Completeness by Forcing

Daniel Găină and Marius Petria

Japan Advanced Institute of Science and Technology
University of Edinburgh
daniel@jaist.ac.jp, Marius.Petria@ed.ac.uk

Abstract. The completeness of the infinitary language $\mathcal{L}_{\omega_1,\omega}$ was proved by Carol Karp in 1964. We express and prove the completeness of infinitary first-order logics in the institution-independent setting by using forcing, a powerful method for constructing models. As a consequence of this abstraction, the completeness theorem becomes available for the infinitary versions of many "first order" logical systems that appear in the area of logic or computer science.

1 Introduction

The goal of this paper is to prove the completeness theorem for infinitary first order logics in the framework of institutions and then apply the general results to a couple of different concrete institutions. Although we emphasize the results obtained for infinitary logics, we will also obtain completeness results for finitary logics. However our results for the finitary case is weaker than the known completeness results for finitary first order logic [17] as it requires a countable number of of symbols in a signature. A paper with similar objectives dedicated to finitary logics that captures the case of uncountable signatures is [21].
The institutional approach to solving completeness problems has two main motivations. Firstly, it allows us to obtain in an uniform way completeness results for the infinitary versions of many specific logics like first order logic, order sorted logic, or logics for partial algebras.
Secondly, the pattern of institutional reasoning is to find categorical definitions of the conditions that are sufficient for proving the desired results. The small and natural set of conditions that we identify for the proof of Theorem 2 helps in understanding at an abstract level "why" an infinitary logic is complete. As in [21], we separate the deduction rules into two sets, rules that deal with the specific syntax of the atomic sentences and rules that deal with first order connectives and with quantification. This is a first step towards the idea that "first order logics" freely extend their "atomic" subpart in all regards: syntactically, semantically and as proof systems, and soundness and completeness are preserved by this free extension. A way to formalize this by means of institution morphisms is left to future investigation.
The present paper introduces the forcing technique in institution-independent model theory, applies it for proving the completeness result, and points out some particular cases. One of the most important contribution of our study is the formalization of forcing in abstract model theory, thus providing an efficient tool for obtaining new results and showing the significance of the top-down approach towards model theory. Forcing is a technique invented by Paul Cohen, for proving consistency and independence results in set theory [4, 5]. A. Robinson [23]

developed an analogous theory of forcing in model theory, and Barwise [2] extended Robinson's theory to infinitary logic and used it to give a new proof of Omitting Types Theorem. A general treatment of the Omitting Types Theorem may be found also in [19].

In Section 2 we present the notions of institution and entailment system. These will constitute the base for expressing the soundness and completeness condition for a logic, i.e. the semantic entailment is the same as the syntactic entailment. After presenting some well known examples of institutions we give the definitions of some important concepts that are used in this paper. Section 3 shows how logical connectives and their meaning can be attached to the definition of institution. In Section 4 we present an institutional version of the definition of a forcing relation. The main theorem of this section says that every generic set of properties has a model. In Section 5 we do the last step in setting the framework for expressing the completeness: we define what is a first order entailment system. The completeness theorem will be proved in the next section, i.e. Section 6. The last section relevant to this result, Section 7, lists a number of examples and shows how the abstract theorem of completeness can be instantiated to concrete cases.

2 Preliminaries

The theory of institutions [13] is a categorical abstract model theory which formalizes the intuitive notion of logical system, including syntax, semantics, and the satisfaction between them.

The concept of institution arose within computing science (algebraic specification) in response to the population explosion among logics in use there, with the ambition of doing as much as possible at a level of abstraction independent of commitment to any particular logic [13]. Besides its extensive use in specification theory (it has become the most fundamental mathematical structure in algebraic specification theory), there have been several substantial developments towards an "institution-independent" (abstract) model theory [24, 7, 9, 16, 22, 12]. A textbook dedicated to this topic is [11]. Apart from reformulation of standard concepts and results in a very general setting, thus applicable to many logical systems, institution-independent model theory has already produced a number of new significant results in classical model theory [9, 16].

Definition 1. *An institution [13] consists of:*

1. *a category* $\mathbb{S}ig$*, whose objects are called* signatures.
2. *a functor* $\mathbb{S}en : \mathbb{S}ig \to \mathbb{S}et$*, providing for each signature a set whose elements are called* (Σ-)sentences.
3. *a functor* $\mathbb{M}od : \mathbb{S}ig \longrightarrow Cat^{op}$*, providing for each signature* Σ *a category whose objects are called* (Σ-)models *and whose arrows are called* (Σ-)morphisms.
4. *a relation* $\models_\Sigma \subseteq |\mathbb{M}od(\Sigma)| \times \mathbb{S}en(\Sigma)$ *for each* $\Sigma \in |\mathbb{S}ig|$*, called* ($\Sigma$-)satisfaction*, such that for each morphism* $\varphi : \Sigma \longrightarrow \Sigma'$ *in* $\mathbb{S}ig$*, the* satisfaction condition

$$M' \models_{\Sigma'} \mathbb{S}en(\varphi)(e) \text{ iff } \mathbb{M}od(\varphi)(M') \models_\Sigma e$$

holds for all $M' \in |\mathbb{M}od(\Sigma')|$ *and* $e \in \mathbb{S}en(\Sigma)$.

The proof theoretic aspect of a logic can be captured by the notion of entailment system.

Definition 2. *[20] An entailment system $\vdash$ for a sentence system $(\mathbb{S}ig, \mathbb{S}en)$ is a family of relations $\{\vdash_\Sigma\}_{\Sigma \in \mathbb{S}ig}$ between sets of sentences with the following properties:*

- *anti-monotonicity: $E_1 \vdash_\Sigma E_2$ if $E_2 \subseteq E_1$ for all signatures Σ ,*
- *transitivity: $E_1 \vdash_\Sigma E_3$ if $E_1 \vdash_\Sigma E_2$ and $E_2 \vdash_\Sigma E_3$ for all signatures Σ ,*
- *unions: $E_1 \vdash_\Sigma E_2 \cup E_3$ if $E_1 \vdash_\Sigma E_2$ and $E_1 \vdash_\Sigma E_3$.for all signatures Σ,*
- *translation: $E \vdash_\Sigma E'$ implies $\varphi(E) \vdash_{\Sigma'} \varphi(E')$ for all $\varphi : \Sigma \longrightarrow \Sigma'$*

Notice that the satisfaction relation of any institution can be seen as an entailment system, called the semantic entailment system.
An entailment system $(\mathbb{S}ig, \mathbb{S}en, \vdash)$ is *sound* (resp. *complete*) for an institution $(\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ if $\Gamma \vdash_\Sigma \rho$ implies $\Gamma \models_\Sigma \rho$ (resp. $\Gamma \models_\Sigma \rho$ implies $\Gamma \vdash_\Sigma \rho$) for every signature Σ, all sets of Σ-sentences Γ and any Σ-sentence ρ.

