(|v|^2+it)G(|v|^2+it)\exp(-i\nu t)\,dt\,\sigma_{\nu,\omega}(v,0)\xi\,dv\\
&=2\pi^{q/2+1/2}\Gamma(\beta)/\Gamma(\gamma)\int_{\mathfrak{v}}\int_{\mathbb{R}^+} f(x)\overline{g(x+\nu)}\exp(-(2x+\nu)|v|^2)\,dx\,\sigma_{\nu,\omega}(v,0)\xi\,dv\\
&=2\pi^{q/2+1/2}\Gamma(\beta)/\Gamma(\gamma)\int_{\mathbb{R}^+} f(x)\overline{g(x+\nu)}\int_{\mathfrak{v}}\exp(-(2x+\nu)|v|^2)\sigma_{\nu,\omega}(v,0)\xi\,dv\,dx\\
&=2\pi^{q/2+1/2}\Gamma(\beta)/\Gamma(\gamma)\int_{\mathbb{R}^+} f(x)\overline{g(x+\nu)}\pi^p(2x)^d(2x+2\nu)^{-d-p}\xi\,dx\\
&=2^{1-p}\pi^{(2p+q+1)/2}[\Gamma(\beta)\Gamma(\gamma)]^{-1}\int_{\mathbb{R}^+}\exp(-(2x+\nu))x^{\beta+d-1}(x+\nu)^{\beta-d-p-1}\,dx\,\xi\\
&=2^{1-p}\pi^{(2p+q+1)/2}[\Gamma(\beta)\Gamma(\gamma)]^{-1}\nu^\lambda L(\nu,\beta+d,\,-\beta+d+p+1)\xi,
\end{aligned}$$

and

$$\sigma_{\nu,\omega}(u_\lambda)\xi=\frac{2^r\Gamma\left(\frac{2p+q+1}{2}\right)}{\Gamma(\beta)\Gamma(\gamma)}\nu^\lambda L\left(\nu,\frac{2d+\lambda+p+1}{2},\frac{2d-\lambda+p+1}{2}\right)\xi.$$

as required. □

Later progress will depend on knowing the functional equation for the function T defined above (3.10).

Proposition 3.6. *Suppose $a>1/2$ and $b>1/2$. Then for any μ in $\mathbb{R}^+$,*

$$\frac{\mu^a}{\Gamma(a)}\int_0^\infty\frac{t^{a-1}e^{-\mu t}}{(1+t)^b}\,dt=\frac{\mu^b}{\Gamma(b)}\int_0^\infty\frac{s^{b-1}e^{-\mu s}}{(1+s)^a}\,ds. \tag{3.11}$$

These expressions continue analytically to entire functions of a and b. Consequently,

$$\frac{(2v)^a}{\Gamma(a)} L(v,a,b) = \frac{(2v)^b}{\Gamma(b)} L(v,b,a) \quad \forall a,b \in \mathbb{C}.$$

and

$$(v/2)^{\lambda/2} \frac{\Gamma\left(\frac{-\lambda+p+1}{2}\right)\Gamma\left(\frac{-\lambda+r}{2}\right)}{\Gamma\left(\frac{2d-\lambda+p+1}{2}\right)} T(v,-\lambda,d)$$

$$= (v/2)^{-\lambda/2} \frac{\Gamma\left(\frac{\lambda+p+1}{2}\right)\Gamma\left(\frac{\lambda+r}{2}\right)}{\Gamma\left(\frac{2d+\lambda+p+1}{2}\right)} T(v,\lambda,d) \tag{3.12}$$

for all v in $\mathbb{R}^+$ *and d in* $\mathbb{N}_0$, *as an identity of entire functions.*

Proof. The equality (3.11) is known ([1], 6.5(2) and 6.5(6)), but we offer a proof for completeness. We shall first prove (3.11) for $a, b > 1/2$. For a fixed small positive ε, we define functions $f, g\colon \mathbb{R} \to \mathbb{C}$ by the formulae

$$f(s) = \begin{cases} s^{a-1} e^{-(\varepsilon+i)\mu s}/\Gamma(a) & \forall s \in \mathbb{R}^+ \\ 0 & \forall s \in \mathbb{R}\backslash\mathbb{R}^+ \end{cases},$$

$$g(t) = \begin{cases} t^{b-1} e^{-t}/\Gamma(b) & \forall t \in \mathbb{R}^+ \\ 0 & \forall t \in \mathbb{R}\backslash\mathbb{R}^+ \end{cases}.$$

By Plancherel's theorem,

$$\int_{\mathbb{R}} f(s)\hat{g}(s)\,ds = \int_{\mathbb{R}} \hat{f}(t) g(t)\,dt,$$

so

$$\frac{1}{\Gamma(a)} \int_{\mathbb{R}^+} \frac{s^{a-1} e^{-(\varepsilon+i)\mu s}}{(1+is)^b}\,ds = \frac{1}{\Gamma(b)} \int_{\mathbb{R}^+} \frac{t^{b-1} e^{-t}}{((\varepsilon+i)\mu+it)^a}\,dt.$$

We multiply both sides by $((\varepsilon+i)\mu)^a$, and change variables – in the left hand integral, we use contour integration, and in the right hand integral we put $t = \mu s$ – to obtain the equality

$$\frac{\mu^a}{\Gamma(a)} \int_{\mathbb{R}^+} \frac{t^{a-1} e^{-\mu t}}{(1+it/(\varepsilon+i))^b}\,dt = \frac{\mu^b}{\Gamma(b)} \int_{\mathbb{R}^+} \frac{s^{b-1} e^{-\mu s}}{(1+is/(\varepsilon+i))^a}\,ds.$$

We now let ε tend to 0 to finish the proof of (3.11).

To see the analytic continuation of the left hand side of (3.11), we write

$$\frac{\mu^a}{\Gamma(a)} \int_0^\infty \frac{t^{a-1} e^{-\mu t}}{(1+t)^b}\,dt = \frac{\mu^a}{\Gamma(a)} \int_0^{1/2} \frac{t^{a-1} e^{-\mu t}}{(1+t)^b}\,dt + \frac{\mu^a}{\Gamma(a)} \int_{1/2}^\infty \frac{t^{a-1} e^{-\mu t}}{(1+t)^b}\,dt.$$

The last integral continues analytically to an entire function of a and b; by writing the integrand of the second as a convergent series of the form $\sum_{m \in \mathbb{N}_0} c_m(b)\, t^{a+m-1}$, it is clear that this integral continues meromorphically with simple poles when $a \in -\mathbb{N}_0$, which are cancelled by the zeros of $\Gamma(a)^{-1}$. The other functional equations follow immediately. □

It is worthwhile pointing out that, if we define $u_{\lambda,\delta}$ *by the formula*

$$u_{\lambda,\delta}(v,z)=((\delta+|v|^2)^2+|z|^2)^{-(\lambda+r)/2} \quad \forall (v,z)\in N$$

for δ in $\mathbb{R}^+$ and λ in $\mathbb{C}$, then a calculation like the proof of Theorem 3.5, or an application of homogeneity arguments, shows that, for homogeneous ξ of degree d in $\mathfrak{H}_{\nu,\omega}$,

$$\sigma_{\nu,\omega}(u_{\lambda,\delta})\xi=\frac{2^r\Gamma\left(\frac{2p+q+1}{2}\right)}{\Gamma\left(\frac{\lambda+r}{2}\right)\Gamma\left(\frac{\lambda+p+1}{2}\right)}\nu^\lambda L\left(\delta\nu,\frac{2d+\lambda+p+1}{2},\frac{2d-\lambda+p+1}{2}\right)\xi.$$

When δ tends to 0, $u_{\lambda,\delta}$ tends distributionally to the distribution $n^{-(\lambda+r)}$, given by the locally integrable function $(v,z)\to(|v|^4+|z|^2)^{-(\lambda+r)/2}$ for λ with $\mathrm{Re}(\lambda)<0$, and by meromorphic continuation otherwise. When N is the nilpotent component of a real rank one Lie group G, this distribution is the kernel of the so-called intertwining operator of A.W. Knapp and E.M. Stein [23]; its Fourier transform, at least formally, should be given by the rule

$$\begin{aligned}\sigma_{\nu,\omega}(n^{-(\lambda+r)})\xi&=\frac{2^r\Gamma\left(\frac{2p+q+1}{2}\right)}{\Gamma\left(\frac{\lambda+r}{2}\right)\Gamma\left(\frac{\lambda+p+1}{2}\right)}\nu^\alpha L\left(0,\frac{2d+\lambda+p+1}{2},\frac{2d-\lambda+p+1}{2}\right)\xi,\\&=\frac{2^r\Gamma\left(\frac{2p+q+1}{2}\right)\Gamma\left(\frac{2d+\lambda+p+1}{2}\right)\Gamma(-\lambda)}{\Gamma\left(\frac{\lambda+r}{2}\right)\Gamma\left(\frac{\lambda+p+1}{2}\right)\Gamma\left(\frac{2d-\lambda+p+1}{2}\right)}\nu^\lambda\xi,\end{aligned}$$

for homogeneous ξ of degree d in $\mathfrak{H}_{\nu,\omega}$. This formula agrees with that of Cowling [8], after the different definitions of p, r and λ, and the different normalisation of Haar measure are taken into account. Moreover, if we define the meromorphic function c by the rule

$$c(\lambda)=2^{(r-\lambda)}\frac{\Gamma(\lambda)\Gamma\left(\frac{2p+q+1}{2}\right)}{\Gamma\left(\frac{\lambda+r}{2}\right)\Gamma\left(\frac{\lambda+p+1}{2}\right)},$$

then, formally, we see that, for homogeneous ξ in $\mathfrak{H}_{\nu,\omega}$ of degree d,

$$\sigma_{\nu,\omega}(c(\lambda)^{-1}n^{\lambda-r})\xi=(\nu/2)^{-\lambda}\frac{\Gamma\left(\frac{\lambda+p+1}{2}\right)\Gamma\left(\frac{\lambda+r}{2}\right)\Gamma\left(\frac{2d-\lambda+p+1}{2}\right)}{\Gamma\left(\frac{-\lambda+p+1}{2}\right)\Gamma\left(\frac{-\lambda+r}{2}\right)\Gamma\left(\frac{2d+\lambda+p+1}{2}\right)}\xi.$$

