J. Math. Soc. Japan
Vol. 33, No. 3, 1981

Completely positive maps in the tensor products of von Neumann algebras

By Masaru NAGISA and Jun TOMIYAMA

(Received Dec. 5, 1979)

Let M_i and N_i be von Neumann algebras and τ_i be completely positive maps from M_i to N_i ($i=1, 2$). Then there exists a completely positive map $\tau_1\otimes\tau_2$ (called the product map of τ_1 and τ_2) from the spatial C^*-tensor product $M_1\otimes_\alpha M_2$ to the spatial C^*-tensor product $N_1\otimes_\alpha N_2$ such that $\tau_1\otimes\tau_2(a\otimes b)=\tau_1(a)\otimes\tau_2(b)$. Moreover, when N_1 and N_2 are von Neumann subalgebras of M_1 and M_2 and both τ_1 and τ_2 are projections of norm one to N_1 and N_2 it is known ([**6**], [**9**]) that the map $\tau_1\otimes\tau_2$ can further be extended, without the normality of τ_1 and τ_2, to the von Neumann tensor product $M_1\bar{\otimes}M_2$ so that the resulting extension τ becomes a projection of norm one of $M_1\bar{\otimes}M_2$ to the von Neumann subalgebra $N_1\bar{\otimes}N_2$. This result is used to show a basic fact for injective von Neumann algebras; if M and N are injective von Neumann algebras, then $M\bar{\otimes}N$ is injective ([**6**], [**9**]).

In Section 2 we shall show that the completely positive extension of $\tau_1\otimes\tau_2$ to the algebra $M_1\bar{\otimes}M_2$ always exists for a couple of completely positive maps (τ_1, τ_2). In the proof, regarding the positive elements of $M_1\bar{\otimes}M_2$ as completely positive maps from M_{1*} to M_2, we construct it.

Next in Section 3, we determine the normal part (σ-weakly continuous part) of the above extention in terms of the normal part of τ_1 and τ_2. The corresponding result for projections of norm one ([**11**; Theorem 3.1]) is used to classify certain types of maximal abelian subalgebras.

1. Preliminaries.

Let M and N be von Neumann algebras. We denote by M^* and M_*, the dual of M and the predual of M, and by $M\otimes N$ and $M\bar{\otimes}N$, the algebraic tensor product of M and N and the von Neumann tensor product of M and N respectively. For each $\phi\in M_*$ (resp. $\psi\in N_*$) we can define a σ-weakly continuous linear map R_ϕ of $M\bar{\otimes}N$ to N (resp. L_ψ of $M\bar{\otimes}N$ to M) which we call the right slice map (resp. left slice map) such that

$$R_\phi(a\otimes b)=\langle a, \phi\rangle b$$

$$(\text{resp. } L_\psi(a\otimes b)=\langle b, \psi\rangle a)\,.$$

These two kinds of maps are related in the following way,

$$\langle x, \phi\otimes\psi\rangle=\langle R_{\phi}(x), \psi\rangle=\langle L_{\psi}(x), \phi\rangle .$$

As we shall often use this relation we call this the Fubini type principle of slice maps. Using the right slice map (resp. left slice map), for an element x of $M\overline{\otimes}N$ we define a linear map $r(x): M_{*}\rightarrow N$ (resp. $l(x): N_{*}\rightarrow M$) such that

$$r(x)(\phi)=R_{\phi}(x) \qquad (\text{resp. } l(x)(\psi)=L_{\psi}(x)) .$$

Furthermore for each $\phi\in M^{*}$, we can define a bounded linear map R_{ϕ} of $M\overline{\otimes}N$ to N:

$$\langle R_{\phi}(x), \psi\rangle=\langle L_{\psi}(x), \phi\rangle \qquad \text{for all } \psi\in N_{*} ,$$

which we call the generalized right slice map. Similarly we can define the generalized left slice map for each $\psi\in N^{*}$. Using them, for each $x\in M\overline{\otimes}N$ we can also define a linear map $r_{g}(x): M^{*}\rightarrow N$ (resp. $l_{g}(x): N^{*}\rightarrow M$) such that

$$r_{g}(x)(\phi)=R_{\phi}(x) \qquad (\text{resp. } l_{g}(x)(\psi)=L_{\psi}(x)) .$$

We note that $r_{g}(x)$ (resp. $l_{g}(x)$) is an extension of $r(x)$ (resp. $l(x)$), which is continuous for the weak* and σ-weak topologies in M^{*} and N.

LEMMA 1. *The transposed map* ${}^{t}(r(x))$ *of* $r(x)$ *is equal to* $l_{g}(x)$. *Similarly* ${}^{t}(l(x))$ *is equal to* $r_{g}(x)$.

PROOF. For $\phi\in M_{*}$ and $\psi\in N^{*}$, we have

$$\begin{aligned}\langle l_{g}(x)(\psi), \phi\rangle&=\langle L_{\psi}(x), \phi\rangle=\langle R_{\phi}(x), \psi\rangle\\&=\langle r(x)(\phi), \psi\rangle=\langle {}^{t}(r(x))(\psi), \phi\rangle ,\end{aligned}$$

hence $l_{g}(x)={}^{t}(r(x))$.

Let E and F be either C^{*}-algebras or closed subspaces of duals of C^{*}-algebras. We denote by $M_{n}(E)$ the $n\times n$ matrix space over E. A map τ of E to F is said to be completely positive if the map $\tau_{n}; [a_{i,j}]\in M_{n}(E)\rightarrow[\tau(a_{i,j})]\in M_{n}(F)$ is positive for every positive integer n. Let $L(M_{*}, N)$ (resp. $L(M^{*}, N)$) be the space of all linear maps of M_{*} to N (resp. M^{*} to N) in which we consider the order induced by the cone $L(M_{*}, N)^{\oplus}$ (resp. $L(M^{*}, N)^{\oplus}$) of all completely positive maps. We can regard that r (resp. r_{g}) is a linear map from $M\overline{\otimes}N$ into $L(M_{*}, N)$ (resp. $L(M^{*}, N)$). The following characterization of the von Neumann tensor product $M\overline{\otimes}N$ is proved by E. G. Effros in [4]. We includes here the proof for readers' convenience.