Definition 3. *Let $(\mathbb{S}ig, \mathbb{S}en, \vdash)$ be an entailment system. We say that $\vdash_\Sigma$ is* compact *for a set of Σ-sentences Γ if for any set of sentences $E \subseteq \Gamma$ and any sentence $e \in \Gamma$ if $E \vdash e$ then there is $E_f \subseteq E$ finite such that $E_f \vdash e$. We say that the entailment system is* compact *if $\vdash_\Sigma$ is compact for $\mathbb{S}en(\Sigma)$ for every $\Sigma \in \mathbb{S}ig$.*

*Example 1 (First order logic (***FOL***)).* Its signatures (S, F, P) consist of a set of sorts S, a set F of function symbols, and a set P of relation symbols. Each function or relation symbol comes with a string of argument sorts, called its *arity*, and for functions symbols, a result sort.
Simple signature morphisms map the three components in a compatible way. In order to treat substitutions as signature morphisms we will work in this paper with a more powerful version of signature morphisms. A *generalized* **FOL**-*morphism* between (S, F, P) and (S', F', P') is a simple signature morphism between (S, F, P) and $(S', F' + T_{F'}, P')$, i.e. constants can be mapped to terms.
Models M are first order structures interpreting each sort symbol s as a set M_s, each function symbol σ as a total function M_σ from the product of the interpretations of the argument sorts to the interpretation of the result sort, and each relation symbol π as a subset M_π of the product of the interpretations of the argument sorts. A model morphism $h : M \to N$ is a family of functions $\{h_s : M_s \to N_s\}_{s \in S}$ indexed by the sets of sorts of the signature such that: $h_s(M_\sigma(m)) = N_\sigma(h_w(m))$ for each $\sigma : w \to s$ and each $m \in M_w$ and $h_w(M_\pi) \subseteq N_\pi$ for each $\pi : w$.
Note that each sort interpretation M_s is non-empty since it contains the interpretation of at least one term.
Sentences are the usual closed first order formulas built from equational and relational atoms by iterative application of logical connectives (negation and disjunction), and existential quantifiers over a finite number of variables. We assume that the quantification is done over variables with non-empty sorts, i.e. sorts which contain at least one term. For example if (S, F, P) is a first order signature and $(\exists x : s)e$ is a (S, F, P)-sentence then there exists a term $t \in (T_F)_s$ with the sort s.
Sentence translations rename the sort, function, and relation symbols. For each signature morphism φ, the reduct $M' \upharpoonright_\varphi$ of a model M' is defined by $(M' \upharpoonright_\varphi)_x = M'_{\varphi(x)}$ for each sort, function, or relation symbol x from the domain signature of φ.
The satisfaction of sentences by models is the usual Tarskian satisfaction defined inductively on the structure of sentences.

*Example 2 (First order equational logic (**FOEQL**)).* Obtained from **FOL** by discarding both the relation symbols and their interpretation in models.

*Example 3 (Quantifier-free first order logic (**QfFOL**)).* This institution is the restriction of **FOL** to the sentences formed without quantifiers.

Example 4. Let **EQLN** be the fragment of **FOEQL**, allowing sentences obtained from atoms and negations of atoms through only one round of quantification, either universal or existential, over a set of variables. More precisely, all sentences have the form $(Q\,X)t_1 \; k \; t_2$ where $Q \in \{\exists, \forall\}$ and $k \in \{=, \neq\}$.

*Example 5 (Preorder algebra (**POA**)).* The **POA** signatures are just the ordinary algebraic signatures. The **POA** models are *preordered algebras* which are interpretations of the signatures into the category of preorders $\mathbb{P}re$ rather than the category of sets $\mathbb{S}et$. This means that each sort gets interpreted as a preorder, and each operation as a preorder functor, which means a preorder-preserving (i.e. monotonic) function. A *preordered algebra homomorphism* is just a family of preorder functors (preorder-preserving functions) which is also an algebra homomorphism.
The sentences have two kinds of atoms: equations and *preorder atoms*. A preorder atom $t \leq t'$ is satisfied by a preorder algebra M when the interpretations of the terms are in the preorder relation of the carrier, i.e. $M_t \leq M_{t'}$. Full sentences are constructed from equational and preorder atoms by using Boolean connectives and first order quantification.

*Example 6 (Order sorted algebra (**OSA**)).* An order-sorted signature $(S, \leq, F)$ consists of an algebraic signature (S, F), with a partial ordering $(S, \leq)$ such that the following *monotonicity condition* is satisfied

$$\sigma \in F_{w_1 \to s_1} \cap F_{w_2 \to s_2} \text{ and } w_1 \leq w_2 \text{ imply } s_1 \leq s_2$$

A morphism of **OSA** signatures $\varphi : (S, \leq, F) \to (S', \leq', F')$ is just a morphism of algebraic signatures $(S, F) \to (S', F')$ such that the ordering is preserved, i.e. $\varphi(s_1) \leq' \varphi(s_2)$ whenever $s_1 \leq s_2$.
Given an order-sorted signature $(S, \leq, F)$, an order-sorted $(S, \leq, F)$-algebra is a (S, F)-algebra M such that

- $s_1 \leq s_2$ implies $M_{s_1} \subseteq M_{s_2}$, and
- $\sigma \in F_{w_1 \to s_1} \cup F_{w_2 \to s_2}$ and $w_1 \leq w_2$ imply $M_\sigma^{w_1, s_1} = M_\sigma^{w_2, s_2}$ on M_{w_1}.

Given order-sorted $(S, \leq, F)$-algebras M and N, an order-sorted $(S, \leq, F)$-morphism $h : M \to N$ is a (S, F)-morphism such that $s_1 \leq s_2$ implies $h_{s_1} = h_{s_2}$ on M_{s_1}.
An order-sorted signature $(S, \leq, F)$ is *regular* iff for each $\sigma \in F_{w_1 \to s_1}$ and each $w_0 \leq w_1$ there is a unique least element in the set $\{(w, s) \mid \sigma \in F_{w \to s} \text{ and } w_0 \leq w\}$.

Remark 1. For regular signatures $(S, \leq, F)$, any F-term t has a least sort $LS(t)$ and the initial $(S, \leq, F)$-algebra can be defined as a term algebra, cf. [15].

Proof. We proceed by induction on the structure of the term t. If $t \in F_{\to s_1}$ then by regularity with $w_0 = w_1 = \lambda$ there is a least $s \in S$ such that $t \in F_{\to s}$; this is the least sort of t. If $t = \sigma(t_1, \ldots, t_n) \in (T_F)_s$ then by induction hypothesis each t_i has a least sort, say s_i; let $w_0 = s_1 \ldots s_n$. Then $\sigma \in F_{w' \to s'}$ for some pair $(w', s') \in S^* \times S$ with $s' \leq s$ and $w_0 \leq w'$. By regularity, there exists least pair $(w'', s'') \in S^* \times S$ such that $\sigma \in F_{w'' \to s''}$; this s'' is the desired least sort of t.

