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COMPLETELY REDUCIBLE OPERATORS THAT COMMUTE WITH COMPACT OPERATORS

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ABSTRACT. It is shown that if T is a completely reducible operator on a Banach space and $TK = KT$, where K is an injective compact operator with a dense range, then T is a scalar type spectral operator. Other related results are also obtained.

Let $\mathcal{A}$ be an algebra of bounded linear operators on a Banach space X. $\operatorname{lat}\mathcal{A}$ is the lattice of (closed) invariant subspaces of $\mathcal{A}$. We say that $\mathcal{A}$ is *completely reducible* if for every $M \in \operatorname{lat}\mathcal{A}$ there is $N \in \operatorname{lat}\mathcal{A}$ with $M \dotplus N = X$ (that is, $M \cap N = 0$ and the algebraic sum $M + N$ coincides with X). An operator T is *completely reducible* if the algebra generated by T is. It is unknown whether a weakly closed unital completely reducible algebra must be reflexive; that is, must contain every operator which leaves invariant its invariant subspaces. Some partial solutions of this problem can be found in [**1**, **6**, **7**].

In this paper we show that every completely reducible operator commuting with an injective compact operator with a dense range is a scalar type spectral operator. In particular, the weakly closed unital algebra generated by such an operator must be reflexive. This result seems to be unknown even for operators on a Hilbert space. Also, we show that every compact completely reducible operator must be a scalar type spectral operator. This answers a question raised by E. Azoff and A. Lubin (see the last page of [1]) and, independently, by V. Lomonosov. Finally, our result generalizes the results of Loginov and Šul′man [**2**] and Rosenthal [**5**] on reductive Hilbert space operators that commute with compact operators.

The following theorem is the central result of the author's paper [**4**], where it was stated in a slightly different form:

THEOREM 1. *Let $\mathcal{A}$ be a commutative operator algebra on a Banach space X. If the commutant of $\mathcal{A}$ is completely reducible and the ranges of compact operators in $\mathcal{A}$ span X, then every operator in $\mathcal{A}$ is a scalar type spectral operator. If, in addition, $\mathcal{A}$ is a weakly closed unital completely reducible algebra, then $\mathcal{A}$ is generated, as a uniformly closed algebra, by a complete totally atomic Boolean algebra of projections. Moreover, $\mathcal{A}$ is reflexive and admits spectral synthesis* (*i.e., every invariant subspace of $\mathcal{A}$ is spanned by its one-dimensional invariant subspaces*).

Thus, in order to prove the result described above, it suffices to show that if $\mathcal{A}$ is a commutative completely reducible algebra which has enough hyperinvariant subspaces, then the commutant of $\mathcal{A}$ is also completely reducible. This will be done in Theorem 8 below. The above result then follows easily, a sufficient supply of

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hyperinvariant subspaces being provided by Lomonsov's theorem. It can be shown, by a slight variation of the proof of Theorem 8, that the word "hyperinvariant" in its statement can be replaced with "invariant."

Let us introduce some definitions and notation. For Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the collection of all bounded linear operators from X to Y; $\mathcal{L}(X,X)$ is denoted by $\mathcal{L}(X)$. X^* means the conjugate space of the Banach space X. For $M \subseteq X$, $M^\perp$ is an annihilator of M in X^*. An operator E in $\mathcal{L}(X)$ is a projection if $E^2 = E$. If E and F are projections, we write $E \leq F$ provided $EF = FE = E$. Clearly, $E \leq F$ if and only if $E(X) \subseteq F(X)$ and $\operatorname{Ker} E \supseteq \operatorname{Ker} F$. If E is a projection, we write $E^\perp$ for $I - E$. If $\mathcal{A}$ is a subalgebra of $\mathcal{L}(X)$, then $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$; that is, the set of all operators in $\mathcal{L}(X)$ that commute with every operator in $\mathcal{A}$. Hyperinvariant subspaces of $\mathcal{A}$ are those invariant for $\mathcal{A}'$. We will write $\mathcal{P}(\mathcal{A})$ for the family of all projections in $\mathcal{A}'$, and $\mathcal{P}_0(\mathcal{A})$ for the set of those projections in $\mathcal{P}(\mathcal{A})$ whose range is hyperinvariant for $\mathcal{A}$. Finally, for E in $\mathcal{P}(\mathcal{A})$, we define $\operatorname{int}_E \mathcal{A}$ as the set of all $T \in \mathcal{L}(E^\perp(X), E(X))$ such that