THEOREM 2. *Set*

$$(M\overline{\otimes}N)_{1}^{+}=\{x\in M\overline{\otimes}N: 0\leqq x\leqq 1\otimes 1\} ,$$

then the map r *is an order isomorphism between the set* $(M\overline{\otimes}N)_{1}^{+}$ *and the set of completely positive maps* $\{\tau\in L(M_{*}, N): 0\leqq\tau\leqq r(1\otimes 1)\}$. *Similar result also holds for the map* l.

Moreover, the map r_g and l_g are order isomorphisms from $(M\overline{\otimes}N)_1^+$ into the sets $\{\tau\in L(M^*, N): 0\leqq\tau\leqq r_g(1\otimes 1)\}$ and $\{\rho\in L(N^*, M): 0\leqq\rho\leqq l_g(1\otimes 1)\}$ respectively.

PROOF. Assume that the von Neumann algebras M and N act standardly on the space H and K (cf. [5]), then one easily verifies that the algebras $M_n(M)$ and $M_n(N)$ act also standardly on the n^2-fold copy spaces H^{n^2} and K^{n^2} of H and K. Let ϕ and ψ be normal states of $M_n(M)$ and $M_n(N)$, then by [**5**; Lemma 2.10] they are vector states i.e. $\phi=\omega(\xi)$, $\psi=\omega(\eta)$ for unit vectors $\xi=(\xi_{i,j})\in H^{n^2}$ and $\eta=(\eta_{i,j})\in K^{n^2}$. Write $\phi=[\phi_{i,j}]$ and $\psi=[\psi_{i,j}]$, then $\phi_{i,j}=\sum_s \omega(\xi_{j,s}, \xi_{i,s})$ and $\psi_{i,j}=\sum_s \omega(\eta_{j,s}, \eta_{i,s})$ $(i, j, s=1, \cdots, n)$. For every element x in $M\overline{\otimes}N$ we have

$$\langle r(x)_n(\phi), \psi\rangle=\langle[r(x)(\phi_{i,j})], [\psi_{i,j}]\rangle=\sum_{i,j}\langle R_{\phi_{i,j}}(x), \psi_{i,j}\rangle$$

$$=\sum_{i,j}\langle x, \phi_{i,j}\otimes\psi_{i,j}\rangle=\sum_{i,j}\sum_{s,t}(x(\xi_{j,t}\otimes\eta_{j,s})|\xi_{i,t}\otimes\eta_{i,s})$$

$$=\sum_{s,t}\Big(x\Big(\sum_i\xi_{i,t}\otimes\eta_{i,s}\Big)\Big|\sum_i\xi_{i,t}\otimes\eta_{i,s}\Big).$$

Hence, $x\geqq 0$ if and only if $r(x)_n$ is a positive map for every positive integer n, that is, $r(x)\geqq 0$ in $L(M_*, N)$. It follows that $0\leqq x\leqq 1\otimes 1$ is equivalent to $0\leqq r(x)\leqq r(1\otimes 1)$ in $L(M_*, N)$.

We assert next that the map r is an onto map. Let τ be a map in $L(M_*, N)$ such that $0\leqq\tau\leqq r(1\otimes 1)$ in $L(M_*,N)$. Then the product form $[\ |\ ]_\tau$ on the algebraic tensor product $H\otimes K$

$$\Big[\sum_{i=1}^n\xi_i\otimes\eta_i\Big|\sum_{i=1}^n\xi_i'\otimes\eta_i'\Big]_\tau=\sum_{i,j}\langle\tau(\omega(\xi_i, \xi_j')), \omega(\eta_i, \eta_j')\rangle$$

$$=\langle\tau_n([\omega(\xi_i, \xi_j')]), [\omega(\eta_i, \eta_j')]\rangle$$

is well-defined and positive definite by the assumption for τ. Moreover, we get the inequality;

$$0\leqq\Big[\sum_{i=1}^n\xi_i\otimes\eta_i\Big|\sum_{i=1}^n\xi_i\otimes\eta_i\Big]_\tau$$

$$=\langle\tau_n([\omega(\xi_i, \xi_j)]), [\omega(\eta_i, \eta_j)]\rangle$$

$$\leqq\langle r(1\otimes 1)_n([\omega(\xi_i, \xi_j)]), [\omega(\eta_i, \eta_j)]\rangle$$

$$=\sum_{i,j}\langle R_{\omega(\xi_i,\xi_j)}(1\otimes 1), \omega(\eta_i, \eta_j)\rangle$$

$$=\sum_{i,j}(\xi_i|\xi_j)(\eta_i|\eta_j)$$

$$=\left\| \sum_{i=1}^{n} \xi_i \otimes \eta_i \right\|^2 .$$

Then the form $[\ |\ ]_\tau$ can be extended to the completion $H \otimes_\sigma K$ of $H \otimes K$ and there exists an operator $a \in B(H \otimes_\sigma K)$ such that $0 \leqq a \leqq 1$ and $[\zeta | \kappa]_\tau = (a\zeta | \kappa)$ for every $\zeta, \kappa \in H \otimes_\sigma K$. It follows that we have,

$$\begin{aligned} (\tau(\omega(\xi, \xi') | M)\eta | \eta') &= [\xi \otimes \eta | \xi' \otimes \eta']_\tau \\ &= (a(\xi \otimes \eta) | \xi' \otimes \eta') \\ &= (R_{\omega(\xi, \xi')}(a)\eta | \eta') \end{aligned}$$

where $R_{\omega(\xi, \xi')}$ is the right slice map in $B(H) \overline{\otimes} B(K)$ with respect to the vector functional $\omega(\xi, \xi')$ on $B(H)$. Hence, $R_{\omega(\xi, \xi')}(a) = \tau(\omega(\xi, \xi') | M) \in N$.

