COMPLETELY REDUCIBLE OPERATORS

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1. **Introduction.** Let H be a separable complex Hilbert space. A (closed) subspace of H is *nontrivial* if it is different from $\{0\}$ and H. A bounded linear operator A on H is *completely reducible* if whenever M is a reducing subspace of A of dimension greater than 1, the operator $A|M$ has a nontrivial reducing subspace. The spectral theorem implies that every normal operator is completely reducible. If H is finite dimensional then every completely reducible operator is normal. This is not the case in general, however, as the example given below shows.

Our main results are sufficient conditions that a completely reducible operator have a *reducing eigenvector*, i.e. an eigenvector that is also an eigenvector of the adjoint of the operator. For normal operators, of course, every eigenvector is a reducing eigenvector. Andô [1] has shown that a compact operator which has the property that every invariant subspace is reducing must be normal. Our techniques are similar to Andô's, and Corollary 3 below generalizes his result to the case of polynomially compact operators.

2. **An example.** Let B denote the Hermitian operator consisting of multiplication by the independent variable on $L^2(0, 1)$. P. R. Halmos has shown (unpublished) that every reducing subspace of the operator

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix},$$

acting on the space $L^2(0, 1) \oplus L^2(0, 1)$, is of the form $N \oplus N$, where N is a reducing subspace of B. It follows that

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

is a completely reducible operator, although it is clearly not normal.

A slight modification of Halmos's example shows that a completely reducible operator can have a spanning set of eigenvectors

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and still be nonnormal. We show that every reducing subspace of the operator

$$A = \begin{pmatrix} 0 & B \\ 0 & 1 \end{pmatrix}$$

is of the form $N \oplus N$, where N reduces B. Our argument is essentially the same as Halmos's.

First note that

$$A^* = \begin{pmatrix} 0 & 0 \\ B & 1 \end{pmatrix} \quad \text{and} \quad A^*A = \begin{pmatrix} 0 & 0 \\ 0 & B^2 + 1 \end{pmatrix}.$$

Now let M be any reducing subspace of A. Let N be the reducing subspace of B generated by the set of vectors f in $L^2(0, 1)$ such that there exists a g in $L^2(0, 1)$ with $\langle f, g\rangle$ or $\langle g, f\rangle$ in M. Then clearly M is contained in $N \oplus N$. To prove the reverse inclusion, let $\langle h, k\rangle$ be any element of M. Then the fact that $(A^*A)^n\langle h, k\rangle$ is in M for every positive integer n implies that $\{0\} \oplus \bigvee_{n=1}^{\infty}\{(B^2+1)^n k\}$ is contained in M. But polynomials without constant term in the variable (x^2+1) are uniformly dense in the space of continuous functions on $[0, 1]$. Therefore $\langle 0, k\rangle$ is in M, and then so is $\langle h, 0\rangle$. If we apply A and A^* to these vectors we see that $\{0\} \oplus \bigvee_{n=0}^{\infty}\{B^{2n}k\}$, $\{0\} \oplus \bigvee_{n=0}^{\infty}\{B^{2n+1}h\}$, $\bigvee_{n=0}^{\infty}\{B^{2n}h\} \oplus \{0\}$, and $\bigvee_{n=0}^{\infty}\{B^{2n+1}k\} \oplus \{0\}$ are each contained in M.

Now if $\langle f_1, f_2\rangle$ is in $N \oplus N$ then the foregoing shows us how to prove that $\langle f_1, 0\rangle$ and $\langle 0, f_2\rangle$ are in M. Hence $N \oplus N$ is contained in M.

3. **Results.** The above example shows that a completely reducible operator need not have a reducing eigenvector even if it has a spanning set of eigenvectors and is polynomially compact; ($A^2 - A = 0$ in the example). Our results show that a slight strengthening of either of these conditions does imply that a completely reducible operator has a reducing eigenvector.