Let $(S, \leq, F)$ be an order-sorted signature. We say that the sorts s_1 and s_2 are in the same *connected component* of S iff $s_1 \equiv s_2$, where $\equiv$ is the least equivalence on S that contains $\leq$. A partial ordering $(S, \leq)$ is *filtered* iff for all $s_1, s_2 \in S$, there is some $s \in S$ such that $s_1 \leq s$ and $s_2 \leq s$. A partial ordering is *locally filtered* iff every connected component of it is filtered. An order-sorted signature $(S, \leq, F)$ is *locally filtered* iff $(S, \leq)$ is locally filtered, and it is *coherent* iff it is both locally filtered and regular. Hereafter we assume that all **OSA** signatures are coherent.
The atoms of the signature $(S, \leq, F)$ are equations of the form $t_1 = t_2$ such that the least sort of the terms t_1 and t_2 are in the same connected component. The sentences are closed formulas built by application of Boolean connectives and quantification to the equational atoms. Order-sorted algebras were extensively studied in [14, 15].

Example 7. [Partial algebra (**PA**)] [3]. A partial algebraic signature (S, F) consists of a set S of sorts and a set F of partial operations. We assume that there is a distinguished constant on each sort $\perp_s : s$. Signature morphisms map the sorts and operations in a compatible way, preserving $\perp_s$; we also allow that constants can be mapped to terms.
A partial algebra is just like an ordinary algebra but interpreting the operations of F as partial rather than total functions; $\perp_s$ is always interpreted as undefined. A *partial algebra homomorphism* $h : A \to B$ is a family of (total) functions $\{h_s : A_s \to B_s\}_{s \in S}$ indexed by the set of sorts S of the signature such that $h_s(A_\sigma(a)) = B_\sigma(h_w(a))$ for each operation $\sigma : w \to s$ and each string of arguments $a \in A_w$ for which $A_\sigma(a)$ is defined.
We consider one kind of "base" sentences: *existence equality* $t \stackrel{e}{=} t'$. The existence equality $t \stackrel{e}{=} t'$ holds when both terms are defined and are equal. The definedness predicate and strong equality can be introduced as notations: $def(t)$ stands for $t \stackrel{e}{=} t$ and $t \stackrel{s}{=} t'$ stands for $(t \stackrel{e}{=} t') \vee (\neg def(t) \wedge \neg def(t'))$.
The sentences are formed from these "base" sentences by logical connectives and quantification over variables.
We consider the atomic sentences in $\mathbb{S}en(S, F)$ to be the atomic existential equalities that do not contain $\perp_s$.

Example 8 (Infinitary logic $\mathbf{FOL}_{\omega_1,\omega}$*).* This is the infinitary version of first order logic allowing conjunctions of arbitrary sets of sentences.
Similarly we may define $\mathbf{QfFOL}_{\omega_1}$, $\mathbf{POA}_{\omega_1,\omega}$, $\mathbf{OSA}_{\omega_1,\omega}$ and $\mathbf{PA}_{\omega_1,\omega}$.

Definition 4. *A set of sentences $E \subseteq Sen(\Sigma)$ is called* basic *[6] if there exists a Σ-model M_E, called a* basic model*, such that for all Σ-models M, $M \models E$ iff there exists a homomorphism $M_E \to M$. A set of sentences Γ is called* part-basic *if all its subsets $E \subseteq \Gamma$ are basic.*

The concept of basic sentence constitute the best institution-independent approximation of the actual atoms of the logic.

Lemma 1. *Any set of atomic sentences in* **FOL***,* **POA***,* **OSA** *and* **PA** *is basic.*

Proof. In **FOL** the basic model M_E for a set E of atomic (S, F, P)-sentences is constructed a follows: on the quotient $(T_F)/_{\equiv_E}$ of the term model T_F by the congruence generated by the equational atoms of E, we interpret each relation symbol $\pi \in P$ by $(M_E)_\pi = \{(t_1/_{\equiv_E}, \ldots, t_n/_{\equiv_E}) \mid \pi(t_1, \ldots, t_n) \in E\}$. A similar argument as the preceding holds for **POA** and **OSA**.

In **PA** for a set of atomic sentences E we define T_E to be the set of sub-terms appearing in E. Note that T_E is a partial algebra. The basic model M_E will be the quotient of T_E by the partial congruence induced by the equalities from E.

Definition 5. *Consider two signature morphisms* $\chi_1 : \Sigma \to \Sigma_1$ *and* $\chi_2 : \Sigma \to \Sigma_2$. *A signature morphisms* $\theta : \Sigma_1 \to \Sigma_2$ *such that* $\chi_1;\theta = \chi_2$ *is called a substitution morphism between* χ_1 *and* χ_2.

As implied by the choice of the signature morphisms in the example we plan to treat substitutions as morphisms in a comma category of signature morphisms. A more general treatment of substitutions may be found in [8].

Definition 6. *Let* $\mathcal{D}$ *be a subcategory of signature morphism. We say that a* Σ-*model* M *is* $\mathcal{D}$-*reachable if for each signature morphism* $\chi : \Sigma \to \Sigma'$ *in* $\mathcal{D}$, *each* χ-*expansion* M' *of* M *determines a substitution* $\theta : \chi \to 1_\Sigma$ *such that* $M \upharpoonright_\theta = M'$.

In concrete examples of institutions $\mathcal{D}$ consists of signature morphisms used for quantifications, i.e. extensions of signatures with finite number of constants. A model M is reachable if the elements of M are exactly the interpretations of the terms.

Proposition 1. *For any set* E *of atomic sentences in* **FOL**, **POA**, **OSA** *and* **PA**, *the model* M_E *is* $\mathcal{D}$-*reachable where* $\mathcal{D}$ *is the class of signature morphisms used for quantification, i.e. signature extensions with finite number of constants.*

Proof. In **FOL** consider an extension M' of a basic model M_E along a signature morphism in $\mathcal{D}$. Note that M_E is a quotient of the algebra of terms Each constant added by the signature morphism from $\mathcal{D}$ is interpreted by an element of M_E; this map between constants and terms builds a generalized signature morphism as needed. This kind of argument may be replicated for **POA** and **OSA** too.
In **PA** consider an extension M' of a basic model M_E along a signature morphism in $\mathcal{D}$. For any added constant x we must find a mapping into terms. If M'_x is defined then the value of its interpretation is an isomorphism class (modulo the equations from E) of terms from S_E. We can map x to any of the terms from this isomorphism class. Otherwise, if M'_x is undefined we map x to $\perp_s$.

The main result is obtained under the assumption that signatures morphisms used for quantifications do not add constant symbols on "void sorts". We express this requirement abstractly by the following definition.

Definition 7. *A signature morphism* $\chi : \Sigma \to \Sigma'$ *is non-void if there exists a substitution* $\theta : \chi \to 1_\Sigma$.