Next let ϕ be a σ-weakly continuous functional on M and let $\phi = |\phi| v$ be the polar decomposition of ϕ (cf. [**3**; Chap. 1 § 4.7]). There is a vector ξ in H such that $|\phi| = \omega(\xi) | M$, then $\phi = \omega(\xi, v^*\xi) | M$ and

$$\|\phi\| = \| |\phi| \| = \|\xi\|^2 = \|\xi\| \|v^*\xi\| .$$

Therefore,

$$\begin{aligned} \|\tau(\phi)\| &= \|\tau(\omega(\xi, v^*\xi) | M)\| \\ &= \|R_{\omega(\xi, v^*\xi)}(a)\| \\ &\leqq \|\xi\| \|v^*\xi\| \\ &\leqq \|\phi\| \end{aligned}$$

i. e. $\|\tau\| \leqq 1$.

Thus, τ is norm continuous and we see that $\tau(\phi | M) = R_\phi(a) \in N$ for every $\phi \in B(H)_*$. Moreover, if we consider the transposed map ${}^t\tau$ from N^* to M, we have, for every vector state $\omega(\eta)$ on $B(K)$,

$$\begin{aligned} (L_{\omega(\eta)}(a)\xi | \xi) &= (a(\xi \otimes \eta) | \xi \otimes \eta) \\ &= \langle \tau(\omega(\xi) | M), \omega(\eta) | N \rangle = \langle {}^t\tau(\omega(\eta) | N), \omega(\xi) | M \rangle \\ &= ({}^t\tau(\omega(\eta) | N)\xi | \xi) \end{aligned}$$

where we distinguish the state $\omega(\eta)$ from its restriction $\omega(\eta) | N$ to the von Neumann algebra N. Thus, $L_{\omega(\eta)}(a) = {}^t\tau(\omega(\eta) | N)$ and, by the norm continuity of both members, $L_\psi(a) = {}^t\tau(\psi | N) \in M$ for every $\psi \in B(K)_*$. It follows from [**10**; Theorem 2.1] that the operator a belongs to $M \overline{\otimes} N$ and apparently we have $r(a) = \tau$. This completes the proof.

For the assertions for the maps r_g and l_g, they are immediate consequences of Lemma 1.1 and the fact that the transposed map of a completely positive map is also completely positive.

It is noticed that the image of $(M\bar{\otimes}N)_1^+$ by the map r_g is the set of completely positive map τ's which are continuous for the weak* and σ-weak topology and satisfy $0\leqq\tau\leqq r_g(1\otimes 1)$. Hence, r_g may not be an onto map to the set $\{\tau: M^*\to N: 0\leqq\tau\leqq r_g(1\otimes 1)\}$ in general.

2. Product maps of completely positive maps in the tensor product of von Neumann algebras.

In this section we prove the existence theorem of product completely positive maps by using Theorem 2.

Let M_i and N_i $(i=1, 2)$ be von Neumann algebras and $\tau_i: M_i\to N_i$ be completely positive maps. In order to avoid rather complicated notations such as $r_{M_1\bar{\otimes}M_2}$, $r_{M_1\bar{\otimes}N_2}$ etc., we use, in the following arguments, the same notation of the map r (also l, r_g and l_g) to mean the canonical representation of the tensor product algebras in Theorem 2. We also employ the same way to denote identities of various von Neumann algebras, i.e. just writing as l in place of l_{M_1}, l_{N_1} etc.

THEOREM 3. *In the above setting there exists a completely positive map τ from $M_1\bar{\otimes}M_2$ to $N_1\bar{\otimes}N_2$ such that*

$$\tau(a\otimes b)=\tau_1(a)\otimes\tau_2(b)$$

for every $a\in M_1$ and $b\in M_2$.

PROOF. We assume first that τ_1 and τ_2 are unital. We shall show that there exists a completely positive map ρ_2 from $M_1\bar{\otimes}M_2$ to $M_1\bar{\otimes}N_2$ such that

$$\rho_2(a\otimes b)=a\otimes\tau_2(b)\,.$$

Take an element x of $(M_1\bar{\otimes}M_2)_1^+$, then by Theorem 2, $\tau_2\circ r(x)$ is a completely positive map of M_{1*} to N_2 and we have

$$0\leqq\tau_2\circ r(x)\leqq\tau_2\circ r(1\otimes 1)=r(1\otimes 1) \qquad (\text{in } M_1\bar{\otimes}N_2)$$

in the order of completely positive maps. Hence by Theorem 2, there exists a unique element $\rho_2(x)$ of $(M_1\bar{\otimes}N_2)_1^+$ such that $r(\rho_2(x))=\tau_2\circ r(x)$. Let x be an element of $M_1\bar{\otimes}M_2$ and write $x=\sum_{j=1}^{4} i^j x_j$ where $x_j\in(M_1\bar{\otimes}M_2)^+$ and $i=\sqrt{-1}$. We put

$$\rho_2(x)=\sum_{j=1}^{4} i^j\rho_2(x_j)\,.$$

If x is written in another form $\sum_{j=1}^{4} i^j x_j'$ $(x_j'\in(M_1\bar{\otimes}M_2)^+)$, we have