From Theorem 3.5 and Proposition 3.6, we would expect that

$$c(\lambda)^{-1} n^{\lambda-r} * u_\lambda = u_{-\lambda}. \tag{3.13}$$

On the other hand, the general theory of intertwining operators implies that, for a suitable normalisation factor c, (3.13) holds, and that the Plancherel measure $\mu(\lambda)$ associated to the class one principal series representation π_λ of G is given, up to a constant, by $|c(\lambda)|^{-2}$. We would therefore predict that the Plancherel measure should be given by

$$\mu(\lambda) = \text{const.} \left| \frac{\Gamma\left(\frac{\lambda+r}{2}\right)\Gamma\left(\frac{\lambda+p+1}{2}\right)}{\Gamma(\lambda)} \right|^2 .$$

This agrees with the results of Harish-Chandra. See S. Helgason [18] (Chapter IV) for more on c-functions and the Plancherel measure.

Our later development depends on a careful study of the function T. Most of the facts we shall need are summarised in the following result.

Propsoition 3.7. *Fix v in $\mathbb{R}^+$ and d in $\mathbb{N}_0$. The function $\lambda \to T(v,\lambda,d)$ is entire. Furthermore, if $\lambda = \beta + i\gamma$, where $0 \leqq \beta < r$ and $\gamma \in \mathbb{R}$, then*

$$\left(\int_{\mathbb{R}^+} |v^{-\lambda/2} T(v,\lambda,d)|^2 v^{r-1} dv \right)^{1/2} \leqq C_1(p,q)(r-\beta)^{-1/2} e^{|\gamma|}, \tag{3.14}$$

where $C_1(p,q)$ depends only on p and q, and

$$\lim_{\beta \to r-} (r-\beta)^{1/2} \left(\int_{\mathbb{R}^+} |v^{-\beta/2} T(v,\beta,d)|^2 v^{r-1} dv \right)^{1/2} = 1. \tag{3.15}$$

Proof. It is easy to see that $\alpha \to T(v,\alpha,d)$ is holomorphic if $\operatorname{Re}(\lambda) > -p-1$. The functional equation (3.12) then implies that T is holomorphic if $\operatorname{Re}(\lambda) < p+1$, which establishes that T is entire.

To prove (3.14), we shall first prove the following inequality:

$$|T(v,\lambda,d)| \leqq C_2(p,q)\,[v^{(\beta-1)/2} + (v+1)^{(\beta-1)/2}]\, e^{-v} e^{|\gamma|}. \tag{3.16}$$

Write

$$T(v,\lambda,d) = Q(\lambda) \int_{\mathbb{R}^+} \frac{\exp(-2x-v)\, x^{(2d+\lambda+p-1)/2}}{(x+v)^{(2d-\lambda+p+1)/2}}\, dx,$$

where

$$Q(\lambda) = \frac{2^r \Gamma\left(\frac{2p+q+1}{2}\right)}{\Gamma\left(\frac{\lambda+p+1}{2}\right)\Gamma\left(\frac{\lambda+r}{2}\right)}.$$

Because $x^{(2d+p)/2} \leqq (x+v)^{(2d+p)/2}$, then, if $0 \leqq \beta \leqq r$,

$$|T(v,\lambda,d)| \leqq |Q(\lambda)| \int_{\mathbb{R}^+} \frac{\exp(-2x-v)\, x^{(\beta-1)/2}}{(x+v)^{(-\beta+1)/2}}\, dx.$$

If $0 \leqq \beta \leqq 1$, then $(x+v)^{(\beta-1)/2} \leqq v^{(\beta-1)/2}$, so

$$\begin{aligned}|T(v,\lambda,d)| &\leqq |Q(\lambda)|\, v^{(\beta-1)/2} e^{-v} \int_{\mathbb{R}^+} \exp(-2x) x^{(\beta-1)/2} dx \\ &\leqq C_3(p,q)\, v^{(\beta-1)/2} e^{-v} e^{2|\gamma|},\end{aligned}$$

from the known asymptotic behaviour of the Γ-function (see, e.g. Titchmarsh [31], 1.87). If $1 \leqq \beta \leqq r$, then, because $(x+v)^{(\beta-1)/2} \leqq [(x+1)(v+1)]^{(\beta-1)/2}$, we have similarly

$$\begin{aligned}|T(v,\lambda,d)| &\leqq |Q(\lambda)|(v+1)^{(\beta-1)/2} e^{-v} \int_{\mathbb{R}^+} \exp(-2x)(x(x+1))^{(\beta-1)/2} dx \\ &\leqq C_4(p,q)(v+1)^{(\beta-1)/2} e^{-v} e^{2|\gamma|}.\end{aligned}$$

We therefore have the estimate (3.16); (3.14) follows immediately.

To prove (3.15), we first show that, if $\beta > 0$, then

$$|T(v,\lambda,d)| \leqq C_5(p,q)(1+\beta^{-1}), \tag{3.17}$$

$$|T(v,\lambda,d) - T(v,r,d)| \leqq C_6(p,q)(1+\beta^{-2})|\lambda - r|. \tag{3.18}$$

and

$$\lim_{v \to 0+} T(v,r,d) = 1. \tag{3.19}$$

To obtain (3.17), it is easiest to remember that, if ξ is homogeneous of degree d in $\mathfrak{H}_{v,\omega}$, then

$$\sigma_{v,\omega}(u_\lambda)\xi = T(v,\lambda,d)\xi.$$

Therefore

$$\begin{aligned}|T(v,\lambda,d)| &\leqq \|u_\lambda\|_1 = k(p,q)^{-1} \int_{\mathfrak{v}} \int_{\mathfrak{z}} ((1+|v|^2)^2 + |z|^2)^{-(\beta+r)/2} dz dv \\ &= 2^{(r-\beta)} \frac{\Gamma\left(\frac{2p+q+1}{2}\right)\Gamma(\beta)}{\Gamma\left(\frac{\beta+p+1}{2}\right)\Gamma\left(\frac{\beta+r}{2}\right)},\end{aligned}$$

by a calculation like Lemma 3.1; (3.17) follows. The inequality (3.18) is obtained using Cauchy's integral formula to estimate $\partial T(v,\lambda,d)/\partial\lambda$, and then estimating the difference in terms of the derivative. More precisely, by integrating along the line segment γ joining λ to r, we see that

$$T(v,\lambda,d) - T(v,r,d) = \int_\gamma \partial T(v,\mu,d)/\partial\mu\, d\mu,$$

whence

$$|T(v,\lambda,d) - T(v,r,d)| \leqq |\lambda - r| \sup\{|\partial T(v,\mu,d)/\partial\mu| : \mu \in \gamma\};$$

by integrating around the circle κ of centre μ and radius $\beta/2$, we see that

$$\partial T(v,\mu,d)/\partial\mu = (2\pi i)^{-1} \int_\kappa T(v,\zeta,d)(\zeta-\mu)^{-2} d\zeta,$$

whence

$$|\partial T(v,\mu,d)/\partial\mu| \leqq (2/\beta)\sup\{|T(v,\zeta,d)| : \zeta\in\kappa\}.$$

The points ζ which arise as we consider different circles κ corresponding to different centres μ all have real part at least $\beta/2$, and (3.18) follows from the last two inequalities and (3.17). Finally,

$$T(v,r,d)=\frac{2^r\Gamma\left(\frac{2p+q+1}{2}\right)}{\Gamma(r)\Gamma\left(\frac{2p+q+1}{2}\right)}\int_{\mathbb{R}^+}\frac{\exp(-2x-v)x^{(2d+2p+q-1)/2}}{(x+v)^{(2d-q+1)/2}}\,dx.$$

If $0<v<1$, the integrand is dominated by the integrable function

$$x\to\exp(-2x)x^{(2p+q-1)/2}(x+1)^{(q-1)/2}$$

so by the dominated convergence theorem,

$$\begin{aligned}\lim_{v\to 0+} T(v,r,d) &= \frac{2^r}{\Gamma(r)}\int_{\mathbb{R}^+}\exp(-2x)x^{p+q-1}\,dx\\ &=1.\end{aligned}$$