$$EAET = TE^\perp AE^\perp \quad \text{for each } A \in \mathcal{A}$$

or, equivalently,

$$(A|E(X))T = T(A|E^\perp(X)) \quad \text{for each } A \in \mathcal{A}.$$

Clearly, an operator algebra $\mathcal{A}$ is completely reducible if and only if for every M in $\operatorname{lat} \mathcal{A}$ there is a projection in $\mathcal{P}(\mathcal{A})$ with range M. Note also that for $\mathcal{A}$ completely reducible and M in $\operatorname{lat} \mathcal{A}$, the restriction of $\mathcal{A}$ to M, $\mathcal{A}|M$, is also completely reducible.

We shall need some very elementary lemmas. The first is well known.

LEMMA 2. *Let X be a Banach space and let X_1 and X_2 be subspaces of X with $X_1 \dotplus X_2 = X$. Then X is isomorphic to the exterior direct sum $X_1 \oplus X_2$ defined as a vector space of ordered pairs (x_1, x_2), $x_i \in X_i$, endowed with the norm $\|(x_1, x_2)\| = \|x_1\| + \|x_2\|$.*

LEMMA 3. *Let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(X)$ and $E \in \mathcal{P}(\mathcal{A})$. Then $E^\perp(X)$ is in* $\operatorname{lat} \mathcal{A}'$ *if and only if* $\operatorname{int}_E \mathcal{A} = 0$.

PROOF. Suppose $E^\perp(X) \in \operatorname{lat} \mathcal{A}'$. For each $T \in \operatorname{int}_E \mathcal{A}$, $ETE^\perp$ is in $\mathcal{A}'$, so that $ETE^\perp = 0$ and $T = 0$. Conversely, for each $B \in \mathcal{A}'$, $EBE^\perp|E^\perp(X)$ is in $\operatorname{int}_E \mathcal{A}$, and $\operatorname{int}_E \mathcal{A} = 0$ implies $EBE^\perp = 0$, so that $E^\perp(X)$ is invariant under B.

LEMMA 4. *Let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(X)$ and $F \in \mathcal{P}_0(\mathcal{A})$. Let $X_1 = F(X)$ and* $X_2 = \operatorname{Ker} F$.

(i) *For each $M \in \operatorname{lat} \mathcal{A}'$, $F(M) \in \operatorname{lat} \mathcal{A}'$.*

(ii) *A subspace Y which contains X_1 belongs to* $\operatorname{lat} \mathcal{A}'$ *if and only if* $Y = X_1 \dotplus Y_1$, *where $Y_1 \subseteq X_2$ and $Y_1 \in \operatorname{lat}(\mathcal{A}|X_2)'$.*

PROOF. (i) For each $B \in \mathcal{A}'$, $BF(M) \subseteq F(X)$, since $F(X)$ is hyperinvariant, and $BF(M) \subseteq M$, since M is hyperinvariant and $F \in \mathcal{A}'$. Hence, $BF(M) \subseteq F(X) \cap M = F(M)$.

(ii) Let $Y \supseteq X_1$ and $Y \in \operatorname{lat} \mathcal{A}'$. In particular, Y is invariant under F, so that $Y = X_1 \dotplus Y_1$ for some $Y_1 \subseteq X_2$. For each $C \in (\mathcal{A}|X_2)'$, $F^\perp CF^\perp \in \mathcal{A}'$, and $C(Y_1) = F^\perp CF^\perp(X_1 \dotplus Y_1) \subseteq F^\perp(Y) = Y_1$; that is, $Y_1 \in \operatorname{lat}(\mathcal{A}|X_2)'$. Conversely, if $Y_1 \in \operatorname{lat}(\mathcal{A}|X_2)'$ and $B \in \mathcal{A}'$, then $F^\perp BF^\perp|X_2$ is in $(\mathcal{A}|X_2)'$ and $F^\perp BF = 0$. It follows that $B(Y) = (FB + F^\perp BF^\perp)(Y) \subseteq X_1 \dotplus Y_1 = Y$.