$$r\Big(\sum_{j=1}^{4} i^j\rho_2(x_j)\Big)=\sum_{j=1}^{4} i^j r(\rho_2(x_j))=\sum_{j=1}^{4} i^j\tau_2\circ r(x_j)$$

$$=\tau_2\circ r\Big(\sum_{j=1}^{4} i^j x_j\Big)=\tau_2\circ r\Big(\sum_{j=1}^{4} i^j x_j'\Big)$$

$$=\sum_{j=1}^{4} i^j \tau_2\circ r(x_j')=\sum_{j=1}^{4} i^j r(\rho_2(x_j'))$$

$$=r\Big(\sum_{j=1}^{4} i^j \rho_2(x_j')\Big).$$

Hence, $\sum_{j=1}^{4} i^j\rho_2(x_j)=\sum_{j=1}^{4} i^j\rho_2(x_j')$ and the above map ρ_2 is well defined. The map ρ_2 is clearly linear and since we have the relation $r(\rho_2(x))=\tau_2\circ r(x)$ for $x\in M_1\overline{\otimes}M_2$, one may easily verify that

$$\rho_2(a\otimes b)=a\otimes\tau_2(b)$$

for every $a\in M_1$ and $b\in M_2$.

Let $x=[x_{i,j}]$ be a positive element of $M_n(M_1\overline{\otimes}M_2)$. We note first that under the canonical identification of $M_n(M_1\overline{\otimes}M_2)$ with $M_1\overline{\otimes}M_n\overline{\otimes}M_2=M_1\overline{\otimes}(M_n(M_2))$ we have

$$r(x)(\phi)=R_\phi([x_{i,j}])=[R_\phi(x_{i,j})] \qquad \text{for} \quad \phi\in(M_1)_*.$$

Now in order to show that ρ_2 is completely positive it suffices to prove that the map $r([\rho_2(x_{i,j})])$ is a completely positive map of M_{1*} to $M_n(N_2)$ for every positive integer n. Thus, let $[\phi_{k,l}]$ be a positive element in $M_m(M_{1*})$. Then

$$(r((\rho_2)_n(x))_m([\phi_{k,l}])=[r((\rho_2)_n(x))(\phi_{k,l})]$$

$$=[R_{\phi_{k,l}}((\rho_2)_n(x))]=[R_{\phi_{k,l}}(\rho_2(x_{i,j}))]$$

(regarding as a matrix with respect to the indexes i, j, k and l)

$$=[\tau_2(R_{\phi_{k,l}}(x_{i,j}))]=(\tau_2)_{mn}[r(x)(\phi_{k,l})]$$

$$=((\tau_2)_n)_m\circ(r(x))_m([\phi_{k,l}])\geqq 0$$

where we use the same notation $R_{\phi_{k,l}}$ applying different algebras, $M_n(M_1\overline{\otimes}M_2)$, $M_1\overline{\otimes}N_2$ and $M_1\overline{\otimes}M_2$. Therefore, the map $r((\rho_2)_n(x))=r([\rho_2(x_{i,j})])$ is completely positive.

Similarly we can define a completely positive map ρ_1 from $M_1\overline{\otimes}N_2$ to $N_1\overline{\otimes}N_2$ satisfying $l(\rho_1(x))=\tau_1\circ l(x)$ for every $x\in M_1\overline{\otimes}M_2$ with the property, $\rho_1(a\otimes b)=\tau_1(a)\otimes b$ for every $a\in M_1$ and $b\in N_2$. Then the map $\tau=\rho_1\circ\rho_2$ is a required extension.

When τ_1 and τ_2 are not necessarily unital, we put $\tau_1(1)=h$ and $\tau_2(1)=k$. By [**2**, Lemma 2.2], there exist unital completely positive maps τ_1' and τ_2' such that

$$\tau_1(x)=h^{1/2}\tau_1'(x)h^{1/2} \qquad \text{and} \quad \tau_2(x)=k^{1/2}\tau_2'(x)k^{1/2}.$$

Then there is a unital completely positive map τ' satisfying

$$\tau'(a\otimes b)=\tau_1'(a)\otimes\tau_2'(b)$$

for every $a\in M_1$ and $b\in M_2$. Therefore the map $\tau(x)=(h\otimes k)^{1/2}\tau'(x)(h\otimes k)^{1/2}$ is a completely positive map satisfying

$$\tau(a\otimes b)=\tau_1(a)\otimes\tau_2(b)$$

for every $a\in M_1$ and $b\in M_2$. This completes the proof.

In the above construction, if we take $\phi\in M_{1*}$, $\psi\in N_{2*}$ and an element $x\in M_1\overline{\otimes}M_2$, we have by the Fubini type principle

$$\langle l(\rho_2(x))(\psi),\ \phi\rangle=\langle r(\rho_2(x))(\phi),\ \psi\rangle=\langle\tau_2\circ r(x)(\phi),\ \psi\rangle$$
$$=\langle r(x)(\phi),\ {}^t\tau_2(\psi)\rangle=\langle l_g(x)({}^t\tau_2(\psi)),\ \phi\rangle\ ,$$

whence $l(\rho_2(x))=l_g(x)\circ{}^t\tau_2|N_{2*}$. Therefore

$$l(\tau(x))=l(\rho_1\circ\rho_2(x))=\tau_1\circ l(\rho_2(x))=\tau_1\circ l_g(x)\circ{}^t\tau_2|N_{2*}\ .$$

Thus we denote this map τ by $\tau_1\underset{l}{\overline{\otimes}}\tau_2$ and call the left product map of τ_1 and τ_2. On the other hand there is another similar way to get a completely positive map of $M_1\overline{\otimes}M_2$ to $N_1\overline{\otimes}N_2$ for the pair (τ_1,τ_2). Namely, we define completely positive maps $\rho_1':M_1\overline{\otimes}M_2\to N_1\overline{\otimes}M_2$ and $\rho_2':N_1\overline{\otimes}M_2\to N_1\overline{\otimes}N_2$ in a similar way as above and put $\tau'=\rho_2'\circ\rho_1'$. Then the map τ' satisfies the relation;

$$r(\tau'(x))=\tau_2\circ r(\rho_1'(x))=\tau_2\circ r_g(x)\circ{}^t\tau_1|N_{1*}\ .$$

Write this map as $\tau'=\tau_1\underset{r}{\overline{\otimes}}\tau_2$ and we call the right product map of τ_1 and τ_2. These two maps might be different in general but we have

PROPOSITION 4. *With the same notation as above,*

$$\tau_1\underset{r}{\overline{\otimes}}\tau_2=\tau_1\underset{l}{\overline{\otimes}}\tau_2$$

if τ_1 or τ_2 is normal.