Now we prove (3.15). By (3.16), if $\delta>0$, then

$$(r-\beta)\int_\delta^\infty |v^{-\beta/2}T(\lambda,\beta,d)|^2 v^{r-1}\,dv\to 0$$

as $\beta\to r-$, and by (3.18)

$$(r-\beta)\int_0^\delta |v^{-\beta/2}[T(v,\beta,d)-T(v,r,d)]|^2 v^{r-1}\,dv\to 0,$$

as $\beta\to r-$. Further,

$$\begin{aligned}\limsup_{\beta\to r-}\,(r-\beta)\int_0^\delta |v^{-\beta/2}[T(v,r,d)-1]|^2 v^{r-1}\,dv&\\ \leqq \sup\{|T(v,r,d)-1|^2 : 0<v<\delta\},&\end{aligned}$$

and by (3.19) we can make this arbitrarily small by choosing δ small. Finally,

$$\lim_{\beta\to r-}\,(r-\beta)\int_0^\delta |v^{-\beta/2}|^2 v^{r-1}\,dv=1\,;$$

hence (3.15) follows. □

4. Harmonic analysis on AN

The group $S=AN$ is the semidirect product of the vector group A with the normal nilpotent H-type group N. The elements of the group S may be written in the form

$a_s(v,z)$, where $a_s \in A$ ($s \in \mathbb{R}^+$) and $(v,z) \in N$. These multiply according to the rule

$$\begin{aligned} a_{s'}(v',z')a_s(v,z) &= a_{s's}a_{s^{-1}}(v',z')a_s(v,z) \\ &= a_{s's}(s^{-1/2}v', s^{-1}z')(v,z). \\ &= a_{s's}(s^{-1/2}v'+v, s^{-1}z'+z+2s^{-1/2}[v',v]). \end{aligned}$$

Mackey theory may be applied to describe the unitary dual $\hat{S}$ of S. The irreducible unitary representations of S fall into two classes: those which are trivial on the centre of N, of no interest to us here, and those which are nontrivial on the centre of N, which are parametrised by the A-orbits in $\mathfrak{z}^*$. Because these involve integrals of the representations $\sigma_{\nu,\omega}$ of N as ν runs over $\mathbb{R}^+$, it will be convenient to use equivalent representations $\tau_{\nu,\omega}$ of N which all act on the same Hilbert space, as ν runs over $\mathbb{R}^+$.

For ν in $\mathbb{R}^+$, define $I_\nu : \mathfrak{H}_{1,\omega} \to \mathfrak{H}_{\nu,\omega}$ by the formula

$$(I_\nu \xi)(v) = \nu^{p/2}\xi(\nu^{1/2}v) \quad \forall \xi \in \mathfrak{H}_{1,\omega},\ \forall v \in \mathfrak{v};$$

I_ν is an invertible isometry of Hilbert spaces (whose inverse is effectively $I_{\nu^{-1}}$). Let $\tau_{\nu,\omega}$ be the unitary representation $I_\nu^{-1}\sigma_{\nu,\omega}I_\nu$. Define, for s in $\mathbb{R}^+$, the isometric isomorphism $\delta_s : L^1(N) \to L^1(N)$ by

$$(\delta_s f)(v,z) = s^{-r}f(s^{-1/2}v, s^{-1}z) \quad \forall v \in \mathfrak{v},\ \forall z \in \mathfrak{z}.$$

Lemma 4.1. *Fix s and ν in $\mathbb{R}^+$, v in $\mathfrak{v}$, z in $\mathfrak{z}$, ω in S_3, and f in $L^1(N)$. Then*

$$\tau_{\nu,\omega}(s^{1/2}v, sz) = \tau_{s\nu,\omega}(v,z)$$

and

$$\tau_{\nu,\omega}(\delta_s f) = \tau_{s\nu,\omega}(f).$$

Proof. These results follow by changes of variable: first, one shows

$$\tau_{\nu,\omega}(v,z) = \sigma_{1,\omega}(\nu^{1/2}v, \nu z),$$

then we deduce that $\tau_{\nu,\omega}(s^{1/2}v, sz) = \tau_{s\nu,\omega}(v,z)$. Finally, we note that

$$\begin{aligned} \int_N (\delta_s f)(n)\tau_{\nu,\omega}(n)\,dn &= \int_N f(n)\tau_{\nu,\omega}(n_s)\,dn, \\ &= \int_N f(n)\tau_{s\nu,\omega}(n)\,dn \end{aligned}$$

where $(v,z)_s = (s^{1/2}v, sz)$. □

For ω in S_3, we define the Hilbert space $\mathfrak{H}_\omega$, to be $L^2(\mathbb{R}^+; \mathfrak{H}_{1,\omega})$; more precisely, $\mathfrak{H}_\omega$ is the space of (equivalence classes of) measurable functions $\Xi : \mathbb{R}^+ \to \mathfrak{H}_{1,\omega}$, with the property that $\|\Xi\|_\omega$

$$\|\Xi\|_\omega = \{c(p,q) \int_{\mathbb{R}^+} \|\Xi(\nu)\|^2_{\mathfrak{H}_{1,\omega}} \nu^{r-1}\,d\nu\}^{1/2} \tag{4.1}$$

is finite (where the equivalence is equality almost everywhere); here $c(p,q)$ is given by (3.4):

$$c(p,q) = \frac{2^{2-2q-p}\pi^{1/2}}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{2p+q+1}{2}\right)}.$$

We define the unitary representation τ_ω of S on $\mathfrak{H}_\omega$ by the formula

$$[\tau_\omega(a_s(v,z))\Xi](v) = s^r \tau_{sv,\omega}(v,z)(\Xi(sv)) \quad \forall \Xi \in \mathfrak{H}_\omega,\ \forall v \in \mathbb{R}^+ \tag{4.2}$$

for any element $a_s(v,z)$ of S. It is routine to check that τ_ω is indeed a unitary representation, using Lemma 4.1.

We denote by $\Xi_{j,\lambda,\omega} : \mathbb{R} \to \mathfrak{H}_{1,\omega}$ and $H_{j,\lambda,\omega} : \mathbb{R} \to \mathfrak{H}_{1,\omega}$ ($j \in \mathbb{N}_0^p$, $\lambda \in \mathbb{C}$) the functions

$$\Xi_{j,\lambda,\omega}(v) = v^{-\lambda/2} T(v, \lambda, |j|) e_{j,1,\omega} \tag{4.3}$$

and

$$H_{j,\lambda,\omega}(v) = [v^{\lambda/2} T(v, -\lambda, |j|)]^- e_{j,1,\omega}, \tag{4.4}$$

where the $e_{j,1,\omega}$ are the basis elements of $\mathfrak{H}_{1,\omega}$ given by normalised monomials, as in (3.3).

Proposition 4.2. *Fix j in $\mathbb{N}_0^p$, and let $\phi_{j,\lambda} : S \to \mathbb{C}$ be the function given by*

$$\phi_{j,\lambda}(a_s(v,z)) = \int_{S_3} \langle \tau_\omega(a_s(v,z))\Xi_{j,\lambda,\omega}, H_{j,\lambda,\omega}\rangle d\omega .$$

Then if $\lambda \in i\mathbb{R}$, $\phi_{j,\lambda} \in B(S)$; moreover, the function $\lambda \to \phi_{j,\lambda}$ extends to an analytic $B(S)$-valued function in the strip $\{\lambda \in \mathbb{C} : |\mathrm{Re}(\lambda)| < r\}$, and, if $\lambda = \beta + i\gamma$, with β in $[-R, R]$ and γ in $\mathbb{R}$, where $p < R < r$,

$$\|\phi_{j,\lambda}\|_B \leqq C_7(p,q)(1+|j|)^{-R}(r-R)^{-1} e^{6|\gamma|}, \tag{4.5}$$

while

$$\|\phi_{j,\beta}\|_B \leqq C_8(p,q)(1+|j|)^{-|\beta|}, \tag{4.6}$$

and,

$$\lim_{\beta \to r-} \|\phi_{j,\lambda}\|_B = \frac{2^{1-q}\pi^{1/2}\Gamma(r)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{2p+q+1+2|j|}{2}\right)\Gamma\left(\frac{q+1-2|j|}{2}\right)}.$$