LEMMA. *If $\{M_\alpha\}$ is a totally ordered family of subspaces of a separable Hilbert space, and if $N = \bigcap_\alpha M_\alpha$, then there is a countable subfamily $\{M_{\alpha_i}\}$ such that $N = \bigcap_i M_{\alpha_i}$ and $M_{\alpha_{i+1}} \subset M_{\alpha_i}$ for each i.*

PROOF.[2] Taking set-theoretic complements, we get $N' = \bigcup_\alpha M'_\alpha$. Since N' has the Lindelöf property, there is a countable subcover $\{M'_{\alpha_i}\}$. Then $N = \bigcap_i M_{\alpha_i}$. To get $M_{\alpha_{i+1}} \subset M_{\alpha_i}$ we simply discard $M_{\alpha_{i+1}}$ if $M_{\alpha_{i+1}} \not\subset M_{\alpha_i}$.

THEOREM 1. *If A is completely reducible, and if the subspace $E_\lambda = \{x : Ax = \lambda x\}$ is finite dimensional, then E_λ reduces A.*

[2] This proof is due to C. R. MacCluer.

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PROOF. Let $\mathfrak{F}$ denote the family of subspaces of H that reduce A and meet E_λ in a subspace other than $\{0\}$. By the Hausdorff maximal principle there is a maximal chain $\{M_\alpha\}$ in $\mathfrak{F}$. If we let N be $\bigcap_\alpha M_\alpha$, then N clearly reduces A. We will show that N is in $\mathfrak{F}$ and that N is one dimensional.

Choose a countable subfamily of $\{M_\alpha\}$ by the Lemma. Then for each i choose an x_i in M_{α_i} such that $\|x_i\|=1$ and $Ax_i=\lambda x_i$. Some subsequence of $\{x_i\}$ converges, to some x, since E_λ is finite dimensional. Then x is in N, since the sequence $\{x_i\}$ is eventually in each M_{α_i}. Also $\|x\|=1$ and $Ax=\lambda x$. Thus N is in $\mathfrak{F}$ and the dimension of N is at least 1.

If the dimension of N were greater than 1 then $A|N$ would have a nontrivial reducing subspace L. Then at least one of L and $L^\perp \cap N$ would be in $\mathfrak{F}$, contradicting the fact that N is the intersection of the subspaces in a maximal chain in $\mathfrak{F}$.

Thus the dimension of N is 1 and N contains a reducing eigenvector x in E_λ. If the dimension of E_λ is greater than 1 we can apply the above proof to $A|\{x\}^\perp$ and get another reducing eigenvector in E_λ. If we repeat this process finitely many times we will get an orthonormal basis for E_λ consisting of reducing eigenvectors.

COROLLARY 1. *If A is completely reducible and has a spanning set of eigenvectors each of which corresponds to an eigenvalue of finite multiplicity, then A is normal.*

PROOF. Each eigenvector in the spanning set is reducing, by Theorem 1, and thus A can be written as a diagonal matrix.

THEOREM 2. *If A is completely reducible, and if there exists a nonzero compact operator T such that every reducing subspace of A reduces T, then A has a reducing eigenvector.*

PROOF. The proof is similar to the proof of Theorem 1. Here we let $\mathfrak{F}$ denote the family of subspaces M that reduce A and that have the property that $\|T|M\|=\|T\|$. Let N be the intersection of the subspaces in a maximal chain in $\mathfrak{F}$ and write $N=\bigcap_i M_{\alpha_i}$ by the Lemma. We will be done if we show that the dimension of N is 1.

A compact operator attains its norm; thus for each i there is an x_i in M_{α_i} such that $\|x_i\|=1$ and $\|Tx_i\|=\|T\|$. Choose a subsequence of $\{x_i\}$ that converges weakly to some x. Then $\|Tx\|=\|T\|$ and thus $\|x\|=1$. Also, x is in N since $\{x_i\}$ is eventually in each M_{α_i}. Thus N is in $\mathfrak{F}$. If the dimension of N were greater than 1 there would be a nontrivial reducing subspace L of A properly contained in N. But the fact that T attains its norm on N implies that T attains its norm on L

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or on $L^{\perp}\cap N$. Thus at least one of L and $L^{\perp}\cap N$ is in $\mathfrak{F}$, contradicting the definition of N.