PROOF. We assume that τ_1 is normal, then for $\phi\in N_{1*}$ we get ${}^t\tau_1(\phi)\in M_{1*}$ and

$$r(\tau_1\underset{r}{\overline{\otimes}}\tau_2(x))(\phi)=\tau_2\circ r_g(x)\circ{}^t\tau_1(\phi)$$
$$=\tau_2\circ r(x)\circ{}^t\tau_1(\phi)\ .$$

Hence, $r(\tau_1\underset{r}{\overline{\otimes}}\tau_2(x))=\tau_2\circ r(x)\circ{}^t\tau_1|N_{1*}$. Using Lemma 1,

$$l(\tau_1\underset{r}{\overline{\otimes}}\tau_2(x))=l_g(\tau_1\underset{r}{\overline{\otimes}}\tau_2(x))|N_{2*}$$
$$={}^t(r(\tau_1\underset{r}{\overline{\otimes}}\tau_2(x))|N_{2*}$$
$$=\tau_1\circ l_g(x)\circ{}^t\tau_2|N_{2*}$$

$$=l(\tau_1 \underset{l}{\overline{\otimes}} \tau_2(x)).$$

Hence, $\tau_1 \underset{r}{\overline{\otimes}} \tau_2 = \tau_1 \underset{l}{\overline{\otimes}} \tau_2$.

PROPOSITION 5. *Let M_i, N_i and R_i be von Neumann algebras and $\sigma_i : M_i \to N_i$ and $\tau_i : N_i \to R_i$ be completely positive maps* ($i=1, 2$). *Then*

$$(\tau_1 \underset{l}{\overline{\otimes}} \tau_2)\circ(\sigma_1 \underset{l}{\overline{\otimes}} \sigma_2) = (\tau_1 \circ \sigma_1) \underset{l}{\overline{\otimes}} (\tau_2 \circ \sigma_2)$$

if τ_2 is normal. Similar assertion holds for right slice maps if τ_1 is normal.

PROOF. For $x \in M_1 \overline{\otimes} M_2$, by the normality of τ_2

$$l((\tau_1 \underset{l}{\overline{\otimes}} \tau_2)\circ(\sigma_1 \underset{l}{\overline{\otimes}} \sigma_2)(x)) = \tau_1 \circ l_g(\sigma_1 \underset{l}{\overline{\otimes}} \sigma_2(x)) \circ {}^t\tau_2 | R_{2*}$$

$$= \tau_1 \circ l(\sigma_1 \underset{l}{\overline{\otimes}} \sigma_2(x)) \circ {}^t\tau_2 | R_{2*} = \tau_1 \circ \sigma_1 \circ l_g(x) \circ {}^t\sigma_2 \circ {}^t\tau_2 | R_{2*}$$

$$= (\tau_1 \circ \sigma_1) \circ l_g(x) \circ {}^t(\tau_2 \circ \sigma_2) | R_{2*} = l((\tau_1 \circ \sigma_1) \underset{l}{\overline{\otimes}} (\tau_2 \circ \sigma_2)(x)).$$

Hence, $(\tau_1 \underset{l}{\overline{\otimes}} \tau_2)\circ(\sigma_1 \underset{l}{\overline{\otimes}} \sigma_2) = (\tau_1 \circ \sigma_1) \underset{l}{\overline{\otimes}} (\tau_2 \circ \sigma_2)$.

It is worth to notice that Theorem 3 and the method cover well-known results of product maps of completely positive maps. For instance, one may conclude the existence of the product maps of completely positive maps on the C^*-tensor products once we apply the theorem to the von Neumann tensor product of the universal enveloping von Neumann algebras of relevant C^*-algebras for a couple of double transposed maps of starting completely positive maps and restrict the result to the C^*-tensor product. One can also easily recognize that if N_i are von Neumann subalgebras of M_i and τ_i are projections of norm one from M_i to N_i ($i=1, 2$), then for $x \in N_1 \overline{\otimes} N_2$, we have

$$l(\tau_1 \underset{l}{\overline{\otimes}} \tau_2(x)) = \tau_1 \circ l_g(x) \circ {}^t\tau_2 | N_{2*} = l(x)$$

so that $\tau_1 \underset{l}{\overline{\otimes}} \tau_2(x) = x$ i.e. $\tau_1 \underset{l}{\overline{\otimes}} \tau_2$ is a projection of norm one, too. Next we consider the case that τ_1 and τ_2 are normal with the same notations of Proposition 4. Then there exists a unique normal product map of completely positive maps τ, which is equal to $\tau_1 \underset{l}{\overline{\otimes}} \tau_2$ and $\tau_1 \underset{r}{\overline{\otimes}} \tau_2$. It suffices to see that $\tau_1 \underset{l}{\overline{\otimes}} \tau_2$ is normal. For $x \in M_1 \overline{\otimes} M_2$, $\phi \in N_{1*}$ and $\psi \in N_{2*}$ by the normalities of τ_1 and τ_2,