Proof. By definition of τ_ω, $\phi_{j,\lambda} \in B(S)$ and

$$\begin{aligned}
\|\phi_{j,\lambda}\|_B &\leqq \int_{S_3} \|\langle \tau_\omega(.)\Xi_{j,\lambda,\omega}, H_{j,\lambda,\omega}\rangle\|_B d\omega \\
&\leqq \int_{S_3} \|\Xi_{j,\lambda,\omega}\|\ \|H_{j,\lambda,\omega}\| d\omega \\
&\leqq c(p,q) \int_{S_3} \left\{\int_{\mathbb{R}^+} \|\Xi_{j,\lambda,\omega}(v)\|_\omega^2 v^{r-1} dv\right\}^{1/2} \\
&\qquad \left\{\int_{\mathbb{R}^+} \|H_{j,\lambda,\omega}(v)\|_\omega^2 v^{r-1} dv\right\}^{1/2} d\omega,
\end{aligned}$$

as long as the last expression is finite. Let N_j be defined thus:

$$N_j(\lambda) = \left\{\int_{\mathbb{R}^+} |v^{-\lambda/2} T(v, \lambda, |j|)|^2 v^{r-1} dv\right\}^{1/2} \quad \forall \lambda \in \mathbb{C}; \tag{4.7}$$

then our inequality may be more briefly written

$$\|\phi_{j,\lambda}\|_B \leqq c(p,q) N_j(\lambda) N_j(-\lambda).$$

Further when these integrals converge, $\phi_{j,\lambda}$ will be holomorphic as $\lambda \to \Xi_{j,\lambda,\omega}$ is then a holomorphic $\mathfrak{H}_\omega$-valued function and $\lambda \to H_{j,\lambda,\omega}$ is anti-holomorphic. By definition of $\Xi_{j,\lambda,\omega}$ and $H_{j,\lambda,\omega}$, this boils down to the question of when $\lambda \to v^{-\lambda/2}\,T(v,\lambda,|j|)$ is a holomorphic $L^2(\mathbb{R}^+; v^{r-1}dv)$-valued function, and when $N_j(\lambda)N_j(-\lambda)$ is finite. Since $\alpha \to T(v,\lambda,|j|)$ is entire for each v in $\mathbb{R}^+$ and j in $\mathbb{N}_0^p$, by Proposition 3.7, it will suffice to find good norm estimates for $N_j(\lambda)N_j(-\lambda)$. Clearly we may suppose $\operatorname{Re}(\lambda) \geqq 0$. According to (3.14), if $\lambda = \beta + i\gamma$, with $0 \leqq \beta < r$, then

$$N_j(\lambda) \leqq C_1(p,q)\,(r-\beta)^{-1/2} e^{2|\gamma|}\,;$$

further, from (3.12)

$$\begin{aligned} N_j(-\lambda) &= 2^{\beta/2} \left| \frac{\Gamma\left(\frac{\lambda+r}{2}\right)\Gamma\left(\frac{\lambda+p+1}{2}\right)\Gamma\left(\frac{2|j|-\lambda+p+1}{2}\right)}{\Gamma\left(\frac{-\lambda+r}{2}\right)\Gamma\left(\frac{-\lambda+p+1}{2}\right)\Gamma\left(\frac{2|j|+\lambda+p+1}{2}\right)} \right| N_j(\lambda) \\ &\leqq 2^\beta \left| \frac{\Gamma\left(\frac{\lambda+r}{2}\right)}{\Gamma\left(\frac{-\lambda+r}{2}\right)} \right| \prod_{k=1}^{|j|} \left| \frac{2k-\lambda+p-1}{2k+\lambda+p-1} \right| C_1(p,q)\,(r-\beta)^{-1/2} e^{2|\gamma|}, \end{aligned} \tag{4.8}$$

so we have established that

$$N_j(\lambda)N_j(-\lambda) \leqq 2^{\beta/2} \left| \frac{\Gamma\left(\frac{\lambda+r}{2}\right)}{\Gamma\left(\frac{-\lambda+r}{2}\right)} \right| \prod_{k=1}^{|j|} \left| \frac{2k-\lambda+p-1}{2k+\lambda+p-1} \right| C_1(p,q)^2 (r-\beta)^{-1} e^{4|\gamma|}. \tag{4.9}$$

Certainly, then if $\lambda = \beta + i\gamma$, and $-r \leqq \beta < r$, then

$$N_j(\lambda)N_j(-\lambda) \leqq C_9(p,q)\,(r-|\beta|)^{-1} e^{6|\gamma|}.$$

Thus $\phi_{j,\lambda} \in B(S)$, and

$$\|\phi_{j,\lambda}\|_B \leqq C_{10}(p,q)\,(r-|\beta|)^{-1} e^{6|\gamma|},$$

for λ, p and γ as above. However, we may improve on this estimate by working with (4.9) and using the Banach-space valued version of the three lines theorem.

We claim that, if $\lambda = \beta + i\gamma$, with β in $[p,r]$, then

$$\prod_{k=1}^{d} \left| \frac{2k-\lambda+p-1}{2k+\lambda+p-1} \right| \leqq C_{11}(p,q)\,(1+|\gamma|)^\beta d^{-\beta} \quad \forall d \in \mathbb{N}, \tag{4.10}$$

for some constant $C_{11}(p,q)$ depending only on p and q. This can be seen by noting first that if $k \geqq p+q+|\gamma|$, then

$$\begin{aligned} \log\left| \frac{2k-\lambda+p+1}{2k+\lambda+p-1} \right| &= \operatorname{Re}\left\{ \log\left(1 + \frac{-\lambda+p-1}{2k}\right) - \log\left(1 + \frac{\lambda+p-1}{2k}\right) \right\} \\ &= \operatorname{Re} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \frac{(-\lambda+p-1)^m - (\lambda+p-1)^m}{(2k)^m}, \end{aligned}$$

so

$$\sum_{k=d_0}^{d} \log\left|\frac{2d-\lambda+p-1}{2k+\lambda+p-1}\right|$$

$$=\sum_{k=d_0}^{d} \operatorname{Re}\left(\frac{-\lambda}{k}\right)+\sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{m}\left[\operatorname{Re}((-\lambda+p-1)^m-(\lambda+p-1)^m)\right] \sum_{k=d_0}^{d} \frac{1}{(2k)^m},$$

where d_0 is an integer between $p+q+|\gamma|$ and $p+q+|\gamma|+1$. The first of these sums is equal to

$$-\beta\left[\log(d/d_0)+E\right],$$

for some E with $|E|<1$, while the double sum can be dominated by

$$\sum_{m=2}^{\infty} \frac{1}{m(m-1)}\left(|-\lambda+p-1|^m+|\lambda+p-1|^m\right)(2d_0-1)^{1-m} \leqq 2 \sum_{m=2}^{\infty} \frac{1}{m(m-1)}=2\,;$$

(4.10) follows.

By using (4.10), we improve (4.9). First, if $p\leqq\beta\leqq r$, we have the estimate

$$\|\phi_{j,\lambda}\|_B \leqq C_{12}(p,q)(r-\beta)^{-1}(1+\lfloor j\rfloor)^{-\beta} e^{6|\gamma|},$$

and similarly, when $-r\leqq\beta\leqq -p$, we have that

$$\|\phi_{j,\lambda}\|_B \leqq C_{12}(p,q)(r+\beta)^{-1}(1+\lfloor j\rfloor)^{-\beta} e^{6|\gamma|}.$$

Consider the analytic $B(S)$-valued function $\psi:\lambda\to\cos(\lambda/r)^{-6r}\phi_{j,\lambda}$ on the strip $\{\lambda\in\mathbb{C}:|\operatorname{Re}(\lambda)|<r\}$. This is bounded in $B(S)$-norm on each closed substrip $\{\lambda\in\mathbb{C}:|\operatorname{Re}(\lambda)|\leqq R\}$, when $p<R<r$, and satisfies the conditions

$$\|\psi(\lambda)\|_B \leqq C_{13}(p,q)(r-R)^{-1}(1+\lfloor j\rfloor)^{-R},$$

when $\operatorname{Re}(\lambda)=\pm R$. By the three lines theorem for Banach spaces, this estimate also holds inside the strip, and so, if $p\leqq R<r$,and $\lambda=\beta+i\gamma$ with β in $[-R,R]$,

$$\|\phi_{j,\lambda}\|_B \leqq C_7(p,q)(r-R)^{-1}(1+\lfloor j\rfloor)^{-R} e^{6|\gamma|},$$

as required.

The rest of the proof requires us to study the behaviour of $N_j(\beta)N_j(-\beta)$ for β in $(-r,r)$. From (3.14) and (4.8) we have, for β in $(0,r)$,

$$N_j(\beta)N_j(-\beta)$$

$$=2^{\beta}\left|\frac{\Gamma\left(\frac{\beta+r}{2}\right)\Gamma\left(\frac{\beta+p+1}{2}\right)\Gamma\left(\frac{2|j|-\beta+p+1}{2}\right)}{\Gamma\left(\frac{-\beta+r}{2}\right)\Gamma\left(\frac{-\beta+p+1}{2}\right)\Gamma\left(\frac{2\lfloor j\rfloor+\beta+p+1}{2}\right)}\right| \int_{\mathbb{R}^+} |T(v,\beta,\lfloor j\rfloor)|^2 v^{r-\beta-1}\,dv$$

$$\leqq 2^{\beta}\left|\frac{\Gamma\left(\frac{\beta+r}{2}\right)}{\Gamma\left(\frac{-\beta+r}{2}\right)}\right| \sum_{k=1}^{|j|}\left|\frac{2k-\beta+p-1}{2k+\beta+p-1}\right| C_1(p,q)(r-\beta)^{-1},$$

$$\leqq C_{14}(p,q)(1+\lfloor j\rfloor)^{-\beta}$$

from (4.10), whence

$$\|\phi_{j,\beta}\|_B \leqq C_8(p,q)(1+|j|)^{-\beta},$$

as required. Finally, from (3.15), and the preceding equality,

$$\lim_{\beta\to r-} N_j(\beta)N_j(-\beta) = \lim_{\beta\to r-} 2^\beta \left|\frac{\Gamma\left(\frac{\beta+r}{2}\right)}{\Gamma\left(\frac{-\beta+r}{2}\right)}\right| \prod_{k=1}^{|j|} \left|\frac{2k-\beta+p-1}{2k+\beta+p-1}\right| (r-\beta)^{-1}$$

$$= 2^{r-1}\Gamma(r) \prod_{k=1}^{|j|} \left|\frac{2k-q-1}{2k+2p+q-1}\right|,$$

whence

$$\limsup_{\beta\to r-} \|\phi_{j,\beta}\|_B \leqq \frac{2^{1-q}\pi^{1/2}\Gamma(r)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{2p+q+1}{2}\right)} \prod_{k=1}^{|j|} \left|\frac{2k-q-1}{2k+2p+q-1}\right|$$

$$= 2^{1-q}\pi^{1/2}\Gamma(r)\Gamma\left(\frac{q+1}{2}\right) \Big/ \left[\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{2p+q+1+2|j|}{2}\right)\right.$$

$$\left.\Gamma\left(\frac{q+1-2|j|}{2}\right)\right]. \quad \square$$

It should be noted that, since q is an odd integer, the limit is 0 if $|j| \geqq (q+1)/2$.