The following two lemmas will enable us to reduce the proof of the main result to the case when the completely reducible commutative algebra has no nonzero finite-dimensional invariant subspaces.

LEMMA 5. *Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a completely reducible algebra such that the one-dimensional subspaces in* lat $\mathcal{A}$ *span X. Then $\mathcal{A}'$ is completely reducible.*

PROOF. Clearly, $\mathcal{A}$ is commutative. We claim that $\mathcal{A}$ admits spectral synthesis. Indeed, let $M \in \operatorname{lat}\mathcal{A}$. Then there exists $F \in \mathcal{P}(\mathcal{A})$ such that $E(X) = M$. Since E transforms every one-dimensional invariant subspace of $\mathcal{A}$ into an invariant subspace of $\mathcal{A}$ of dimension no greater than one, $E(X)$ is spanned by one-dimensional elements of lat $\mathcal{A}$.

Now suppose $X_1 \in \operatorname{lat}\mathcal{A}'$. Since $\mathcal{A}' \supseteq \mathcal{A}$, X_1 is also in lat $\mathcal{A}$, and one can find $X_2 \in \operatorname{lat}\mathcal{A}$ with $X_1 \dotplus X_2 = X$. We sill show that X_2 is also in lat $\mathcal{A}'$. Suppose not. Denote by E the projection onto X_1 along X_2. Then, by Lemma 3, $\operatorname{int}_E \mathcal{A} \neq 0$, and, by our claim above, there exist such $T \in \operatorname{int}_E \mathcal{A}$ and one-dimensional $N \in \operatorname{lat}(\mathcal{A}|X_2)$ such that $M = T(N) \neq 0$. It is very easy to see that $M \in \operatorname{lat}\mathcal{A}$ and the algebra $\mathcal{A}|(M \dotplus N)$ consists only of multiples of the identity. Denote by S an operator which maps M into N and is identically zero on some invariant complement to $M \dotplus N$. Then $S \in \mathcal{A}'$, but X_1 is not invariant for S, a contradiction.

LEMMA 6. *Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a completely reducible algebra. Suppose X_1 is spanned by all one-dimensional subspaces in* lat $\mathcal{A}$ *and X_2 is in* lat $\mathcal{A}$ *with $X_1 \dotplus X_2 = X$. Then both X_1 and X_2 are in* lat $\mathcal{A}'$.

PROOF. Obviously, X_1 lies in lat $\mathcal{A}'$. Suppose X_2 does not. Then, denoting by E a projection onto X_1 along X_2, we conclude that $\operatorname{int}_E \mathcal{A} \neq 0$. Choose nonzero $T \in \operatorname{int}_E \mathcal{A}$. Since $\operatorname{cl} T(X_2) \in \operatorname{lat}(\mathcal{A}|X_1)$ and, as has been noted in the proof of the previous lemma, $\mathcal{A}|X_1$ admits spectral synthesis, we can find a one-dimensional $P \in \mathcal{P}(\mathcal{A}|X_1)$ such that $PT \neq 0$. However, $PT \in \operatorname{int}_E \mathcal{A}$, so that $\operatorname{Ker} PT \in \operatorname{lat}(\mathcal{A}|X_2)$. On the other hand, $\operatorname{codim}\operatorname{Ker} PT = 1$ and, since $\mathcal{A}|X_2$ is completely reducible, $\mathcal{A}|X_2$ has a one-dimensional invariant subspace, which contradicts the definition of X_1 and therefore completes the proof.