In **FOL** with generalized signature morphisms the non-void quantification translates into accepting extensions of signatures with finite number of constants of non-empty sorts. If we accept only signatures with non-empty sorts (for each sort there exists at least one term), signatures which are sensible [18], then all the extensions of signatures with constants are non-void.

Definition 8. *In any institution a* Σ-*sentence* ρ *is* finitary *iff it can be written as* $\phi(\rho_f)$ *where* $\phi : \Sigma_f \to \Sigma$ *is a signature morphism such that* Σ_f *is a finitely presented signature* [1] *and* ρ_f *is a* Σ_f *sentence.*
An institution has finitary sentences *when all its sentences are finitary.*

[1] An object A in a category $\mathbb{C}$ is called *finitely presented* ([1]) if

This condition usually means that the sentences contain only a finite number of symbols. This is the case of **FOL**, **QfFOL**, **EQLN**, **POA**, **OSA** and **PA**.

Definition 9. *We say that a signature morphism* $\varphi : \Sigma \to \Sigma'$ *is finitary if it is finitely presented in the category* $\Sigma/\mathbb{S}ig$.

In typical institutions the extension of signatures with finitely numbers of symbols are finitary.

3 Internal logic

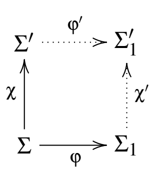
The logical connectives and quantification can be defined generically in any institution.

Definition 10. *[24] In any institution*

1. *a sentence* $\rho \in \mathbb{S}en(\Sigma)$ *is called a semantic negation of a sentence* $\rho_0 \in \mathbb{S}en(\Sigma)$ *when for every model* $M \in \mathbb{M}od(\Sigma)$ *we have* $M \models \rho$ *iff* $M \not\models \rho_0$.
2. *a sentence* $\rho \in \mathbb{S}en(\Sigma)$ *is called a semantic disjunction of two sentences* $\rho_0, \rho_1 \in \mathbb{S}en(\Sigma)$ *when for every model* $M \in \mathbb{M}od(\Sigma)$ *we have* $M \models \rho$ *iff* $M \models \rho_0$ *or* $M \models \rho_1$. *The extension to the infinitary case is straightforward. A sentence* $\rho \in \mathbb{S}en(\Sigma)$ *is called a semantic disjunction of the set* $E \subseteq \mathbb{S}en(\Sigma)$ *when for every model* $M \in \mathbb{M}od(\Sigma)$ *we have* $M \models \rho$ *iff* $M \models e$ *for some* $e \in E$.
3. *a sentence* $\rho \in \mathbb{S}en(\Sigma)$ *is called a semantic existential quantification of a sentence* $\rho' \in \mathbb{S}en(\Sigma')$ *over the signature morphism* $\chi : \Sigma \to \Sigma'$ *when for every model* $M \in \mathbb{M}od(\Sigma)$ *we have* $M \models \rho$ *iff there exists a* χ*-expansion* M' *of* M *(i.e.* $M'\upharpoonright_{\chi} = M$*) that satisfies* ρ'.
   A similar definition can be given for universal quantification.

Distinguished negation $\neg\_$, disjunction $\bigvee\_$ [2], and existential quantification $(\exists\chi)\_$ are called *first order constructors* and they have the semantical meaning defined above.
Throughout this paper we assume the following commutativity of first order constructors with the signature morphisms, i.e. for every signature morphism $\varphi : \Sigma \to \Sigma_1$ and each Σ-sentence

1. $\neg e$, $\varphi(\neg e) = \neg\varphi(e)$,
2. $\bigvee E$, $\varphi(\bigvee E) = \bigvee \varphi(E)$, and
3. $(\exists\chi)e'$, there exists a pushout



such that $\varphi((\exists\chi)e') = (\exists\chi_1)\varphi'(e')$.

- for each directed diagram $D : (J, \leq) \to \mathbb{C}$ with co-limit $\{Di \xrightarrow{\mu_i} B\}_{i\in J}$, and for each morphism $A \xrightarrow{g} B$, there exists $i \in J$ and $A \xrightarrow{g_i} Di$ such that $g_i;\mu_j = g$,
- for any two arrows g_i and g_j as above, there exists $i \leq k, j \leq k \in J$ such that $g_i;D(i \leq k) = g_j;D(j \leq k) = g$.

[2] we will use the symbol $\bigvee$ to represent the most general kind of disjunction from a sentence system even if it is finitary.

Very often quantification is considered only for a restricted class of signature morphisms. For example, quantification in **FOL** considers only the finitary signature extensions with constants. Based on these connectives we can also define the other first order constructers like $\bigwedge$, *false*, $(\forall\chi)\_$ using the classical definitions.

4 Forcing and Generic Models

Forcing is a method of construction models satisfying some properties. In this paper we introduce the notion of forcing in institution-independent model theory and we study completeness in various logics. For this we assume an institution $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ with a subcategory $\mathcal{D} \subseteq \mathbb{S}ig$ of signature morphisms.

First order fragments. By a *$\mathcal{D}$-first order fragment* (or simply *fragment*) over a set Γ of Σ-sentences we mean an extension $\mathcal{L}$ of Γ ($\Gamma \subseteq \mathcal{L}$) such that

1. every sentence of $\mathcal{L}$ is constructed from the sentences of Γ by means of negations, (infinitary) disjunctions and existential quantification over the signature morphisms in $\mathcal{D}$, and
2. $\mathcal{L}$ has the following properties:
   (a) $\mathcal{L}$ is closed under negations, i.e. if $e \in \mathcal{L}$ then $\neg e \in \mathcal{L}$.
   (b) $\mathcal{L}$ is closed to the "sub-sentence" relation, i.e.
   - if $\neg e \in \mathcal{L}$ then $e \in \mathcal{L}$,
   - if $\bigvee E \in \mathcal{L}$ then $e \in \mathcal{L}$ for all $e \in E$, and
   - if $(\exists\chi)e' \in \mathcal{L}$ and $\theta : \chi \to 1_\Sigma$ then $\theta(e') \in \mathcal{L}$.

Note that the closure of $\mathcal{L}$ to "sub-sentence" relation enable us to apply induction on the structure of the sentences.

Our definition of fragments is slightly different from the one in [19]. We do not assume the closure of $\mathcal{L}$ to

- disjunctions, i.e. $e_1,\ e_2 \in \mathcal{L}$ implies $e_1 \vee_2 \in \mathcal{L}$, or
- existential quantifications, i.e. $\theta(e') \in \mathcal{L}$ implies $(\exists\chi)e' \in \mathcal{L}$ in case there exists a substitution $\theta : \chi \to 1_\Sigma$.

Definition 11. *Let Γ be a set of sentences. A forcing property for Γ is a tuple $\mathbb{P} = \langle P, \leq, f\rangle$ such that:*

1. *$\langle P, \leq\rangle$ is a partially ordered set with a least element* 0*.*
2. *f is a function which associates with each $p \in P$ a set $f(p)$ of sentences in Γ.*
3. *Whenever $p \leq q$, $f(p) \subseteq f(q)$.*
4. *For each set of sentences $E \subseteq \Gamma$ and any sentence $e \in \Gamma$ if $E \subseteq f(p)$ and $E \models e$ then there is $q \geq p$ such that $e \in f(q)$.*

The elements of P are called *conditions* of $\mathbb{P}$.