We may now prove the key estimates for the spherical functions.

Theorem 4.3. *Suppose that G is $SU(1,n)$, $\mathrm{Sp}(1,n)$, or $F_{4(-20)}$. When* $\mathrm{Re}(\lambda)=0$, $\phi_\lambda|_S \in B(S)$, *and* $\|\phi_\lambda|_S\|_B = 1$. *The family $\phi_\lambda|_S$ of $B(S)$-functions continues analytically into the strip $\{\lambda\in\mathbb{C}: \mathrm{Re}(\lambda)\in(-r,r)\}$ and, when $\lambda=\beta+i\gamma$, with β in $(-r,r)$ and γ in $\mathbb{R}$, satisfies*

$$\|\phi_\lambda|_S\|_B \leqq C_{15}(p,q)(r-|\beta|)^{-1}e^{6|\gamma|};$$

further

$$\limsup_{\beta\to r-} \|\phi_\beta|_S\|_B \leqq \begin{cases} 1 & \text{when } G=SU(1,n) \\ 2n-1 & \text{when } G=\mathrm{Sp}(1,n). \\ 21 & \text{when } G=F_{4(-20)} \end{cases}$$

Proof. When $\mathrm{Re}(\lambda)=0$, the principal series representation π_λ of G is unitary, so the spherical function ϕ_λ is in $B(G)$, and a fortiori, $\phi_\lambda|_S$ is in $B(S)$. We shall now Fourier analyse $\phi_\lambda|_S$.

We recall that the dimension $D(p,d)$ of the space of complex homogeneous polynomials on $\mathbb{C}^p$ of degree d is given by

$$D(p,d) = \binom{p+d-1}{d}.$$

It may be helpful to recall the definition of δ_s:

$$(\delta_s f)(v,z) = s^{-r} f(s^{-1/2}v, s^{-1}z) \quad \forall v\in\mathfrak{v}, \forall z\in\mathfrak{z}.$$

By Theorem 2.5(a), and a change of variables,

$$\begin{aligned}\phi_\lambda(a_s n) &= s^{-(\lambda+r)/2} \int_N u_\lambda(n^{-1}a_s^{-1}n'a_s)u_{-\lambda}(n')dn' \\ &= s^{(-\lambda+r)/2} \int_N u_\lambda(n^{-1}n'')u_{-\lambda}(a_s n'' a_s^{-1})dn'' \\ &= s^{-(\lambda+r)/2} \int_N u_\lambda(n^{-1}n'')\delta_{s^{-1}}u_{-\lambda}(n'')dn''.\end{aligned}$$

By the Plancherel formula for N, we deduce that

$$\begin{aligned}\phi_\lambda(a_s n) &= s^{-(\lambda+r)/2}c(p,q) \int_{S_3}\int_{\mathbb{R}^+} \operatorname{tr}(\tau_{v,\omega}(n)\tau_{v,\omega}(u_\lambda)\tau_{v,\omega}(\delta_{s^{-1}}u_{-\lambda}))v^{r-1}dvd\omega \\ &= s^{-(\lambda+r)/2}c(p,q) \int_{S_3}\int_{\mathbb{R}^+} \operatorname{tr}(\tau_{v,\omega}(n)\tau_{v,\omega}(u_\lambda)\tau_{s^{-1}v,\omega}(u_{-\lambda}))v^{r-1}dvd\omega \\ &= s^{-(\lambda-r)/2}c(p,q) \int_{S_3}\int_{\mathbb{R}^+} \operatorname{tr}(\tau_{sv,\omega}(n)\tau_{sv,\omega}(u_\lambda)\tau_{v,\omega}(u_{-\lambda}))v^{r-1}dvd\omega.\end{aligned}$$

With the basis $\{e_{j,1,\omega}: j\in\mathbb{N}_0^p\}$ of (3.3) we have, from Theorem 3.5, that

$$\begin{aligned}&\operatorname{tr}(\tau_{sv,\omega}(n)\tau_{sv,\omega}(u_\lambda)\tau_{v,\omega}(u_{-\lambda})) \\ &\quad= \sum_{j\in\mathbb{N}_0^p} \langle\tau_{sv,\omega}(n)\tau_{sv,\omega}(u_\lambda)\tau_{v,\omega}(u_{-\lambda})e_{j,1,\omega}, e_{j,1,\omega}\rangle \\ &\quad= \sum_{j\in\mathbb{N}_0^p} \langle\tau_{sv,\omega}(n)e_{j,1,\omega}, e_{j,1,\omega}\rangle T(sv,\lambda,|j|)T(v,-\lambda,|j|),\end{aligned}$$

and now it is a matter of chasing through Proposition 4.2 and the preceding definitions to see that

$$\phi_\lambda(a_s n) = \sum_{j\in\mathbb{N}_0^p} \phi_{\lambda,j}(a_s n), \tag{4.11}$$

at least formally; the estimate (4.5) justifies the convergence. Indeed, from above, on one hand,

$$\begin{aligned}\phi_\lambda(a_s n) = \sum_{j\in\mathbb{N}_0^p} & s^{-(\lambda-r)/2}c(p,q) \\ &\int_{S_3}\int_{\mathbb{R}^+} \langle\tau_{\delta v,\omega}(n)e_{j,1,\omega}, e_{j,1,\omega}\rangle T(sv,\lambda,|j|)T(v,-\lambda,|j|)v^{r-1}dvd\omega;\end{aligned}$$

on the other hand, from the definition of $\phi_{j,\lambda}$ (Proposition 4.2), of $\mathfrak{H}_\omega$ (4.1) of τ_ω (4.2), and of $\Xi_{j,\lambda,\omega}$ (4.3)and $H_{j,\lambda,\omega}$ (4.4),

$$\begin{aligned}\phi_{j,\lambda}(a_s n) &= \int_{S_3} \langle\tau_\omega(a_s n)\Xi_{j,\lambda,\omega}, H_{j,\lambda,\omega}\rangle d\omega \\ &= \int_{S_3} c(p,q) \int_{\mathbb{R}^+} \langle(\tau_\omega(a_s n)\Xi_{j,\lambda,\omega})(v), H_{j,\lambda,\omega}(v)\rangle v^{r-1}dvd\omega \\ &= \int_{S_3} c(p,q) \int_{\mathbb{R}^+} s^{r/2}\langle(\tau_{sv,\omega}(n)(\Xi_{j,\lambda,\omega}(sv)), H_{j,\lambda,\omega}(v)\rangle v^{r-1}dvd\omega \\ &= c(p,q)s^{(r-\lambda)/2} \int_{S_3}\int_{\mathbb{R}^+} \langle(\tau_{sv,\omega}(n)e_{j,1,\omega}, e_{j,1,\omega}\rangle \\ &\qquad\qquad T(sv,\lambda,|j|)T(v,-\lambda,|j|)v^{r-1}dvd\omega.\end{aligned}$$

When λ is allowed to vary with $\mathrm{Re}(\lambda)$ in $(-r,r)$, both sides of this expression extend analytically; the estimate (4.5) shows that the sum $\sum_{j\in\mathbb{N}_0^p} \phi_{\lambda,j}$ converges in $B(S)$, as

$$\sum_{j\in\mathbb{N}_0^p} \|\phi_{\lambda,j}\|_B \leqq C_7(p,q) \sum_{j\in\mathbb{N}_0^p} (1+|j|)^{-R}(r-R)^{-1}e^{6|\gamma|},$$

when $\lambda=\beta+i\gamma$, with β in $[-R,R]$, and R in (p,r):

$$\sum_{j\in\mathbb{N}_0^p} (1+|j|)^{-R} = \sum_{d\in\mathbb{N}_0} D(p,d)(1+d)^{-R} < \infty,$$

as $D(p,d)\leqq(p+d-1)^{p-1}$. Then equality (4.11) holds for λ with $\mathrm{Re}(\lambda)$ in $(-r,r)$, and the norm estimate for $\|\phi_\lambda|_S\|_B$ follows.