LEMMA 7. *Suppose $\mathcal{A} \subseteq \mathcal{L}(X)$ is a commutative completely reducible algebra which has the following property: for every nonzero $M \in \operatorname{lat}\mathcal{A}'$, there is $N \in \operatorname{lat}\mathcal{A}'$ such that $N \subseteq M$, $N \neq 0$, $N \neq M$. Let $M_1, M_2, \ldots, M_n, \ldots$ be an infinite sequence of nonzero subspaces in* lat $\mathcal{A}'$. *Then there exists such an $F \in \mathcal{P}_0(\mathcal{A})$ that $F^\perp(M_1) \neq 0$ and $F(M_n) \neq 0$ for infinitely many n.*

PROOF. Choose hyperinvariant $N \subseteq M_1$, $N \neq 0$, $N \neq M_1$. Since $\mathcal{A}' \supseteq \mathcal{A}$, $N \in \operatorname{lat}\mathcal{A}$, and there exists $P \in \mathcal{P}_0(\mathcal{A})$ with range N. Now consider two cases.

Case 1. $P(M_n) \neq 0$ for only finitely many n.

Then there is an infinite set of positive integers J such that $M_n \subseteq P^\perp(X)$ for all n in J. Let F denote a projection of $\mathcal{P}_0(\mathcal{A})$ onto a subspace $\bigvee_{n \in J} M_n$ such that $F \leq P^\perp$. Then $F(M_n) = M_n \neq 0$ for every $n \in J$ and $F^\perp(M_1) \supseteq N \neq 0$.

Case 2. $P(M_n) \neq 0$ for infinitely many n.

Then take $F = P$. Clearly, $F(M_n) \neq 0$ for infinitely many n. On the other hand, $F^\perp(M_1) \neq 0$, since $N \neq M_1$.

Now we are ready for the proof of our main result.

THEOREM 8. *Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a commutative completely reducible algebra. Suppose that for every hyperinvariant subspace M of $\mathcal{A}$ of dimension and codimension greater than 1, there exist nontrivial hyperinvariant subspaces of $\mathcal{A}$, M_1 and M_2, other than M, such that $M_1 \subseteq M \subseteq M_2$. Then $\mathcal{A}'$ is completely reducible.*

PROOF. Choose X_1 in $\operatorname{lat}\mathcal{A}'$. Since $\mathcal{A}' \supseteq \mathcal{A}$, X_1 is also in $\operatorname{lat}\mathcal{A}$ and, since $\mathcal{A}$ is completely reducible, there is in $X_2 \in \operatorname{lat}\mathcal{A}$ such that $X_1 \dotplus X_2 = X$. We claim that X_2 is in $\operatorname{lat}\mathcal{A}'$ and therefore that X_2 is the unique complement to X_1 in $\operatorname{lat}\mathcal{A}$.

The claim will be established by contradiction; suppose X_2 is not in $\operatorname{lat}\mathcal{A}'$. Denote by E the projection onto X_1 along X_2. The proof will be divided into three parts. In the first part, we shall construct two infinite sequences of pairwise orthogonal projections, $\{E_n\}_{n=1}^{\infty}$ in $(\mathcal{A}|X_2)'$ and $\{F_n\}_{n=1}^{\infty}$ in $(\mathcal{A}|X_1)' = \mathcal{A}'|X_1$, and a sequence $\{T_n\}_{n=1}^{\infty}$ in $\operatorname{int}_E \mathcal{A}$ such that $E_nT_nF_n \neq 0$ for all n.

Note that for $T \in \operatorname{int}_E \mathcal{A}$, $B \in \mathcal{A}'|X_1$, and $C \in (\mathcal{A}|X_2)'$, $BTC \in \operatorname{int}_E \mathcal{A}$. For an arbitrary projection G in $\mathcal{L}(X_2)$ let $M(G)$ denote the subspace of X_1 spanned by all $TG(X_2)$ with $T \in \operatorname{int}_E \mathcal{A}$. Clearly, $M(G)$ is always in $\operatorname{lat}(\mathcal{A}'|X_1)$.

