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Completely passive natural convection

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We show that a unique, nontrivial, natural convection state exists under the Boussinesq approximation and completely passive boundary conditions.

1 Introduction

Natural convection is a basic process which is important in a wide variety of practical applications [1,2]. In essence, a heated fluid expands and rises from buoyancy due to decreased density. Numerous papers have been written on natural or mixed convection in vertical ducts heated on the side. The effects of added viscous dissipation heat have been considered [3–8]. The fluid motions in all previous works are driven by either an applied pressure gradient or by applied heating of the walls. We consider a long duct, with no pressure gradient, and with the temperature of the walls the same as the ambient reference temperature. In such a completely passive environment there is no energy input, no temperature difference, and no motion. But is there a possibility of nontrivial fluid motion?

The purpose of the present Note is to investigate whether a completely passive vertical duct (no applied pressure gradient or applied boundary heat) can support sustained fluid motion. The idea is simple. Fluid motion causes dissipative heat, heat causes buoyancy, and buoyancy causes fluid motion.

2 Parallel plates

Fig. 1 shows vertical parallel plates with gap width $2L$. In the fully developed state the momentum and energy equations, under the well-accepted Boussinesq approximation, are [9] (p. 72 and p. 323)

$$\mu \frac{d^2 w'}{dy'^2} + \rho g \beta (T - T_a) = 0, \tag{1}$$

$$k \frac{d^2 T}{dy'^2} + \mu \left(\frac{dw'}{dy'} \right)^2 = 0. \tag{2}$$

Here μ is the viscosity of the fluid, ρ is the density, g is the gravitational acceleration, β is the coefficient of thermal expansion, k is the thermal diffusivity, T is the temperature, and T_a is the same ambient temperature on the walls. Normalize the lateral coordinate y', which is placed at the symmetry axis, by L, the axial velocity w' by $k/(\rho g \beta L^2)$ and drop the primes. A normalized temperature θ is defined by

$$\theta = \frac{T - T_a}{\mu k/(\rho g \beta L^2)^2}. \tag{3}$$

Then Eqs. (1), (2) give

$$\theta + \frac{d^2 w}{dy^2} = 0, \tag{4}$$

$$\frac{d^2 \theta}{dy^2} + \left(\frac{dw}{dy} \right)^2 = 0. \tag{5}$$

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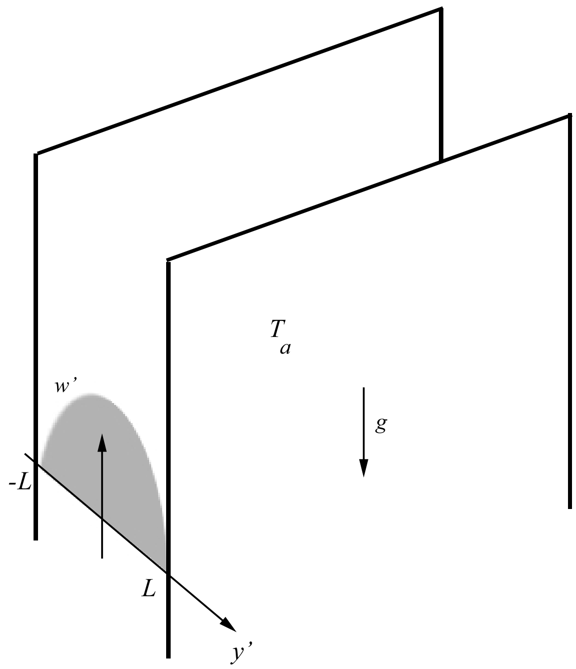

Fig. 1 Flow between parallel plates.

Eliminating θ, we obtain the nonlinear equation

$$\frac{d^4w}{dy^4} = \left(\frac{dw}{dy}\right)^2. \tag{6}$$

w and θ are zero on the wall, hence

$$w(1) = 0, \quad \frac{d^2w}{dy^2}(1) = 0. \tag{7}$$

Symmetry implies

$$\frac{dw}{dy}(0) = 0, \quad \frac{d^3w}{dy^3}(0) = 0. \tag{8}$$

Of course, one solution to Eqs. (6)–(8) is the trivial solution $w = 0$, or no motion. We will now show that Eqs. (6)–(8) have also a unique nontrivial analytic solution!