When λ is real, estimate (4.6) holds, so if $p<b<r$ and $b\leqq\beta\leqq r$,

$$\|\phi_{\beta,j}\|_B \leqq C_8(p,q)(1+|j|)^{-b}.$$

Since

$$\sum_{j\in\mathbb{N}_0^p} (1+|j|)^{-b} = \sum_{d\in\mathbb{N}_0} D(p,d)(1+d)^{-b} < \infty.$$

Lebesgue's dominated convergence theorem implies that

$$\begin{aligned}
\limsup_{\beta\to r-} \|\phi_\beta|_S\|_B &\leqq \lim_{\beta\to r-} \sum_{j\in\mathbb{N}_0^p} \|\phi_{\beta,j}\|_B \\
&= \sum_{j\in\mathbb{N}_0^p} \lim_{\beta\to r-} \|\phi_{\beta,j}\|_B \\
&= \sum_{d\in\mathbb{N}_0} \frac{D(p,d)2^{1-q}\pi^{1/2}\Gamma(r)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{2p+q+1+2d}{2}\right)\Gamma\left(\frac{q+1-2d}{2}\right)} \\
&= \begin{cases} 1 & q=1 \\ 2n-1 & q=3,\ p=2n-2. \\ 21 & q=7,\ p=4 \end{cases} \quad \square
\end{aligned}$$

Remarks: (a) The above sum indexed by $d\in\mathbb{N}_0$ contains only $(q+1)/2$ non-zero terms. For $q=3$, $p=2n-2$ these terms are n (for $d=0$) and $n-1$ (for $d=1$). For $q=7$, $p=4$ the non-zero terms are 6, 9, 5, 1 for $d=0$, 1, 2, 3 respectively.

It is worthwhile to mention, that the sum can be expressed in closed form for arbitrary integers $p,q>0$, namely:

$$\sum_{d\in\mathbb{N}_0} \frac{D(p,d)2^{1-q}\pi^{1/2}\Gamma(p+q)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{2p+q+1+2d}{2}\right)\Gamma\left(\frac{q+1-2d}{2}\right)} = \frac{\pi^{1/2}\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}$$

Indeed, the sum can be rewritten in the form

$$\frac{2^{1-q}\pi^{1/2}\Gamma(p+q)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{2p+q+1}{2}\right)}\sum_{d\in\mathbb{N}_0}(-1)^d\frac{[p]_d\left[\frac{1-q}{2}\right]_d}{d!\left[\frac{2p+q+1}{2}\right]_d}$$
$$=\frac{2^{1-q}\pi^{1/2}\Gamma(p+q)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{2p+q+1}{2}\right)}F\left(p,\frac{1-q}{2};\frac{2p+q+1}{2};-1\right)$$

where $[a]_n=a(a+1)(a+2)\cdot\ldots\cdot(a+n-1)$ and $F(a,b;c;z)$ is the hypergeometric function. Since $\frac{2p+q+1}{2}=1+p-\left(\frac{1-q}{2}\right)$ we can apply [1, Sect. 2.8 (47)]. This yields

$$F\left(p,\frac{1-q}{2};\frac{2p+q+1}{2};-1\right)=\frac{2^{-p}\Gamma\left(\frac{2p+q+1}{2}\right)\pi^{1/2}}{\Gamma\left(\frac{p+q+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}$$

Applying now Legendre's formula for the Γ-function

$$\Gamma(s)=\pi^{1/2}2^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$$

in the case $s=p+q$ we get the stated equality.

(b) It would make the proof of Proposition 4.2 and thus of Theorem 4.3 much easier if we could evaluate $N_j(\lambda)$ (see 4.7)). This boils down to having to calculate

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^+}\frac{x^a}{(1+x)^b}\,\frac{y^{\bar a}}{(1+y)^{\bar b}}\,\frac{1}{(x+y+1)^c}\,dxdy\,.$$

Indeed,

$$N_j(\lambda)=\left\{\int_{\mathbb{R}^+}|v^{-\lambda/2}T(v,\lambda,|j|)|^2v^{r-1}dv\right\}^{1/2},$$

and there is a known meromorphic function, P say, so that

$$v^{-\lambda/2}T(v,\lambda,|j|)=P(\lambda)v^{\lambda/2}L\left(v,\frac{2|j|+\lambda+p+1}{2},\frac{2|j|-\lambda+p+1}{2}\right),$$

where

$$L(c,a,b)=\int_{\mathbb{R}^+}\exp(-c(2x+1))x^{a-1}(x+1)^{-b}dx\,,$$

by Theorem 3.5 and (3.9). Taking $a=(2|j|+\lambda+p+1)/2$, $b=(2|j|-\lambda+p+1)/2$, and $c=r+\mathrm{Re}(\lambda)$, and then performing one integration, we see that

$$N_j(\lambda)=|P(\lambda)|\left\{\int_{\mathbb{R}^+}v^{c-1}\left(\int_{\mathbb{R}^+}\exp(-v(2x+1)x^{a-1}(x+1)^{-b}dx\right).\right.$$

$$\left(\int_{\mathbb{R}^+} \exp(-v(2y+1))y^{\bar{a}-1}(y+1)^{-\bar{b}}dy\right)dv\Bigg\}^{1/2}$$

$$=|P(\lambda)|\left\{\int_{\mathbb{R}^+}\int_{\mathbb{R}^+} x^{a-1}(x+1)^{-b}y^{\bar{a}-1}(y+1)^{-\bar{b}}2^{-c}(x+y+1)^{-c}\Gamma(c)dxdy\right\}^{1/2}.$$

We conclude this section by proving one half of the main theorem.

Corollary 4.4. *If G is either $SU(1,n)$, $\mathrm{Sp}(1,n)$ or $F_{4(-20)}$, then G is weakly amenable, and*

$$\Lambda_G \leqq \begin{cases} 1 & \text{for } SU(1,n) \\ 2n-1 & \text{for } \mathrm{Sp}(1,n). \\ 21 & \text{for } F_{4(-20)} \end{cases}$$

Proof. The calculations of Theorem 3.7 of De Cannière and Haagerup [11] can be easily modified to treat the groups $SU(1,n)$, $\mathrm{Sp}(1,n)$ or $F_{4(-20)}$ rather than $SO_0(1,n)$ (see also [9], Theorem 1.3). We leave the details to the reader. Note that the arguments of [11, p. 484, l.1–9] do not apply to $\mathrm{Sp}(1,n)$ and $F_{4(-20)}$. However by the definition of Λ_G it is sufficient to know, that $\varphi_\sigma \to 1$ uniformly on compacta for $\sigma \to r$ $(0<\sigma<r)$. □

5. The lower bound for Λ_G

We have now established that $\Lambda_G=1$ when $G=SO(1,n)$ or $SU(1,n)$, and that $\Lambda_G \leqq 2n-1$ (respectively 21) when $G=\mathrm{Sp}(1,n)$ $(n \geqq 2)$ (respectively $F_{4(-20)}$. In this section we shall prove that $\Lambda_G \geqq 2n-1$ (respectively 21) for these same groups, and thereby establish the main theorem.

We shall work in the context of H-type groups again. Throughout this section, a denotes $p/2$, which we assume to be a integer. If N is an H-type group, let $\Delta_{\mathfrak{z}}$ be the Laplace operator on N given by the formula

$$\Delta_{\mathfrak{z}} u(v,z) = -\sum_{k=1}^{q} \partial^2/\partial t^2 u(v, z+t\omega_k)|_{t=0} \qquad \forall (v,z) \in N,$$

where $\{\omega_k\}$ is an orthonormal basis of $\mathfrak{z}$. Then it is obvious from (3.2) that, for any v in $\mathbb{R}^+$ and ω in $S_{\mathfrak{z}}$,

$$\sigma_{v,\omega}(\Delta_{\mathfrak{z}})\xi = v^2\xi \qquad \forall u \in \mathfrak{H}_{v,\omega}.$$

Proposition 5.1. *Consider the tempered distribution $\Phi_R * \Delta_{\mathfrak{z}}^a$ on N, where Φ_R is as above (3.5). Then if $R, v \in \mathbb{R}^+$, $\omega \in S_{\mathfrak{z}}$, and ξ is homogeneous of degree d in $\mathfrak{H}_{v,\omega}$,*

$$\sigma_{v,\omega}(\Phi_R * \Delta_{\mathfrak{z}}^a)\xi = (\pi v)^p (R-v)^d (R+v)^{-d-p}\xi.$$

Consequently, if $u \in C_c^\infty(N)$, then

$$\left|\int_{\mathfrak{v}} (\Delta_{\mathfrak{z}}^a u)(v,0)dv\right| \leqq \pi^p \|u\|_B. \tag{5.1}$$

Proof. The Fourier transform formula follows from Proposition 3.3; then

$$\|\sigma_{v,\omega}(\phi_R * \Delta_{\mathfrak{z}}^a)\| \leqq \pi^p,$$

$\| \ \|$ here being the norm of operators on $\mathfrak{H}_{v,\omega}$. Therefore

$$|\langle \phi_R * \Delta_{\mathfrak{z}}^a, u\rangle| \leqq \pi^p \|u\|_B.$$

But $\langle \phi_R * \Delta_{\mathfrak{z}}^a, u\rangle = \langle \phi_R, \Delta_{\mathfrak{z}}^a u\rangle = \int_{\mathfrak{v}} \exp(-R|v|^2)\Delta_{\mathfrak{z}}^a u(v,0)\,dv$; letting R tend to 0 proves (5.1). □

It is worth remarking that these distributions combining differentiation in the $\mathfrak{z}$-variable and integration in the $\mathfrak{v}$-variable have cropped up in the study of boundary value problems associated to pseudo-convex domains, and it was certainly known that such operators can have bounded Fourier transforms (see, e.g. D. Geller and E.M. Stein [13]). However, we have not seen any exact calculations of their transforms published.