Now denote by Y the intersection of the kernels of all operators in $\operatorname{int}_E \mathcal{A}$. By our assumption that X_2 is not in $\operatorname{lat}\mathcal{A}'$ and Lemma 3, it follows that $Y \neq X_2$. On the other hand, Y lies in $\operatorname{lat}(\mathcal{A}|X_2)'$ and, by Lemma 4(ii), $X_1 \dotplus Y$ lies in $\operatorname{lat}\mathcal{A}'$. Let Q be a projection in $\mathcal{P}(\mathcal{A}|X_2)$ onto a subspace which is complementary to Y in X_2. By hypothesis, $X_1 \dotplus Y$ is contained in some larger nontrivial hyperinvariant subspace of $\mathcal{A}$. By lemma 4(ii), this larger subspace has the form $X_1 \dotplus Y \dotplus E_1(X_2)$ for some nonzero $E_1 \in \mathcal{P}(\mathcal{A}|X_2)$, $E_1 \leq Q$, $E_1 \neq Q$. Repeating the same argument, one can find nonzero $E_2 \in \mathcal{P}(\mathcal{A}|X_2)$ with $E_2 \leq Q - E_1$, $E_2 \neq Q - E_1$. Proceeding by induction, we get an infinite sequence $\{E_n\}_{n=1}^{\infty}$ of pairwise orthogonal nonzero projections in $P(\mathcal{A}|X_2)$ with $E_n \leq Q$ for all n. It follows from the definition of Q that $M(E_n) \neq 0$ for all n.

Now Lemma 7 provides $G_1 \in \mathcal{P}_0(\mathcal{A}|X_1)$ such that $G_1^{\perp}M(E_1) \neq 0$ and $G_1M(E_n) \neq 0$ for every n in the infinite set J of positive integers.

Renumber the elements of J_1 by $2, 3, \ldots$. By Lemma 4(i), $G_1M(E_n) \in \operatorname{lat}(\mathcal{A}|X_1)'$ for $n \geq 2$. Since

$$((\mathcal{A}|G_1(X_1))' = \mathcal{A}|G_1(X_1),$$

we may again apply Lemma 7 to the algebra $\mathcal{A}|G_1(X_1)$ and a sequence $G_1M(E_2)$, $G_1M(E_3), \ldots$ of its hyperinvariant subspaces. As a result, we obtain $G_2 \in \mathcal{P}_0(\mathcal{A}X_1)$ such that $G_2 \leq G_1$, $(G_1 - G_2)G_1M(E_2) = (G_1 - G_2)M(E_2) \neq 0$, and $G_2G_1M(E_n) = G_2M(E_n) \neq 0$ for every n from an infinite subset J_2 of J_1.

Proceeding by induction (renumbering the elements of J_n by $n+1, n+2, \ldots$), we get a sequence $I = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n \geq \cdots$, where $G_n \in \mathcal{P}_0(\mathcal{A}|X_1)$ and

$$(G_{n-1} - G_n)M(E_n) \neq 0, \qquad n = 1, 2, \ldots.$$

Let $F_n = G_{n-1} - G_n$, $n = 1, 2, \ldots$. The F_n's are pairwise orthogonal projections in $\mathcal{P}(\mathcal{A}|X_1)$, and $F_nM(E_n) \neq 0$ for $n = 1, 2, \ldots$. Finally, from the definition of $M(E_n)$, it follows that there is a sequence $\{T_n\}_{n=1}^{\infty}$ in $\operatorname{int}_E \mathcal{A}$ such that $F_nT_nE_n \neq 0$ for all n.

In the second part of the proof we will construct a closed unbounded linear transformation T defined on the linear manifold $\mathcal{D} \subseteq X_1$, with range in X_2, such that its graph $\{Tx + x,\ x \in \mathcal{D}\}$ is invariant under $\mathcal{A}$. For this, define an operator $T_0 \in \mathcal{L}(X_2, X_1)$ as follows:

$$T_0 = \sum_{n=1}^{\infty} 2^{-n} \|F_n T_n E_n\|^{-1} F_n T_n E_n,$$

where the series converges in the sense of the norm in $\mathcal{L}(X_2, X_1)$. It is easy to see that $T_0 \in \operatorname{int}_E \mathcal{A}$. Now let $L = \bigcap_{n=1}^{\infty} \operatorname{Ker} F_n$. Then $L \in \operatorname{lat}(\mathcal{A}|X_1)$; let N be in $\operatorname{lat}(\mathcal{A}|X_1)$ with $L \dotplus N = X_1$. Denote by P the projection onto N along L. Let us observe that for every n the operator $PF_n|F_n(X_1)$ is injective; it follows that $PF_nT_nE_n \neq 0$ for $n \geq 1$. Now define $S \in (\mathcal{A}|X_2)'$ as follows:

$$S = \sum_{n=1}^{\infty} 2^{-2n} \|PF_nT_nE_n\|\, \|F_nT_nE_n\|^{-1} \|E_n\|^{-1} E_n$$

(again, the series is convergent in the sense of the norm).

We claim that the subspace

$$M = \operatorname{cl}\{(PT_0x, Sx),\ x \in X_2\} \subseteq X_1 \oplus X_2$$

is a graph of some linear transformation $T\colon \mathcal{D} \to X_1$, $\mathcal{D} \subseteq X_2$.

Indeed, the conjugate space to $X_1 \oplus X_2$ is a linear space of vectors (x_1^*, x_2^*), $x_i^* \in X_i^*$, endowed with the norm

$$\|(x_1^*, x_2^*)\| = \sup(\|x_1^*\|, \|x_2^*\|).$$

It is easy to see that $(x_1^*, x_2^*) \in M^\perp$ if and only if $T_0^*P^*x_1^* + S^*x_2^* = 0$. Note that, by the definition of L,

$$\begin{aligned}\text{weak}^*\text{cl}\left(\bigvee_{n=1}^{\infty} F_n^*(X_1^*)\right) &= \text{weak}^*\text{cl}\left(\bigvee_{n=1}^{\infty} (\operatorname{Ker} F_n)^\perp\right) \\ &= \left(\bigcap_{n=1}^{\infty} \operatorname{Ker} F_n\right)^\perp = L^\perp.\end{aligned}$$

Note also that $L^\perp = P^*(X_1^*)$, $N^\perp = \operatorname{Ker} P^*$, and $L^\perp \dotplus N^\perp = X_1^*$. Let

$$\mathcal{L} = \bigcup_{n=1}^{\infty} \left(\sum_{i=1}^{n} F_i^*(X_1^*)\right).$$

Clearly, $\mathcal{L}$ is weak* dense in $L^\perp$. Now, for $m \geq 1$,

$$\begin{aligned}T_0^*P^*F_m^* = T_0^*F_m^* &= \left(\sum_{n=1}^{\infty} 2^{-n}\|F_nT_nE_n\|^{-1} E_n^*T_n^*F_n^*\right) F_m^* \\ &= 2^{-m}\|F_mT_mE_m\|^{-1} E_m^*T_m^*F_m^*.\end{aligned}$$

Similarly, we conclude that $E_m^*(X_2^*)$ is contained in the range of S^* for $m \geq 1$. It follows that for each $x_1^* \in \mathcal{L} + N^\perp$ there exists $x_2^* \in X_2^*$ such that $T_0^*P^*x_1^* + S^*x_2^* = 0$ (if $x_1^* \in N^\perp$, then $P^*x_1^* = 0$ and we can take $x_2^* = 0$), or $(x_1^*, x_2^*) \in M^\perp$. Now suppose $(x, 0) \in M$ for some $x \in X_1$. Then, for each $x_1^* \in \mathcal{L} + N^\perp$, $x_1^*(x) = 0$, and, since $\mathcal{L} + N^\perp$ is weak* dense in X_1^*, $x = 0$. This proves our claim.