$$\langle \tau_1 \underset{l}{\overline{\otimes}} \tau_2(x), \phi \otimes \psi \rangle = \langle l(\tau_1 \underset{l}{\overline{\otimes}} \tau_2(x))(\psi), \phi \rangle$$

$$= \langle \tau_1 \circ l_g(x) \circ {}^t\tau_2(\psi), \phi \rangle = \langle l(x) \circ {}^t\tau_2(\psi), {}^t\tau_1(\phi) \rangle$$

$$= \langle x, {}^t\tau_1(\phi) \otimes {}^t\tau_2(\psi) \rangle.$$

Hence we have ${}^t(\tau_1 \underset{l}{\overline{\otimes}} \tau_2) = {}^t\tau_1 \otimes {}^t\tau_2$ on the algebraic tensor product $N_{1*} \otimes N_{2*}$.

Since ${}^{t}(\tau_1 \overline{\otimes}_l \tau_2)(N_{1*} \otimes N_{2*})$ is contained in the algebraic tensor product $M_{1*} \otimes M_{2*}$ and ${}^{t}(\tau_1 \overline{\otimes}_l \tau_2)$ is continuous, ${}^{t}(\tau_1 \overline{\otimes}_l \tau_2)(N_1 \otimes N_2)_*$ is contained in $(M_1 \otimes M_2)_*$, that is, $\tau_1 \overline{\otimes}_l \tau_2$ is normal. Then we write $\tau = \tau_1 \overline{\otimes} \tau_2$.

3. Normality and singularity of completely positive product maps.

Let M be a von Neumann algebra on a Hilbert space H and let $M^* = M_* \oplus M_*^{\perp}$ (l^1-sum) be the decomposition of the conjugate space M^* into the predual (normal part) M_* and the singular part $M_*^{\perp}$. The subspace $M_*^{\perp}$ of singular functionals is linearly spanned by positive linear functionals and a positive functional ϕ belongs to $M_*^{\perp}$ if and only if for any non-zero projection p of M there exists a non-zero projection q such that $q \leqq p$ and $\phi(q) = 0$ (cf. [7]). Let τ be a bounded linear map from M to a von Neumann algebra N on a space K. Then τ is uniquely decomposed into the σ-weakly continuous map τ_{nor} (normal part of τ) and the singular map τ_{sin} (singular part of τ) (cf. [8]). Let τ be a completely positive map from M to N. By the theorem of Stinespring, there exist a representation π of M on a Hilbert space K_1 and a bounded linear operator v from K into K_1 such that $\tau(x) = v^* \pi(x) v$ for $x \in M$.

PROPOSITION 6. *Let $\pi = \pi_{\text{nor}} + \pi_{\text{sin}}$ be the decomposition of π into the normal and singular part of π, then*

$$\tau = v^* \pi_{\text{nor}} v + v^* \pi_{\text{sin}} v$$

is the canonical decomposition of τ, that is, $\tau_{\text{nor}} = v^* \pi_{\text{nor}} v$, $\tau_{\text{sin}} = v^* \pi_{\text{sin}} v$. Both normal and singular parts of τ are completely positive.

PROOF. Let $\omega(\xi)$ be a vector state of $B(K)$ for a unit vector ξ in K. Then, as functionals on M, we have

$$\omega(\xi) \circ v^* \pi_{\text{nor}} v = \omega(v\xi) \circ \pi_{\text{nor}} \in M_*$$

$$\text{and} \quad \omega(\xi) \circ v^* \pi_{\text{sin}} v = \omega(v\xi) \circ \pi_{\text{sin}} \in M^* .$$

Hence, $v^* \pi_{\text{nor}} v$ and $v^* \pi_{\text{sin}} v$ are normal and singular maps respectively, and

$$\tau = v^* \pi_{\text{nor}} v + v^* \pi_{\text{sin}} v$$

is the canonical decomposition of τ as a bounded map of M into $B(K)$. Since the decomposition of τ does not depend on those von Neumann algebras which include N, we have $\tau_{\text{nor}} = v^* \pi_{\text{nor}} v$, $\tau_{\text{sin}} = v^* \pi_{\text{sin}} v$, and they are completely positive.

THEOREM 7. *Keep the same notations in Theorem 3 and let τ be another completely positive map from $M_1 \overline{\otimes} M_2$ to $N_1 \overline{\otimes} N_2$ such that $\tau(a \otimes b) = \tau_1(a) \otimes \tau_2(b)$. Then we have;*

$$(\tau_1)_{\text{nor}} \overline{\otimes} (\tau_2)_{\text{nor}} \geqq \tau_{\text{nor}}$$

in the order of completely positive maps. In particular $(\tau_1)_{\mathrm{nor}}\overline{\otimes}(\tau_2)_{\mathrm{nor}}=(\tau_1\overline{\underset{l}{\otimes}}\tau_2)_{\mathrm{nor}}$ $=(\tau_1\overline{\underset{r}{\otimes}}\tau_2)_{\mathrm{nor}}$. *Hence, if either* τ_1 *or* τ_2 *is a singular map then* τ *is singular.*

PROOF. Let ϕ and ψ be normal states on N_1 and N_2 and let $y=\sum_{i=1}^n a_i\otimes b_i$ be a fixed element of $N_1\otimes N_2$. We use the notation ${}_a\phi_b$ for the functional $\langle x, {}_a\phi_b\rangle=\langle bxa, \phi\rangle$. Now consider two families of singular functionals,

$${}^t(\tau_{1\,\mathrm{sin}})({}_{a_i}\phi_{a_j^*})={}_{a_i}\phi_{a_j^*}\circ\tau_{1\,\mathrm{sin}} \qquad i, j=1, 2, \cdots, n$$

and

$${}^t(\tau_{2\,\mathrm{sin}})({}_{b_i}\psi_{b_j^*})={}_{b_i}\psi_{b_j^*}\circ\tau_{2\,\mathrm{sin}} \qquad i, j=1, 2, \cdots, n\,.$$