Proposition 5.2. *Suppose that f is a function in $C_c^\infty(\mathbb{R})$, and that $u: N \to \mathbb{C}$ is defined by*

$$u(v,z) = f(4|v|^2 + |v|^4 + |z|^2) \qquad \forall (v,z) \in N.$$

Then

$$\int_{\mathfrak{v}} (\Delta_{\mathfrak{z}}^a u)(v,0)\,dv = (-1)^a \frac{2^{p+1}\pi^p \Gamma(r/2)}{\Gamma(p)\Gamma(q/2)} \int_{\mathbb{R}^+} f^{(a)}(4t^2+t^4)t^{2p-1}\,dt.$$

Proof. Write s for $4|v|^2+|v|^4$. By Taylor's theorem, with the integral form of the remainder, for any H in $\mathbb{N}$.

$$u(v,z) = \sum_{h=0}^{H} \frac{f^{(h)}(s)|z|^{2h}}{h!} + \frac{1}{H!}\int_0^{|z|^2} (|z|^2-t)^H f^{(H+1)}(s+t^2)\,dt,$$

so

$$\Delta_{\mathfrak{z}}^a u(v,z) = \sum_{h=0}^{H} \frac{f^{(h)}(s)}{h!} \Delta_{\mathfrak{z}}^a |z|^{2h} + \frac{1}{H!} \Delta_{\mathfrak{z}}^a \int_0^{|z|^2} (|z|^2-t)^H f^{(H+1)}(s+t^2)\,dt.$$

It is easy to check that

$$\Delta_{\mathfrak{z}}|z|^{2h} = -2h(2h+q-2)|z|^{2h-2}$$

whence

$$\Delta_{\mathfrak{z}}^a|z|^{2h} = (-1)^a 2^{2p} \frac{\Gamma(h+1)\Gamma(h+q/2)}{\Gamma(h+1-a)\Gamma(h+q/2-a)} |z|^{2h-2a};$$

moreover, if H is at least p, then

$$\Delta_{\mathfrak{z}}^a \int_0^{|z|^2} (|z|^2-t)^H f^{(H+1)}(s+t^2)\,dt = \int_0^{|z|^2} (|z|^2-t)P(z,t)f^{(H+1)}(s+t^2)\,dt,$$

where P is a polynomial. Consequently,

$$(\Delta_{\mathfrak{z}}^a u)(v,0) = (-1)^a f^{(a)}(s) 2^p \Gamma(r/2)/\Gamma(q/2).$$

Substituting this in the integral over $\mathfrak{v}$ and using polar coordinates finishes the proof. □

Proposition 5.3. *Suppose that* $f_i \in C_c^\infty(\mathbb{R})$, *that* $\|f_i\|_\infty \leqq L$, *and that* $\lim_{i\to\infty} f_i = 1$ *locally uniformly. Then*

$$\lim_{i\to\infty} \int_{\mathbb{R}^+} f_i^{(a)}(4t^2+t^4)t^{2p-1}\,dt = \frac{(-1)^a}{4}\,\Gamma(a).$$

Proof. We change variables, putting $s=4+4t^2+t^4$, and integrate by parts to get

$$\begin{aligned}\int_{\mathbb{R}^+} f_i^{(a)}(4t^2+t^4)t^{4a-1}\,dt &= \frac{1}{4}\int_4^\infty f_i^{(a)}(s-4)(s^{1/2}-2)^{p-1}s^{-1/2}\,ds\\ &= \frac{(-1)^a}{4}\int_4^\infty f_i(s-4)g'(s)\,ds,\end{aligned}$$

where $g(s)=(d/ds)^{a-1}[(s^{1/2}-2)^{p-1}s^{-1/2}]$ – note that there are no boundary terms as f_i has compact support, and as the first $p-2$ derivatives of $s\to(s^{1/2}-2)^{p-1}s^{-1/2}$ vanish when $s=4$.

Clearly $g'\in C^\infty(\mathbb{R}^+)$; moreover

$$\begin{aligned}g'(s) &= (d/ds)^a\left[\sum_{h=0}^{p-1}\binom{p-1}{h}s^{(h-1)/2}(-2)^{p-1-h}\right]\\ &= \sum_{h=0}^{p-1}\binom{p-1}{h}[(h-1)/2]_a s^{(h-p-1)/2}(-2)^{p-1-h},\end{aligned}$$

where $\binom{p-1}{h}$ is the usual binomial coefficient and $[b]_a=b(b-1)\dots(b-a+1)$. When $h=p-1$, $[(h-1)/2]_a=0$, and when $h<p-1$, $s^{(h-p-1)/2}$ vanishes at least as fast as $s^{-3/2}$ at infinity. Consequently,

$$\int_4^\infty |g'(s)|\,ds<\infty\,;$$

we may therefore apply the dominated convergence theorem to deduce that

$$\begin{aligned}\lim_{i\to\infty}\int_{\mathbb{R}^+} f_i^{(a)}(4t^2+t^4)t^{2p-1}\,dt &= \frac{(-1)^a}{4}\int_4^\infty g'(s)\,ds\\ &= \frac{(-1)^a}{4}\left[\lim_{s\to\infty} g(s)-g(4)\right]\\ &= \frac{(-1)^a}{4}\lim_{s\to\infty} g(s).\end{aligned}$$

We conclude by observing that

$$g(s)=\sum_{h=0}^{p-1}\binom{p-1}{h}[(h-1)/2]_{a-1}s^{(h-p+1)/2}(-2)^{p-1-h},$$

and as s tends to $+\infty$, all terms where $h<p-1$ tends to zero, whence

$$\lim_{s\to\infty} g(s)=[a-1]_{a-1}.$$

Theorem 5.4. *Suppose that G is isomorphic to* $\mathrm{Sp}(1,n)(n\geqq 2)$ *or to* $F_{4(-20)}$, *and that* (v_i) *is a net of* $C_c^\infty(G)$*-functions so that* $\|v_i\|_{M_0}\leqq L$ *and* $v_i\to 1$ *uniformly on compacta as* $i\to\infty$. *Then* $L\geqq 2n-1$ *if* $G=\mathrm{Sp}(1,n)$ *and* $L\geqq 21$ *if* $G=F_{4(-20)}$.

Proof. As argued in Sect. 1, the existence of such a net of funtions implies the existence of a net (u_i) of $C_c^\infty(K\backslash G/K)|_N$-functions satisfying $\|u_i\|_B\leqq L$ and tending to 1 uniformly on compacta. By Theorem 2.5(b) and Propositions 5.1, 5.2 and 5.3, and by Legendre's formula for the Γ-function,

$$\limsup_i \|u_i\|_B\geqq 2^{p-1}\frac{\Gamma(r/2)\Gamma(p/2)}{\Gamma(q/2)\Gamma(p)}$$

$$=\frac{\pi^{1/2}\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}$$

$$=\begin{cases}2n-1 & \text{if} \quad p=2n-2, q=3\\ 21 & \text{if} \quad p=4, q=7\end{cases}. \qquad \square$$

6. Applications to von Neumann algebras

Let $\mathfrak{A}$ be a C^*-algebra. Following Haagerup [16], we say that $\mathfrak{A}$ has the completely bounded approximation property if there exists C in $\mathbb{R}^+$ and a net of finite-rank operators, $(T_i : i\in I)$ say, on $\mathfrak{A}$ such that

$$\|T_i\|_{cb}\leqq C \qquad \forall i\in I$$

and

$$\lim_i \|T_i x - x\| = 0 \qquad \forall x\in\mathfrak{A},$$

where $\|\ \|_{cb}$ denotes the completely bounded operator norm. We denote by $\Lambda(\mathfrak{A})$ the infimum of all values of C for which there exist such nets. Similarly, a von Neumann algebra $\mathfrak{M}$ is said to have the weak* completely bounded approximation property if there exists C in $\mathbb{R}^+$ and a net of σ-weakly continuous finite-rank operators, $(T_i : i\in I)$ say, on $\mathfrak{M}$ such that

$$\|T_i\|_{cb}\leqq C \qquad \forall i\in I$$

and (6.1)

$$\lim_i (T_i x, y) = (x, y) \qquad \forall x\in\mathfrak{M},\ \forall y\in\mathfrak{M}_*.$$

and we denote $\Lambda(\mathfrak{M})$ the infimum of all values of C for which such nets exist. It is probably worth pointing out that, in both cases, the infimum is attained, but we shall not need this here.