Now we shall show that T is unbounded. Indeed, for every $m \geq 1$, $E_m(X_2) \subseteq S(X_2)$ and therefore $E_m(X_2) \subseteq \mathcal{D}$. Furthermore, $TSE_m = PT_0E_m$, or

$$T(2^{-2m}\|PF_mT_mE_m\|\,\|F_mT_mE_m\|^{-1}\|E_m\|^{-1})E_m = 2^{-m}\|F_mT_mE_m\|^{-1}PF_mT_mE_m.$$

Since $PF_mT_mE_m \neq 0$,

$$TE_m = 2^m\|E_m\|\,\|PF_mT_mE_m\|^{-1}PF_mT_mE_m.$$

Hence, $\|TE_m\| = 2^m\|E_m\|$, which proves that T is unbounded.

Now let M_0 be the closure of $\{PT_0x + Sx,\ x \in X_2\}$ in X. Lemma 2 allows us to identify M_0 with M. That is, we may suppose that $M_0 = \{Tx + x,\ x \in \mathcal{D}\}$; in particular, $M_0 \cap X_1 = 0$ and $(M_0 + X_1 \cap X_2 = \mathcal{D}$. For each A in $\mathcal{A}$, $A = EAE + E^\perp AE^\perp$; it follows, since $PT_0 \in \operatorname{int}_E \mathcal{A}$ and $S \in (\mathcal{A}|X_2)'$, that for $x \in X_2$,

$$\begin{aligned} A(PT_0x + Sx) &= (EAE + E^\perp AE^\perp)(PT_0x + Sx) \\ &= EAEPT_0x + E^\perp AE^\perp Sx = PT_0E^\perp AE^\perp x + SE^\perp AE^\perp x, \end{aligned}$$

which shows that $M_0 \in \operatorname{lat} \mathcal{A}$. This ends the second part of the proof.

In the last part of the proof we obtain a contradiction. To do this, it would suffice to refer to a simple result of Fong [**1**], but we prefer to give a direct proof.

Since $M_0 \in \operatorname{lat} \mathcal{A}$ and $\mathcal{A}$ is completely reducible, one can find $M_1 \in \operatorname{lat} \mathcal{A}$ such that $M_0 \dot{+} M_1 = X$. Let E_0 denote the projection onto M_0 along M_1. Then $E_0 \in \mathcal{A}'$ and hence X_1 is invariant under E_0. This implies that

$$X_1 = X_1 \cap M_0 \dot{+} X_1 \cap M_1.$$

However, as we have seen, $X_1 \cap M_0 = 0$; that means $X_1 \subseteq M_1$. From the fact that $M_0 \dot{+} M_1 = X$ and Lemma 2, it follows that the manifold $M_0 + X_1$ is closed. Then $\mathcal{D} = (M_0 + X_1) \cap X_2$ is also closed. But $\mathcal{D}$ is the domain of definition for a closed unbounded transformation T, and, by the Closed Graph Theorem, cannot be closed. This contradiction completes the proof of the theorem.

THEOREM 9. *Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a commutative unital weakly closed completely reducible algebra. Suppose that the intersection of the kernels of all compact operators in $\mathcal{A}'$ is zero and the subspace spanned by ranges of all compact operators in $\mathcal{A}'$ is X. Then $\mathcal{A}$ is generated, as a uniformly closed algebra, by a complete bounded totally atomic Boolean algebra of projections; in particular, $\mathcal{A}$ is an algebra of scalar type spectral operators. Furthermore, $\mathcal{A}$ is reflexive and admits spectral synthesis.*