They are expressed as linear combinations of finitely many positive singular functionals of M_1 and M_2 respectively. Hence, by Takesaki's theorem mentioned before ([7]), we can find the families of orthogonal projections $\{p_\alpha\}$ in M_1 and $\{q_\beta\}$ in M_2 such that $\sum_\alpha p_\alpha=1_{H_1}$ and $\sum_\beta q_\beta=1_{H_2}$ with

$$\langle p_\alpha, {}^t\tau_{1\,\mathrm{sin}}({}_{a_i}\phi_{a_j^*})\rangle=\langle q_\beta, {}^t\tau_{2\,\mathrm{sin}}({}_{b_i}\psi_{b_j^*})\rangle=0$$

for every pair (i, j), α and β. Thus,

$$\begin{aligned}&\langle p_\alpha\otimes q_\beta, {}^t(\tau_{1\,\mathrm{sin}}\otimes\tau_2)({}_y\phi\otimes\psi_{y^*})\rangle\\ &=\sum_{i,j=1}^n\langle p_\alpha, {}^t\tau_{1\,\mathrm{sin}}({}_{a_i}\phi_{a_j^*})\rangle\langle q_\beta, {}^t\tau_2({}_{b_i}\psi_{b_j^*})\rangle=0\end{aligned}$$

for an arbitrary pair (p_α, q_β). Therefore, we have

$$\langle xp_\alpha\otimes q_\beta, {}^t(\tau_{1\,\mathrm{sin}}\otimes\tau_2)({}_y(\phi\otimes\psi)_{y^*})\rangle=0$$

for every element x in $M_1\otimes M_2$. Similarly we have

$$\langle xp_\alpha\otimes q_\beta, {}^t(\tau_1\otimes\tau_{2\,\mathrm{sin}})({}_y(\phi\otimes\psi)_{y^*})\rangle=0\,.$$

It follows that for an element x in $M_1\otimes M_2$

$$\begin{aligned}&\langle\tau(xp_\alpha\otimes q_\beta), {}_y\phi\otimes\psi_{y^*}\rangle\\ &=\langle xp_\alpha\otimes q_\beta, {}^t(\tau_1\otimes\tau_2)({}_y\phi\otimes\psi_{y^*})\rangle\\ &=\langle xp_\alpha\otimes q_\beta, {}^t(\tau_{1\,\mathrm{nor}}\otimes\tau_{2\,\mathrm{nor}})({}_y\phi\otimes\psi_{y^*})\rangle\\ &=\langle\tau_{1\,\mathrm{nor}}\overline{\otimes}\tau_{2\,\mathrm{nor}}(xp_\alpha\otimes q_\beta), {}_y\phi\otimes\psi_{y^*}\rangle\,.\end{aligned}$$

Let J_1 and J_2 be finite subsets of the index set $\{\alpha\}$ and $\{\beta\}$, and put $J=J_1\times J_2$. Set $p_{J_1}=\sum_{\alpha\in J_1}p_\alpha$, $q_{J_2}=\sum_{\beta\in J_2}q_\beta$ and $r_J=p_{J_1}\otimes q_{J_2}$. Let x be a positive element of $M_1\otimes_\alpha M_2$. By the above arguments

$$\langle\tau_{1\,\mathrm{nor}}\overline{\otimes}\tau_{2\,\mathrm{nor}}(r_Jxr_J), {}_y\phi\otimes\psi_{y^*}\rangle$$

$$=\langle\tau_1\otimes\tau_2(r_J x r_J),\ {}_y\phi\otimes\psi_{y^*}\rangle$$
$$=\langle\tau(r_J x r_J),\ {}_y\phi\otimes\phi_{y^*}\rangle$$
$$\geqq\langle\tau_{\mathrm{nor}}(r_J x r_J),\ {}_y\phi\otimes\psi_{y^*}\rangle\,.$$

As the element x is the σ-weakly limit of $r_J x r_J$ with respect to the index set $\{J\}$, we have

$$\langle\tau_{1\,\mathrm{nor}}\overline{\otimes}\tau_{2\,\mathrm{nor}}(x),\ {}_y\phi\otimes\psi_{y^*}\rangle\geqq\langle\tau_{\mathrm{nor}}(x),\ {}_y\phi\otimes\psi_{y^*}\rangle\,.$$

Therefore we have the same ordering for a positive element in $M_1\overline{\otimes}M_2$. Now the order in $N_1\overline{\otimes}N_2$ is determined by the family of functionals $\{{}_y\phi\otimes\psi_{y^*}\}$ where ϕ, ψ are ranging over all normal states ϕ of N_1, ψ of N_2 and all elements $y=\sum_{i=1}^{n}a_i\otimes b_i$ in $N_1\otimes N_2$. Hence

$$\tau_{1\,\mathrm{nor}}\overline{\otimes}\tau_{2\,\mathrm{nor}}(x)\geqq\tau_{\mathrm{nor}}(x)$$

for every positive element x in $M_2\overline{\otimes}M_2$

Let M_m be the $m\times m$ matrix algebra and consider the map

$$\tau_m: M_m(M_1\overline{\otimes}M_2)\longrightarrow M_m(N_1\overline{\otimes}N_2)$$

defined by

$$\tau_m([x_{i,j}])=[\tau(x_{i,j})]\,.$$

We can easily see that $(\tau_m)_{\mathrm{nor}}=(\tau_{\mathrm{nor}})_m$ and $(\tau_m)_{\mathrm{sin}}=(\tau_{\mathrm{sin}})_m$. With the identification of $M_m(M_1\overline{\otimes}M_2)=M_m\overline{\otimes}M_1\overline{\otimes}M_2$ and $M_m(N_1\overline{\otimes}N_2)=M_m\overline{\otimes}N_1\overline{\otimes}N_2$ it follows from the preceding arguments