We will define the forcing relation $\Vdash \subseteq P \times \mathcal{L}$ associated to a forcing property $\mathbb{P} = (P, f, \leq)$.

Definition 12. *Let $\mathbb{P} = \langle P, f, \leq\rangle$ be a forcing property for a set Γ of Σ-sentences and $\mathcal{L}$ a fragment over Γ. The relation $p \Vdash e$ in $\mathbb{P}$, read p forces e, is defined by induction on e, for $p \in P$ and $e \in \mathcal{L}$, as follows:*

- For $e \in \Gamma$. *$p \Vdash e$ if $e \in f(p)$.*
- For $\neg e \in \mathcal{L}$. *$p \Vdash \neg e$ if there is no $q \geq p$ such that $q \Vdash e$.*
- For $\bigvee E \in \mathcal{L}$. *$p \Vdash \bigvee E$ if $p \Vdash e$ for some $e \in E$.*

– For $(\exists\chi)e \in \mathcal{L}$. *$p \Vdash (\exists\chi)e$ if $p \Vdash \theta(e)$ for some substitution $\theta : \chi \to 1_\Sigma$.*

We say that p weakly forces *e, in symbols $p \Vdash^w e$, iff $p \Vdash \neg\neg e$. Also, we can easily extend the forcing relation to any set of properties $Q \subseteq P$ and any $e \in \mathcal{L}$: $Q \Vdash e$ if there exists $q \in Q$ such that $q \Vdash e$.*

The above definition is a generalization of the forcing studied in [23, 2, 19].

Lemma 2. *Let $\mathbb{P} = (P, f, \leq)$ be a forcing property for a set Γ of Σ-sentences, $\mathcal{L}$ a fragment over Γ and e a sentence in $\mathcal{L}$.*
1. *$p \Vdash^w e$ iff for each $q \geq p$ there is a condition $r \geq q$ such that $r \Vdash e$.*
2. *If $p \leq q$ and $p \Vdash e$ then $q \Vdash e$.*
3. *If $p \Vdash e$ then $p \Vdash^w e$.*
4. *We can not have both $p \Vdash e$ and $p \Vdash \neg e$.*

Proof.
1. $p \Vdash^w e$ iff $p \Vdash \neg\neg e$ iff for each $q \geq p$, $q \nVdash \neg e$ iff for each $q \geq p$, there exists $r \geq q$ such that $r \Vdash e$.
2. By induction on e.
   For $e \in \Gamma$. The conclusion follows from $f(p) \subseteq f(q)$.
   For $\neg e \in \mathcal{L}$. We have $p \Vdash \neg e$. Suppose towards a contradiction $q \nVdash \neg e$, then by definition of forcing there is $q' \geq q$ such that $q' \Vdash e$. Therefore there is $q' \geq p$ such that $q' \Vdash e$, thus $p \nVdash \neg e$, which is a contradiction.
   For $\bigvee E \in \mathcal{L}$. $p \Vdash e$ for some $e \in E$. By induction $q \Vdash e$ which implies $q \Vdash \bigvee E$.
   For $(\exists\chi)e \in \mathcal{L}$. Since $p \Vdash (\exists\chi)e$ then $p \Vdash \theta(e)$ for some substitution $\theta : \chi \to 1_\Sigma$. By induction $q \Vdash \theta(e)$, and by the definition of forcing relation $q \Vdash (\exists\chi)e$.
3. It follows easily from 1 and 2.
4. Obvious.

Definition 13. *Let $\mathbb{P} = (P, f, \leq)$ be a forcing property for a set Γ of Σ-sentences, and $\mathcal{L}$ a fragment over Γ. A subset $G \subseteq P$ is said to be a generic (relatively to the fragment $\mathcal{L}$) iff*
1. *$p \in G$ and $q \leq p$ implies $q \in G$.*
2. *$p, q \in G$ implies that there exists $r \in G$ with $p \leq r$ and $q \leq r$.*
3. *for each sentence $e \in \mathcal{L}$ there exists a condition $p \in G$ such that either $p \Vdash e$ or $p \Vdash \neg e$.*

Lemma 3. *If $\mathcal{L}$ is countable then every p belongs to a generic set.*

Proof. The proof of this lemma is similar to the one in [19]. Since $\mathcal{L}$ is countable let $\{e_n \mid n < \omega\}$ be an enumeration of $\mathcal{L}$. We form a chain of conditions $p_0 \leq p_1 \leq \ldots$ in P as follows. Let $p_0 = p$. If $p_n \Vdash \neg e_n$, let $p_{n+1} = p_n$, otherwise choose $p_{n+1} \geq p_n$ such that $p_{n+1} \Vdash e_n$. The set $G = \{q \in P \mid q \leq p_n$ for some $n < \omega\}$ is generic and contains p.

Definition 14. *Let $\mathbb{P} = \langle P, \leq, f \rangle$ be a forcing property for a set Γ of Σ-sentences and $\mathcal{L}$ a fragment over Γ.*
1. *M is a model for $G \subseteq P$ if for every sentence e*

$$M \models e \text{ iff } G \Vdash e$$

2. *M is a generic model for $p \in P$ if there is a generic set $G \subseteq P$ such that $p \in G$ and M is a model for G.*

Proposition 2. *Assume that*

1. Γ *is part-basic,*
2. *for each* $E \subseteq \Gamma$ *there exists a basic model* M_E *that is* $\mathcal{D}$*-reachable, and*
3. $\models$ *is compact for* Γ.

Then there is a model for every generic set G.