PROOF. Let X_1 denote the subspace spanned by all one-dimensional subspaces in $\operatorname{lat} \mathcal{A}$, and let X_2 be a complement to X_1 in $\operatorname{lat} \mathcal{A}$. By Lemma 6, X_1 and X_2 are in $\operatorname{lat} \mathcal{A}'$. We shall show that for $\mathcal{A}|X_2$ the conditions of the previous theorem are satisfied. Denote by $\mathcal{C}$ the family of all compact operators in $(\mathcal{A}|X_2)'$. Clearly, intersections of the kernels of all operators in $\mathcal{C}$ is zero, and the subspace spanned by all their ranges is X_2. Let M be a nonzero subspace in $\operatorname{lat}(\mathcal{A}|X_2)'$ and E be in $\mathcal{P}_0(\mathcal{A}|X_2)$ with $E(X_2) = M$. Then there is $K_1 \in \mathcal{C}$ such that $K_1|M \neq 0$ and (note that M is infinite-dimensional), by Lomonosov's theorem [**3**], there is a nonzero $M_1 \subseteq M$ such that $M_1 \in \operatorname{lat}(\mathcal{A}|X_2)'$ and $M_1 \neq M$. On the other hand, there exists $K_2 \in \mathcal{C}$ such that $E^\perp K_2E^\perp \neq 0$, for otherwise $E^\perp KE = E^\perp KE^\perp = 0$ for each K in $\mathcal{C}$, hence $K = EK$ and $K(X_2) \subseteq E(X_2)$, which contradicts our hypothesis. Again, by Lomonosov's theorem, the algebra $\mathcal{A}|E^{\perp}(X_2)$ has a nontrivial hyperinvariant subspace, and now Lemma 4(ii) implies that $\mathcal{A}|X_2$ has a nontrivial hyperinvariant subspace M_2 strictly containing M. So the conditions of Theorem 8 are satisfied for $\mathcal{A}|X_2$.

Choose a subspace in lat $\mathcal{A}'$. Clearly, it can be written as $M \dotplus N$, where $M \subseteq X_1$ and $N \subseteq X_2$. By Lemma 5, there exists M_1 in $\operatorname{lat}(\mathcal{A}|X_1)'$ such that $M \dotplus M_1 = X_1$, and, by Theorem 8, we can find $N_1 \in \operatorname{lat}(\mathcal{A}|x_2)'$ such that $N \dotplus N_1 = X_2$. But then $M_1 \dotplus N_1$ lies in lat $\mathcal{A}'$ and $(M \dotplus N) \dotplus (M_1 \dotplus N_1) = X$. It follows that $\mathcal{A}'$ is completely reducible. Now, to conclude the proof of the theorem, it suffices to apply Theorem 1.

The following corollary follows immediately from Theorem 9.

COROLLARY 10. *Let $T \in \mathcal{L}(X)$ be a completely reducible operator. If $TK = KT$, where K is an injective compact operator such that* $\operatorname{cl} K(X) = X$, *then T is a scalar type spectral operator, and spectral synthesis holds for T.*

COROLLARY 11. *Let $\mathcal{A} \subseteq \mathcal{L}(X)$ be a commutative unital weakly closed completely reducible algebra. If the intersection of the kernels of all the compact operators in $\mathcal{A}$ is zero, or if the ranges of all the compact operators in $\mathcal{A}$ span X, then the conclusions of Theorem* 9 *hold for $\mathcal{A}$.*

PROOF. We shall show that the assumpition about the kernels is equivalent to that about the ranges. Then the result would be an immediate consequence of Theorem 9. Let M be intersection of the kernels of all compact operators in $\mathcal{A}$. Clearly, $M \in \operatorname{lat} \mathcal{A}$. Let $N \in \operatorname{lat} \mathcal{A}$ be such that $M \dotplus N = X$. Let N_0 be the subspace spanned by all $K(N)$, where K runs over the set of all compact operators in $\mathcal{A}$. Obviously, $N_0 \in \operatorname{lat} \mathcal{A}$ and $N_0 \subseteq N$. Let $N_1 \in \operatorname{lat} \mathcal{A}$, $N_0 \dotplus N_1 = N$. Then, by the definition of N_0, all the compact operators in $\mathcal{A}$ vanish on N_1; that is, $N_1 \subseteq M \cap N = 0$ and $N_0 = N$. On the other hand, the range of every compact operator in $\mathcal{A}$ is contained in N. It follows that the subspace spanned by the ranges of the compact operators in $\mathcal{A}$ is exactly N. But $M = 0$ implies $N = X$, and vice versa.

To end this paper, we give a characterization of completely reducible compact operators.

COROLLARY 12. *Every compact, completely reducible operator $K \in \mathcal{L}(X)$ is a scalar type spectral operator.*

PROOF. It suffices to note that $\operatorname{Ker} K \dotplus \operatorname{cl} K(X) = X$ [**1**].

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