$$\begin{aligned}(\tau_{1\,\mathrm{nor}}\overline{\otimes}\tau_{2\,\mathrm{nor}})_m(x)&=((\tau_1)_m)_{\mathrm{nor}}\overline{\otimes}(\tau_2)_{\mathrm{nor}}(x)\\&\geqq(\tau_m)_{\mathrm{nor}}(x)\\&=(\tau_{\mathrm{nor}})_m(x)\end{aligned}$$

for every positive element x in $M_m(M_1\overline{\otimes}M_2)$. Thus, $(\tau_1)_{\mathrm{nor}}\overline{\otimes}(\tau_2)_{\mathrm{nor}}-\tau_{\mathrm{nor}}$ is a completely positive map.

Moreover for an element x in $M_1\overline{\otimes}M_2$,

$$l((\tau_1\overline{\underset{l}{\otimes}}\tau_2)(x))=\tau_1\circ l_g(x)\circ{}^t\tau_2\,|\,N_{2*}$$

$$\begin{aligned}&=(\tau_{1\,\mathrm{nor}}\circ l_g(x)\circ{}^t\tau_{2\,\mathrm{nor}}+\tau_{1\,\mathrm{nor}}\circ l_g(x)\circ{}^t\tau_{2\,\mathrm{sin}}+\tau_{1\,\mathrm{sin}}\circ l_g(x)\circ{}^t\tau_{2\,\mathrm{nor}}\\&\quad+\tau_{1\,\mathrm{sin}}\circ l_g(x)\circ{}^t\tau_{2\,\mathrm{sin}})\,|\,N_{2*}\\&=(l((\tau_{1\,\mathrm{nor}}\overline{\underset{l}{\otimes}}\tau_{2\,\mathrm{nor}})(x))+l((\tau_{1\,\mathrm{nor}}\overline{\underset{l}{\otimes}}\tau_{2\,\mathrm{sin}})(x))+l((\tau_{1\,\mathrm{sin}}\overline{\underset{l}{\otimes}}x_{2\,\mathrm{nor}})(x))\\&\quad+l((\tau_{1\,\mathrm{sin}}\overline{\underset{l}{\otimes}}\tau_{2\,\mathrm{sin}})(x))\,|\,N_{2*}\,.\end{aligned}$$

Hence $(\tau_1 \bar{\otimes}_l \tau_2)=\tau_{1\,\mathrm{nor}} \bar{\otimes}_l \tau_{2\,\mathrm{nor}}+\tau_{1\,\mathrm{nor}} \bar{\otimes}_l \tau_{2\,\sin}+\tau_{1\,\sin} \bar{\otimes}_l \tau_{2\,\mathrm{nor}}+\tau_{1\,\sin} \bar{\otimes}_l \tau_{2\,\sin}$, therefore $(\tau_1 \bar{\otimes}_l \tau_2)(x) \geqq \tau_{1\,\mathrm{nor}} \bar{\otimes}_l \tau_{2\,\mathrm{nor}}(x)$ for every positive element x. From the definition of the normal part of $(\tau_1 \bar{\otimes}_l \tau_2)$ (cf. [8]) one sees that

$$(\tau_1 \bar{\otimes}_l \tau_2)_{\mathrm{nor}}(x) \geqq \tau_{1\,\mathrm{nor}} \bar{\otimes}_l \tau_{2\,\mathrm{nor}}(x)$$

for every positive element x. Hence, $(\tau_1 \bar{\otimes}_l \tau_2)_{\mathrm{nor}}=\tau_{1\,\mathrm{nor}} \bar{\otimes} \tau_{2\,\mathrm{nor}}$. The last statement is a trivial consequence. This completes all proofs.

ACKNOWLEDGEMENT. The authors are indebted to Professor E. G. Effros for allowing them to cite his Theorem (Theorem 2) from his unpublished paper with the whole proof.

References

[1] W. Arveson, Subalgebras of C^*-algebras, Acta Math., **123** (1969), 141-224.
[2] M. D. Choi and E. G. Effros, Injectivity and operator algebras, J. Functional Analysis, **24** (1977), 156-209.
[3] J. Dixmier, Les algèbres d'opérateurs dans l'espace hilbertien, Gauthier-Villars, Paris, 1969.
[4] E. G. Effros, Approximation problems for C^*-algebras, unpublished.
[5] U. Haagerup, The standard form of von Neumann algebras, Math. Scand., **37** (1957), 271-283.
[6] J. Hakeda and J. Tomiyama, On some extension properties of von Neumann algebras, Tohoku Math. J., **19** (1967), 305-323.
[7] M. Takesaki, Singularity of positive functionals, Proc. Japan Acad., **35** (1959), 365-366.
[8] J. Tomiyama, On the projection of norm one in W^*-algebras III, Tohoku Math. J., **11** (1959), 125-129.
[9] J. Tomiyama, On the tensor product of von Neumann algebras, Pacific J. Math., **30** (1969), 263-270.
[10] J. Tomiyama, Tensor products and projections of norm one of von Neumann algebras, Lecture note, Univ. of Copenhagen, 1970.
[11] J. Tomiyama, On some types of maximal abelian subalgebras, J. Functional Analysis, **10** (1972), 373-386.

Masaru NAGISA
Department of Applied Mathematics
Faculty of Engineering Science
Osaka University
Toyonaka 560
Japan

Jun TOMIYAMA
Department of Mathematics
Faculty of Science
Niigata University
Niigata 950-21
Japan