Proof. Let T be the set of all sentences of $\mathcal{L}$ which are forced by G. Let $B = \Gamma \cap T$. We prove that for each $e \in \mathcal{L}$ $M_B \models e$ iff $e \in T$ by induction on e.
For $e \in \Gamma$. Suppose $M_B \models e$ then we have $B \models e$ and by the hypothesis there is $B' \subseteq B$ finite such that $B' \models e$. Since G is generic there exists $p \in G$ such that $B' \subseteq f(p)$. Suppose towards a contradiction that $e \notin T$ which because G is generic leads to $\neg e \in T$. Then there is $q \in G$ such that $q \Vdash \neg e$. Since G is generic there is $r \in G$ such that $r \geq p$ and $r \geq q$. We have $B' \subseteq f(r)$ and using Lemma 2(2) we obtain $r \Vdash \neg e$. By the definition of forcing property $r' \Vdash e$ for some $r' \geq r$ and and by Lemma 2(2) $r' \Vdash \neg e$ which is a contradiction. If $e \in T$ then $e \in B$ and $M_B \models e$.
For $\neg e \in \mathcal{L}$. Exactly one of e, $\neg e$ is in T. Since G is generic there is $p \in G$ such that either $p \Vdash e$ or $p \Vdash \neg e$. Therefore $e \in T$ or $\neg e \in T$. Suppose towards a contradiction that e ,$\neg e \in T$, then there exists $p,q \in G$ such that $p \Vdash e$ and $q \Vdash \neg e$. By the definition of generic sets there is $r \in G$ such that $r \geq p$ and $r \geq q$. By Lemma 1(2) $r \Vdash e$ and $r \Vdash \neg e$ which is a contradiction.
Let $\neg e \in T$. Suppose that $M_B \models e$, then by induction we have $e \in T$, which is a contradiction. Therefore $M_B \models \neg e$. Now if $M_B \models \neg e$, then e is not in T, therefore $\neg e \in T$.
For $\bigvee E \in \mathcal{L}$. If $M_B \models \bigvee E$ then $M_B \models e$ for some $e \in E$. By induction $e \in T$. We have $p \Vdash e$ for some $p \in G$ and we obtain $p \Vdash \bigvee E$. Thus, $\bigvee E \in T$. Now if $\bigvee E \in T$ then $e \in T$, for some $e \in E$. Therefore, by induction, $M_B \models e$ and thus $M_B \models \bigvee E$.
For $(\exists\chi)e \in \mathcal{L}$. Assume that $M_B \models (\exists\chi)e$ where $\chi : \Sigma \to \Sigma'$. There exists a χ-expansion N of M_B such that $N \models e$. Because M_B is $\mathcal{D}$-reachable there exists a substitution $\theta : \chi \to 1_\Sigma$ such that $M_B \restriction_\theta = N$. By the satisfaction condition $M_B = N \restriction_\chi \models \theta(e)$. By induction $\theta(e) \in T$ which implies $(\exists\chi)e \in T$. For the converse implication assume that $p \Vdash (\exists\chi)e$ for some $p \in G$. We have that $p \Vdash \theta(e)$ for some substitution $\theta : \chi \to 1_\Sigma$. By induction $M_B \models \theta(e)$ which implies $M_B \restriction_\theta \models e$. Since $(M_B \restriction_\theta) \restriction_\chi = M_B$ we obtain $M_B \models (\exists\chi)e$.

Theorem 1. *(Generic model theorem) Under the conditions of Proposition 2, if* $\mathcal{L}$ *is countable then there is a generic model for each condition* $p \in P$.

Proof. By Lemma 3 there is a set generic set $G \subseteq P$ such that $p \in G$ and by Proposition 2 there is a model M for G.

The following is a corollary of the generic model theorem.

Corollary 1. *Under the condition Theorem 1 for every condition* $p \in P$ *and any sentence* $e \in \mathcal{L}$ *we have that* $p \Vdash^w e$ *iff* $M \models e$ *for each generic model M for p.*

Proof. Suppose $p \Vdash^w e$ and M is a generic model for p. We have $p \Vdash \neg\neg e$ which implies $M \models \neg\neg e$ and $M \models e$. Now for the converse implication suppose that $p \nVdash^w e$. There exists $q \geq p$ such that $q \Vdash \neg e$. By Proposition 2 there is a generic model M for q which implies $M \models \neg e$. But M is also a generic model for p.

5 First Order Institutions and Entailment Systems

Let $I = (\mathbb{S}ig, \mathbb{S}en, \mathbb{M}od, \models)$ be an institution and

- let $\mathbb{S}en_0$ be a sub-functor of $\mathbb{S}en$ (i.e. $\mathbb{S}en_0 : \mathbb{S}ig \to \mathbb{S}et$ such that $\mathbb{S}en_0(\Sigma) \subseteq \mathbb{S}en(\Sigma)$ and $\varphi(\mathbb{S}en_0(\Sigma)) \subseteq \mathbb{S}en_0(\Sigma')$ for each signature morphism $\varphi : \Sigma \to \Sigma'$), and
- $\mathcal{D} \subseteq \mathbb{S}ig$ is a broad subcategory of signature morphisms.

We say that I is a *$\mathcal{D}$-first order institution over* I_0, where $I_0 = (\mathbb{S}ig, \mathbb{S}en_0, \mathbb{M}od, \models)$, when for every signature Σ the set $\mathbb{S}en(\Sigma)$ is a $\mathcal{D}$-first order fragment over $\mathbb{S}en_0(\Sigma)$.

For example **FOL** is a first order institution over **Atomic**(**FOL**), where the quantification class $\mathcal{D}$ of signature morphisms is the class of all signature extensions with finite number of constants. Similarly, the infinitary version $\mathbf{FOL}_{\omega_1,\omega}$ is also an example of first order institution.

We also assume another mild condition, namely that the sentences of I_0 are not obtained by applying the first order constructors. An immediate consequence of this definition is the following.

Remark 2. Let $\varphi : \Sigma \to \Sigma'$ be a signature morphism, $e \in \mathbb{S}en(\Sigma)$ and $e' \in \mathbb{S}en(\Sigma')$ two sentences such that $\varphi(e) = e'$. Then

- $e \in \mathbb{S}en_0(\Sigma)$ iff $e' \in \mathbb{S}en_0(\Sigma')$,
- e is obtained by applying Boolean connectives iff e' is obtained by applying Boolean connectives, and
- e is an existential quantified sentence iff e' is an existential quantified sentence.

A *first order entailment system* of a first order institution satisfies the following properties:

1. Reductio ad absurdum: $\Gamma \cup \{e\} \vdash false$ iff $\Gamma \vdash \neg e$
2. False: $false \vdash \rho$
3. Double negation elimination: $\neg\neg e \vdash e$
4. Disjunction introduction:$e \vdash \bigvee E$ for all $e \in E$,
5. Disjunction elimination: $\Gamma \vdash \rho$ whenever $\Gamma \vdash \bigvee E$, $\Gamma \cup \{e\} \vdash \rho$ for all $e \in E$.
6. Substitutivity: $\theta(e) \vdash (\exists\chi)e$ for all substitutions $\theta : \chi \to 1_\Sigma$
7. Generalization: $\Gamma \vdash \neg(\exists\chi)e'$ iff $\chi(\Gamma) \vdash \neg e'$

Note that these rules are given for both finitary and infinitary case. In the finitary case the disjunction $\bigvee E$ occurring in the Disjunction introduction and Disjunction elimination is finitary, i.e. E is a finite set of sentences. Generally speaking, if one of the first order constructors is missing then the corresponding properties are disregarded. For example in case of **QfFOL** Substitutivity and Generalization are omitted.

We call a set E of sentences *inconsistent* when $E \vdash false$.

The *first order entailment system over* I_0 is the first order entailment system of I freely generated over the semantic entailment system $(\mathbb{S}ig, \mathbb{S}en_0, \models)$ of I_0.

The following result can be proved in the style of [10].

Proposition 3. *The first order entailment of I over I_0 is sound.*

In the next section we will concentrate on proving the completeness of first order entailment system of I.