COROLLARY 2. *A compact completely reducible operator is normal.*

PROOF. Let A be compact and completely reducible, and let $\mathcal{E}$ be a maximal orthonormal set of reducing eigenvectors of A. It suffices to show that $\mathcal{E}^{\perp}$ is $\{0\}$. If $\mathcal{E}^{\perp}$ is not $\{0\}$ then either $A|\mathcal{E}^{\perp}=0$ or Theorem 2 applies, with $T=A$. In either case the maximality of $\mathcal{E}$ is contradicted.

COROLLARY 3. *If A is polynomially compact, (i.e. there is a nonzero polynomial p such that $p(A)$ is compact), and if every invariant subspace of A is reducing, then A is normal.*

PROOF. First, such an operator is completely reducible, by the Bernstein-Robinson invariant subspace theorem for polynomially compact operators [2], [3]. Let $\mathcal{E}$ be a maximal orthonormal set of reducing eigenvectors of A. If $\mathcal{E}^{\perp}\neq\{0\}$, consider $A|\mathcal{E}^{\perp}$. If $p(A)|\mathcal{E}^{\perp}$ is not 0 then A has a reducing eigenvector in $\mathcal{E}^{\perp}$ by Theorem 2, with $T=p(A)$. If $p(A)|\mathcal{E}^{\perp}$ is 0 then $A|\mathcal{E}^{\perp}$ has an eigenvector by the spectral mapping theorem. But every eigenvector of A is reducing by hypothesis.

COROLLARY 4. *If A is completely reducible, and if $p(A)$ is compact and has finite-dimensional nullspace for some polynomial p, then A is normal.*

PROOF. As in the previous corollaries, let $\mathcal{E}$ be a maximal orthonormal set of reducing eigenvectors of A. If $\mathcal{E}^{\perp}$ is finite dimensional then $A|\mathcal{E}^{\perp}$ is normal and the proof is finished. If $\mathcal{E}^{\perp}$ is infinite dimensional then $p(A)|\mathcal{E}^{\perp}$ is not 0 and Theorem 2 applies.

COROLLARY 5. *If the von Neumann algebra generated by a completely reducible operator A contains a nonzero compact operator, then A has point spectrum.*

PROOF. The hypotheses imply that A and the compact operator satisfy the conditions of Theorem 2.

THEOREM 3. *Let A and B be operators on H with disjoint spectra. Then an operator S commutes with $A\oplus B$ on $H\oplus H$ if and only if $S=C\oplus D$, where C commutes with A and D commutes with B.*

PROOF. Let

$$S=\begin{pmatrix} C & E \\ F & D \end{pmatrix}$$

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commute with $A \oplus B$. We must show that $E=0$ and $F=0$. Now

$$S(A \oplus B) = \begin{pmatrix} CA & EB \\ FA & DB \end{pmatrix} \quad \text{and} \quad (A \oplus B)S = \begin{pmatrix} AC & AE \\ BF & BD \end{pmatrix}.$$

Therefore $AE=EB$ and $BF=FA$. A theorem of Rosenblum [4] implies that these equations can hold only if $E=0$ and $F=0$ (because of the fact that the spectra of A and B are disjoint).

COROLLARY 6. *If A and B are completely reducible and have disjoint spectra then $A \oplus B$ is completely reducible.*

PROOF. It follows from Theorem 3 that every projection that commutes with $A \oplus B$ is of the form $P \oplus Q$. Thus every reducing subspace of $A \oplus B$ is of the form $M \oplus N$, where M reduces A and N reduces B.

4. **Remarks.** The above theorems and corollaries all hold for operators on nonseparable spaces too; this follows from the fact that every operator can be written as the direct sum of operators on separable spaces.

It would be interesting to find other sufficient conditions that a completely reducible operator be normal. In particular, if A is completely reducible and every invariant subspace of A is reducing must A be normal?

ADDED IN PROOF. The case of Corollary 3 where $p(z)=z^n$ has been proven by T. Saitô in *Some remarks to Ando's Theorems*, Tôhoku Math. J. (2) **18** (1966).

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