It is convenient to write $\Lambda(\mathfrak{A})=\infty$ or $\Lambda(\mathfrak{M})=\infty$ to indicate that the C^*-algebra $\mathfrak{A}$ or the von Neumann algebra $\mathfrak{M}$ does not have the corresponding approximation property.

We may now rephrase two results from [16]:

Proposition 6.1 [16]. *Let Γ be a discrete group. Then the following conditions are equivalent:*

(a) *$C_r^*(\Gamma)$ has the completely bounded approximation property;*

(b) *$VN(\Gamma)$ has the weak* completely bounded approximation property;*

(c) *Γ is weakly amenable.*

Moreover, $\Lambda(C_r^*(\Gamma))=\Lambda(VN(\Gamma))=\Lambda_\Gamma$.

Proposition 6.2 [16]. *Let Γ be a lattice in a second countable locally compact group G. Then $\Lambda_\Gamma=\Lambda_G$.*

By virtue of Proposition 1.3, this holds for lattices in arbitrary groups, but we shall not need this. We shall, however, need the following result.

Proposition 6.3. *Let $\mathfrak{M}$ be a von Neumann algebra with a finite faithful trace τ. Then for any von Neumann subalgebra $\mathfrak{N}$ of $\mathfrak{M}$,*

$$\Lambda(\mathfrak{N})\leqq\Lambda(\mathfrak{M}).$$

Proof. We may and shall assume that $\Lambda(\mathfrak{M})<\infty$.

Let $\mathfrak{N}$ be a von Neumann subalgebra of $\mathfrak{M}$. Then there exists a weak* continuous trace-preserving completely positive conditional expectation E of $\mathfrak{M}$ onto $\mathfrak{N}$ (in the sense of H. Umegaki [32]), with the property that $\|E\|_{cb}=\|E(I)\|=1$ (see M. Nakamura, M. Takesaki, and H. Umegaki [27]).

Now if $(T_i : i\in I)$ is a net of σ-weakly continuous finite-rank operators on M such that (6.1) holds, then

$$\|ET_i\|_{cb}\leqq\|T_i\|_{cb}\leqq C$$

and

$$\lim_i (ET_ix,y)=(Ex,y)=(x,y)\qquad \forall x\in N,\ \forall y\in M_*,$$

from which the desired conclusion follows. □

We now come to the first of our applications in the theory of von Neumann algebras.

Theorem 6.4. *Let Γ_1 and Γ_2 be lattices in* $\mathrm{Sp}(1,n_1)$ *and* $\mathrm{Sp}(1,n_2)$, *where $n_1<n_2$. Then $C_r^*(\Gamma_1)$ and $C_r^*(\Gamma_2)$ are not isomorphic as C^*-algebras, and $VN(\Gamma_1)$ and $VN(\Gamma_2)$ are not isomorphic as von Neumann algebras; indeed, $VN(\Gamma_2)$ cannot be embedded in $VN(\Gamma_1)$ as a von Neumann subalgebra.*

Proof. By our main theorem, and Propositions 6.1 and 6.2,

$$\begin{aligned}\Lambda(C_r^*(\Gamma_1))&=\Lambda(VN(\Gamma_1))=\Lambda_{\Gamma_1}=2n_1-1\\ &<2n_2-1=\Lambda_{\Gamma_2}=\Lambda(VN(\Gamma_2))=\Lambda(C_r^*(\Gamma_1)),\end{aligned}$$

so neither the two C^*-algebras nor the two von Neumann algebras can be isomorphic. The rest of the theorem follows from Proposition 6.3. □

It is well known that, if Γ is an infinite discrete group, then $VN(\Gamma)$ is a factor (necessarily of type II_1) it and only if all the conjugacy classes of Γ except $\{e\}$ are infinite (see, for instance, S. Sakai [30], p. 182). We write *ICC* for the class of such groups. The following lemma shows that lattices in $\mathrm{Sp}(1,n)$ are "almost" *ICC*-groups.

Lemma 6.5. *Let G be a connected semisimple Lie group with finite centre, and let Γ be a lattice in G. Then $\{y^{-1}xy : y\in\Gamma\}$ has cardinality 1 if x is in the centre of G and is infinite otherwise.*

Proof. By factoring out the centre of G, we may assume that G is algebraic.

We shall show that x is central in G if its Γ-conjugacy class is finite. Indeed, if the conjugacy class is finite –

$$\{y^{-1}xy : y\in\Gamma\} = \{x_1, x_2, \dots, x_n\} \;-$$

then the set

$$\bigcup_{i=1}^{n} \{g\in G : g^{-1}xg = x_i\}$$

is a Zariski-closed subset of G containg Γ, and hence is all of G, by the Borel density theorem (which states that Γ is Zariski dense in G – see [2]). As G is connected, $g^{-1}xg=x$ for all g in G, as required. □

We shall now construct our examples.

As was proved by A Borel and Harish-Chandra [3], every arithmetic subgroup of $\mathrm{Sp}(1,n)$ $(n\in\mathbb{N})$ is a lattice. In particular, we denote by $\mathbb{H}_{\mathrm{int}}$ the quaternionic integers $\mathbb{Z}+\mathbb{Z}i+\mathbb{Z}j+\mathbb{Z}k$; then the subgroup Γ_n of $\mathrm{Sp}(1,n)$ consisting of $(n+1)\times(n+1)$ matrices with entries in $\mathbb{H}_{\mathrm{int}}$ which preserve the bilinear form Q –

$$Q(x,y) = \bar{y}_0 x_0 - \sum_{m=1}^{n} \bar{y}_m x_m .$$

where $x=(x_0, x_1, \dots, x_n)$ and $y=(y_0, y_1, \dots, y_n)$ lie in $\mathbb{H}^{n+1}$ – is a lattice in $\mathrm{Sp}(1,n)$. The centre of $\mathrm{Sp}(1,n)$ consists of two elements $(\pm I)$.

Corollary 6.6. *Let $\Gamma_n^0 = \Gamma_n\backslash(\pm I)$, $n\geqq 2$. Then $\mathfrak{M}_n = VN(\Gamma_n^0)$ is a II_1-factor and $\Lambda(\mathfrak{M}_n) = 2n-1$.*

Proof. Lemma 6.5 applied to $G=\mathrm{Sp}(1,n)/(\pm I)$ shows that Γ_n^0 is *ICC*, so $\mathfrak{M}_n$ is a II_1-factor. Moreover $\Lambda(\mathfrak{M}_n)=2n-1$, because $\Lambda_{\Gamma_n}=\Lambda_{\Gamma_n^0}$ by Proposition 1.3(c). □

Now we write $\mathfrak{M}$ for the II_1-factor $VN(\Gamma_2^0)$, and $\mathfrak{M}\hat{\otimes}^n$ for the n-fold spatial tensor product $\mathfrak{M}\hat{\otimes}\mathfrak{M}\hat{\otimes}\dots\hat{\otimes}\mathfrak{M}$, which is isomorphic to $VN(\Gamma_{(n)})$, the von Neumann algebra of the n-fold product $\Gamma_2^0\times\Gamma_2^0\times\dots\Gamma_2^0$.

Corollary 6.7. *If $\mathfrak{M}\hat{\otimes}^n$ is as just defined, then $\Lambda(\mathfrak{M}\hat{\otimes}^n)=3^n$, and $\mathfrak{M}$, $\mathfrak{M}\hat{\otimes}\mathfrak{M}$, $\mathfrak{M}\hat{\otimes}\mathfrak{M}\hat{\otimes}\mathfrak{M}, \dots$, are all non-isomorphic II_1-factors.*

Proof. By Proposition 6.2 and Corollary 1.5,

$$\Lambda(\mathfrak{M}\hat{\otimes}^n) = \Lambda_{\Gamma_{(n)}} = 3^n . \quad \square$$

Remarks. (a) A. Connes found in 1975 another example of a II_1-factor, such that its tensor powers are all non-isomorphic (cf. [6], Corollaire 5).

(b) B. Kostant [24] showed that, if $n \geqq 2$, then $\mathrm{Sp}(1, n)$ has D. A. Kazhdan's [22] "property T"; it follows that Γ_n and Γ_n^0 also have this property. The von Neumann algebras $\mathfrak{M}_n$ $(n \geqq 2)$ of Corollary 6.6 and $\mathfrak{M}\bar{\otimes}^n$ $(n \geqq 1)$ of Corollary 6.7 therefore have Property T, in the sense of A. Connes and V. F. R. Jones [7].

(c) We do not know whether $\Lambda(\mathfrak{M}\hat{\otimes}\mathfrak{N}) = \Lambda(\mathfrak{M})\Lambda(\mathfrak{N})$ for all von Neumann algebras $\mathfrak{M}$ and $\mathfrak{N}$. By Corollary 1.5 the formula holds when M and N are von Neumann algebras associated with two discrete groups.

(d) By Prasad's extension of Mostow's rigidity theorem [29], lattices in $\mathrm{Sp}(1, 11)$ and $F_{4(-20)}$ cannot be isomorphic. We do not know if their von Neumann algebras are non-isomorphic. Similar comments apply about lattices in any two semisimple Lie groups G_1 and G_2, where $\Lambda_{G_1} = \Lambda_{G_2}$. See, for instance, R. J. Zimmer [33] for more information on rigidity.

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