6 First Order Completeness

Completeness of the first order entailment systems is significantly more difficult than the soundness property and therefore requires more conceptual infrastructure. The first order completeness result below comes both in a finite and in an infinite variant. However the completeness result restricts the category of signatures to the countable case, i.e. signatures Σ with $card(\mathbb{S}en_0(\Sigma)) \leq \omega$.

Definition 15. *Let $\mathcal{D} \subseteq \mathbb{S}ig$ be a subcategory of signature morphisms. We say that $\Sigma \xrightarrow{\chi} \Sigma'$ is a $\mathcal{D}$-extension of Σ if*

1. *χ is non-void, and*
2. *it is the vertex of a directed co-limit $(\chi_i \xrightarrow{\varphi_i} \chi)_{i \in J}$ of a directed diagram $(\chi_i \xrightarrow{\varphi_{i,j}} \chi_j)_{(i \leq j) \in (J, \leq)}$ in $\Sigma/\mathcal{D}$ ($\Sigma \xrightarrow{\chi_i} \Sigma_i \in \mathcal{D}$ for all $i \in J$ and $\Sigma_i \xrightarrow{\varphi_{i,j}} \Sigma_j \in \mathcal{D}$ for all $(i,j) \in (J, \leq)$) such that for all signature morphisms $\Sigma_i \xrightarrow{\psi_i} \Sigma'_i \in \mathcal{D}$ there exists a substitution $\psi_i \xrightarrow{\psi_{i,j}} \varphi_{i,j} \in (\Sigma_i/\mathbb{S}ig)$ which is non-void.*

Throughout this section we assume that the institution I has the following properties

1. every signature Σ has the $\mathcal{D}$-extension property,
2. every signature morphism in $\mathcal{D}$ is non-void and finitary, and
3. every sentence of I_0 is finitary.

The $\mathcal{D}$-extension property is easily fulfilled in concrete examples. Take for example **FOL** and assume that $\mathcal{D}$ is the class of signature extensions with finite number of constants of non-void sorts. For every signature $\Sigma = (S, F, P)$ consider a set C of new constant symbols (C does not contain any symbol from Σ) such that

- C_s is an infinite set for all non-void sorts $s \in S$, and
- $C_s \cap C_{s'} = \emptyset$ for all sorts $s, s' \in S$.

The inclusion $\Sigma \overset{\chi}{\hookrightarrow} \Sigma(C)$ is non-void, where $\Sigma(C) = (S, F \cup C, P)$, and it is the vertex of the directed co-limit $((\Sigma \overset{\chi_i}{\hookrightarrow} \Sigma(C_i)) \overset{\varphi_i}{\hookrightarrow} (\Sigma \overset{\chi}{\hookrightarrow} \Sigma(C)))_{C_i \subseteq C\, finite}$ of the directed diagram $(\chi_i \overset{\varphi_{i,j}}{\hookrightarrow} \chi_j)_{C_i \subseteq C_j \subseteq Y\, finite}$. Since C is infinite, for every signature extension $\psi_i : \Sigma(C_i) \hookrightarrow \Sigma_i(C_i \cup Y)$, where Y is a finite set of new constants of non-void sorts, there exists an injective mapping $\psi_{i,j} : C_i \cup Y \to C_j$ such that the restriction $\psi_{i,j}\,|_{C_i} : C_i \to C_j$ is the inclusion.

In case of first-order institutions with sentences formed without quantifiers we may consider $\mathcal{D}$ the broad subcategory of signature morphisms with $\mathcal{D}(\Sigma, \Sigma) = 1_\Sigma$ and $\mathcal{D}(\Sigma, \Sigma') = \emptyset$ for all signatures $\Sigma \neq \Sigma'$. Note that in this case any signature Σ has a $\mathcal{D}$-extension $\chi = 1_\Sigma$.

6.1 Canonical Forcing Properties

Let $\chi : \Sigma \to \Sigma'$ be a $\mathcal{D}$-extension of Σ as in Definition 15. We have the following consequence of the finiteness of the "atomic" sentences.

Lemma 4. *$Sen_0(\Sigma') = \bigcup_{i \in J} \varphi_i(\mathbb{S}en_0(\Sigma_i))$.*

We denote by $\mathcal{L}_{\Sigma'}$ the set of sentences $\bigcup_{i \in J} \varphi_i(\mathbb{S}en(\Sigma_i))$ and we have the following consequence of Remark 2 and the finiteness of signature morphisms in $\mathcal{D}$.

Lemma 5. *$\mathcal{L}_{\Sigma'}$ is a fragment over $\mathbb{S}en_0(\Sigma)$.*

Now, let $\mathcal{L}$ be an arbitrary fragment over $\mathbb{S}en_0(\Sigma)$. We define the *canonical forcing property* $\mathbb{P} = (P, f, \leq)$ for $\mathbb{S}en_0(\Sigma)$ (relatively to the fragment $\mathcal{L}$).

- $P = \{\varphi_i(p_i) \mid \ p_i \subseteq \mathbb{S}en(\Sigma_i),\ \varphi_i(p_i) \subseteq \mathcal{L}$ and $\varphi_i(p_i)$ is consistent$\}$,
- $f(p) = p \cap \mathbb{S}en_0(\Sigma)$ for all $p \in P$, and
- $\leq$ is the inclusion relation $\subseteq$.

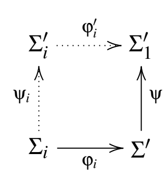
Proposition 4. *$\mathbb{P} = (P, \leq, f)$ is a forcing property.*

Proof. All the conditions of the forcing property, except the last one, obviously hold for $\mathbb{P}$. Assume a condition $p \in P$ and a set of sentences $E \subseteq f(p)$ such that $E \models e$ where $e \in \mathbb{S}en_0(\Sigma)$. We prove that $p \cup \{e\} \in P$.
By the completeness of the proof rules for I_0 we get $E \vdash e$ and moreover $p \vdash e$ which implies $p \cup \{e\}$ consistent. By the definition of $\mathbb{P}$ the condition $p \in P$ may be written as $p = \varphi_i(p_i)$ for some $i \in J$ and $p_i \in \mathbb{S}en(\Sigma_i)$. Since e is a sentence in $\mathbb{S}en_0(\Sigma')$ it may be written as $e = \varphi_j(e_j)$ for some $j \in J$ and $e_j \in \mathbb{S}en_0(\Sigma_j)$. Let $(i \leq k) \in (J, \leq)$ and $(j \leq k) \in (J, \leq)$. We have that $p \cup \{e\} = \varphi_k(\varphi_{i,k}(p_i) \cup \{\varphi_{j,k}(e_j)\})$ is consistent. Therefore $p \cup \{e\} \in P$.