An analytic solution of (6) must be in the form

$$w(y) = \sum_{n=0}^{\infty} a_n y^n \tag{9}$$

and a_n have to satisfy the recurrence relation

$$(n+4)(n+3)(n+2)(n+1)a_{n+4} = \sum_{k=0}^{n}(k+1)(n-k+1)a_{k+1}a_{n-k+1} \quad \text{for } n \geq 0. \tag{10}$$

Eq. (8) implies $a_1 = a_3 = 0$ and hence (10) implies that if $n > 0$ and $a_n \neq 0$ then $n = 4k+2$ for some $k \geq 0$. If $a_2 = 0$ then (10) implies that all $a_n = 0$ for $n > 0$ which forces $w = 0$. Hence assume $\lambda = 2a_2 \neq 0$ and define

$$b_k = (4k+2)\lambda^{-k-1}a_{4k+2} \quad \text{for } k \geq 0. \tag{11}$$

Thus (9) becomes

$$w(y) = a_0 + \sum_{k=0}^{\infty} \frac{b_k\lambda^{k+1}}{4k+2}\, y^{4k+2}, \tag{12}$$

where

$$4n(16n^2-1)b_n=\sum_{k=0}^{n-1}b_kb_{n-1-k}\quad \text{for } n\geq 1;\, b_0=1. \tag{13}$$

In order to satisfy (7) we need to have

$$a_0=-\sum_{k=0}^{\infty}\frac{b_k\lambda^{k+1}}{4k+2},\quad \text{and}\quad F(\lambda)=0 \tag{14}$$

where

$$F(\lambda)=\sum_{k=0}^{\infty}\lambda^k(4k+1)b_k. \tag{15}$$

Using (13) it is easy to find a zero of F to be

$$\lambda=-13.15510836\ldots \tag{16}$$

and hence (14) implies

$$w(0)=a_0=6.111400215\ldots,\qquad w''(0)=2a_2=\lambda=-13.15510836\ldots. \tag{17}$$

This establishes existence of the nontrivial solution of Eqs. (6)–(8). The total flow rate per width, normalized by $k/(\rho g\beta L)$, is

$$q=\int_{-1}^{1}w(y)dy=7.972440121\ldots. \tag{18}$$

To prove uniqueness of the analytic solution, we need to show that F given by Eq. (15) has only one real zero. To do this it is sufficient to show that $F'(\lambda)>0$ for all λ. Eq. (13) implies $b_n>0$ for all $n\geq 0$, hence we need to show $F'(\lambda)>0$ for $\lambda<0$ only.

Let $h(t)=16t^{3/4}F'(-t)$ for $t>0$. Note that Eq. (15) implies

$$h(t)=-16\sum_{n=1}^{\infty}(-1)^n n t^{n-1/4}(4n+1)b_n \tag{19}$$

and hence

$$h'(t)=t^{-1/4}\sum_{n=1}^{\infty}(-t)^{n-1}4n(16n^2-1)b_n. \tag{20}$$

Using (13) we obtain

$$h'(t)=t^{-1/4}\sum_{n=1}^{\infty}(-t)^{n-1}\sum_{k=0}^{n-1}b_kb_{n-1-k} \tag{21}$$

– which can be rewritten as

$$h'(t)=t^{-1/4}\left(\sum_{n=0}^{\infty}(-t)^n b_n\right)^2. \tag{22}$$

Therefore

$$16t^{3/4}F'(-t)=h(t)=\int_0^t s^{-1/4}\left(\sum_{n=0}^{\infty}(-s)^n b_n\right)^2 ds>0 \tag{23}$$

– which proves $F'(\lambda)>0$ for $\lambda<0$ and hence the uniqueness of the nontrivial solution.

3 Circular duct

Consider a circular duct, Fig. 2. A similar normalization gives

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)^2 W=\left(\frac{dW}{dr}\right)^2. \tag{24}$$

The corresponding boundary conditions are

$$\frac{dW}{dr}(0)=0, \quad \frac{d}{dr}\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)W(0)=0, \tag{25}$$

$$W(1)=0, \quad \left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)W(1)=0. \tag{26}$$

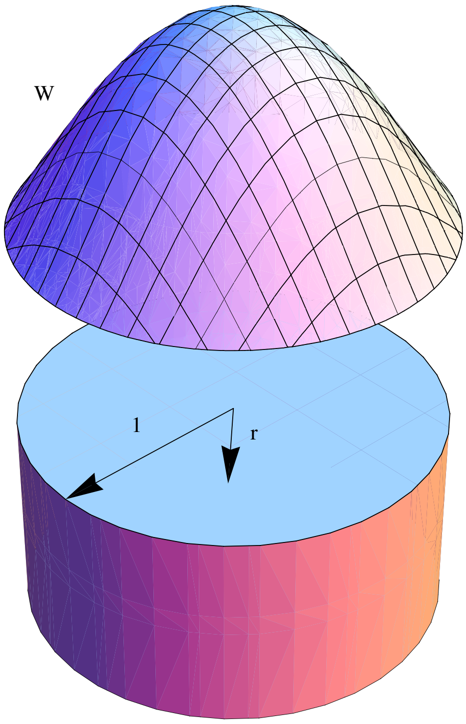

Fig. 2 (online colour at: www.zamm-journal.org) Flow in a circular duct.

Of course, one solution to Eqs. (24)–(26) is the trivial solution $W=0$, or no motion. We will now show, just like in the previous section, that Eqs. (24)–(26) have also a unique nontrivial analytic solution.

An analytic solution of (24) has to be in the form

$$W(r)=\sum_{n=0}^{\infty}A_n r^n \tag{27}$$

and A_n have to satisfy the recurrence relation

$$(n+2)^2(n+4)^2 A_{n+4}=\sum_{k=0}^{n}(k+1)(n-k+1)A_{k+1}A_{n-k+1} \qquad \text{for } n\geq 0. \tag{28}$$

(25) implies $A_1=A_3=0$ and hence (28) implies that if $n>0$ and $A_n\neq 0$ then $n=4k+2$ for some $k\geq 0$. If $A_2=0$ then (28) implies that all $A_n=0$ for $n>0$ which forces $W=0$. Hence assume $\lambda=2A_2\neq 0$ and define B_k as in (11). We have again

$$W(r)=A_0+\sum_{k=0}^{\infty}\frac{B_k\lambda^{k+1}}{4k+2}\,r^{4k+2}, \tag{29}$$

where

$$32n^2(2n+1)B_n = \sum_{k=0}^{n-1} B_k B_{n-1-k} \quad \text{for } n \geq 1;\, B_0 = 1. \tag{30}$$

In order to satisfy (26) we need to have

$$A_0 = -\sum_{k=0}^{\infty} \frac{B_k \lambda^{k+1}}{4k+2} \quad \text{and} \quad G(\lambda) = 0, \tag{31}$$

where

$$G(\lambda) = \sum_{k=0}^{\infty} \lambda^k (2k+1) B_k. \tag{32}$$

Using (30) it is easy to find a zero of G to be

$$\lambda = -38.8358638\ldots \tag{33}$$

which shows existence of the solution for Eqs. (24)–(26) with

$$W(0) = A_0 = 16.9770144\ldots, \qquad W''(0) = 2A_2 = \lambda = -38.8358638\ldots\,. \tag{34}$$

The total flow rate per width, normalized by $k/(\rho g \beta)$, is

$$Q = 2\pi \int_0^1 w(r) r \, dr = 24.79516051\ldots\,. \tag{35}$$

W in Fig. 2 is actually an accurate representation of W.

In order to show uniqueness it is enough to show that $G'(\lambda) > 0$ for all λ. Let $h(\lambda) = 32\lambda G'(\lambda)$ and note that

$$h'(\lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} 32n^2(2n+1)B_n = \sum_{n=1}^{\infty} \lambda^{n-1} \sum_{k=0}^{n-1} B_k B_{n-1-k} = \left(\sum_{n=0}^{\infty} \lambda^n B_n \right)^2 \tag{36}$$

hence

$$32\lambda G'(\lambda) = h(\lambda) = \int_0^{\lambda} \left(\sum_{n=0}^{\infty} s^n B_n \right)^2 ds \tag{37}$$

proving $G'(\lambda) > 0$ for all λ and hence uniqueness.

4 Results and discussion

Fig. 3 shows the unique velocity profile for completely passive flow between parallel plates and in the circular duct. We mention that in order to attain such a state, one must start the flow somehow, either with an initial heat input or with a primer pump.

Our proposed passive pump is related to the thermo-siphon, where heat is utilized for natural convection. The major difference is that the thermo-siphon requires external heat input while our pump is completely passive.

Does the completely passive pump violate the second law of thermodynamics? The answer is no. The system is not closed since fluid enters from the bottom and exits at the top with an irreversible thermal expansion. Granted that the theory may have defects, such as entrance effects and non exactness of the Boussinesq approximation etc, our analysis shows such a perpetual passive state exists, thus can be maintained with little effort even with the defects.

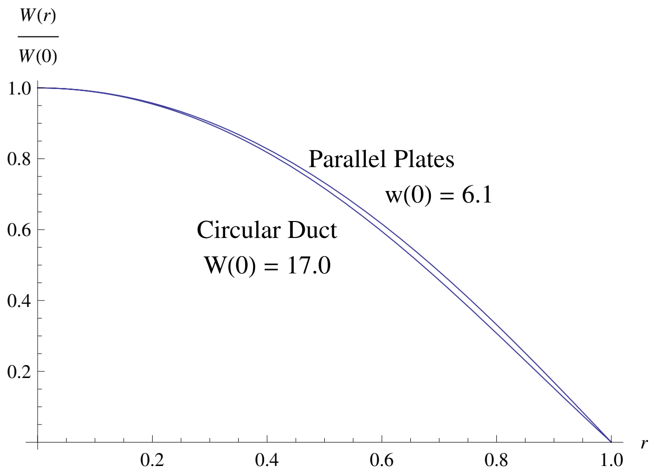

Fig. 3 Comparison of flow in circular duct and parallel plates.

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