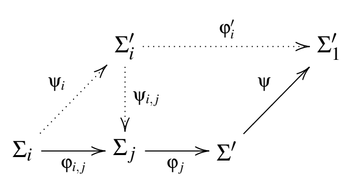
Lemma 6. *$\mathbb{P}$ has the following properties.*

1. *if $p \in P$ and $\bigvee E \in p$ then $p \cup \{e\} \in P$ for some $e \in E$.*
2. *if $p \in P$ and $(\exists \psi)e \in p$ (where $\psi : \Sigma' \to \Sigma'_1$) there exists a substitution $\theta : \psi \to 1_\Sigma$ such that $p \cup \{\theta(e)\} \in P$.*

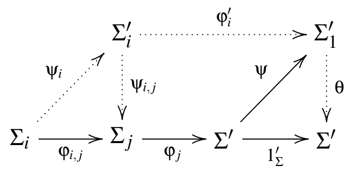
Proof. 1. Suppose towards a contradiction that $p \cup \{e\} \notin P$ for all $e \in E$.
If $e \in E$ then $p \cup \{e\} \in \mathcal{L}$. By Remark 2 there exists $\bigvee E_i \in p_i$ such that $\varphi_i(E_i) = E$. Since $p \cup \{e\} = \varphi_i(p_i \cup \{e_i\})$ for some $e_i \in E_i$, $p \cup \{e\} \subseteq \mathcal{L}$ and $p \cup \{e\} \notin P$ we get $p \cup \{e\}$ not consistent.
Because $p \vdash \bigvee E$ and for every $e \in E$ we have $p \cup \{e\}$ inconsistent by Disjunction elimination property we get p inconsistent which is a contradiction.
2. There exists $p_i \subseteq \mathbb{S}en(\Sigma_i)$ such that $\varphi_i(p_i) = p$. By Remark 2 there is a sentence $(\exists \psi_i)e_i \in p_i$ and a pushout



such that $\varphi_i((\exists \psi_i)e_i) = (\exists \psi)\varphi'_i(e_i)$ and $e = \varphi'_i(e_i)$. By Definition 15 there exists $(i \leq j) \in (J, \leq)$ and a substitution $\psi_{i,j} : \psi_i \to \varphi_{i,j}$ with $\psi_{i,j}$ non-void as a signature morphism.



Because $\{\Sigma'_i \xleftarrow{\psi_i} \Sigma_i \xrightarrow{\varphi_i} \Sigma',\ \Sigma' \xrightarrow{\psi} \Sigma'_1 \xleftarrow{\varphi'_i} \Sigma'_i\}$ is a pushout and $\psi_i;(\psi_{i,j};\varphi_j) = \varphi_i;1_{\Sigma'}$ there exists $\theta : \Sigma'_1 \to \Sigma'$ such that $\varphi'_i;\theta = (\psi_{i,j};\varphi_j)$ and $\psi;\theta = 1_{\Sigma'}$.



We show that $\psi_i(p_i) \cup \{e_i\}$ is consistent. Suppose towards a contradiction that $\psi_i(p_i) \cup \{e_i\}$ is inconsistent. We have that $\psi_i(p_i) \vdash \neg e_i$ and by Generalization we get $p_i \vdash \neg(\exists \psi_i)e_i$ which is a contradiction with the consistency of p_i.
Since $\psi_{i,j}$ is non-void and $\psi_i(p_i) \cup \{e_i\}$ is consistent we have that $\psi_{i,j}(\psi_i(p_i) \cup \{e_i\})$ is consistent. Since φ_j is non-void, we obtain that $\varphi_j(\psi_{i,j}(\psi_i(p_i) \cup \{e_i\})) = p \cup \theta(e)$ is consistent. Therefore $p \cup \{\theta(e)\} \in P$.

Proposition 5. *If $\mathcal{L} \subseteq \mathcal{L}_{\Sigma'}$ then for each sentence $e \in \mathcal{L}$ and each condition $p \in P$*

$$\textit{there exists } q \geq p \textit{ such that } q \Vdash e \textit{ iff } p \cup \{e\} \in P$$

Proof. We proceed by induction on the structure of the sentence e.
For $e \in \mathbb{S}en_0(\Sigma)$. If there is $q \geq p$ such that $q \Vdash e$ then $e \in q$ which implies $p \cup \{e\} \subseteq q$. q is consistent and any subset of q is consistent too which implies $p \cup \{e\}$ is consistent. Therefore $p \cup \{e\} \in P$. For the converse implication take $q = p \cup \{e\}$.
For $\neg e$. By the induction hypothesis, applied to e, for each $q \in P$ we have

$$\text{for each } r \geq q, r \nVdash e \iff q \cup \{e\} \notin P$$

which implies that for each $q \in P$ we have

$$q \Vdash \neg e \iff q \cup \{e\} \notin P$$

We need to prove

$$\text{there exists } q \geq p \text{ such that } q \cup \{e\} \notin P \iff p \cup \{\neg e\} \in P$$

Assume that there is $q \geq p$ such that $q \cup \{e\} \notin P$. Then $q \cup \{e\}$ inconsistent which implies $q \vdash \neg e$. We obtain $q \cup \{\neg e\}$ consistent (suppose $q \cup \{\neg e\}$ is inconsistent we obtain $q \vdash \neg\neg e$, a contradiction with the consistency of q). Since $p \cup \{\neg e\} \subseteq q \cup \{\neg e\}$, we have $p \cup \{\neg e\}$ consistent. Therefore $p \cup \{\neg e\} \in P$. For the converse implication, take $q = p \cup \{\neg e\}$.
For $\bigvee E$. If there is $q \geq p$ such that $q \Vdash \bigvee E$, then there is $e \in E$ such that $q \Vdash e$. By the induction hypothesis, $p \cup \{e\} \in P$. If $p \cup \{e\}$ consistent implies $p \cup \{\bigvee E\}$ consistent then $p \cup \{\bigvee E\} \in P$. Suppose towards a contradiction that $p \cup \{\bigvee E\}$ is not consistent, then $p \cup \{e, \bigvee E\}$ is not consistent. Because $p \cup \{e\} \vdash \bigvee E$ (by Disjunction introduction property) we obtain $p \cup \{e\}$ inconsistent which is a contradiction.
For the converse implication assume that $p \cup \{\bigvee E\} \in P$. By Lemma 6 (1) there is $e \in E$ such that $p \cup \{\bigvee E, e\} \in P$. By induction hypothesis applied to e we have $q \Vdash e$ for some $q \geq p \cup \{\bigvee E\}$. Hence there exists $q \geq p$ such that $q \Vdash \bigvee E$.

For $(\exists\chi)e$. Assume that there is $q \geq p$ such that $q \Vdash (\exists\chi)e$. By the definition of forcing relation there exists a substitution $\chi' : \chi \to 1_\Sigma$ such that $q \Vdash \chi'(e)$. By induction $p \cup \{\chi'(e)\} \in P$. By Substitutivity $p \cup \{\chi'(e)\} \vdash (\exists\chi)e$ which implies $p \cup \{\chi'(e), (\exists\chi)e\}$ consistent. Because $p \cup \{(\exists\chi)e\} \subseteq p \cup \{\chi'(e), (\exists\chi)e\}$ we get $p \cup \{(\exists\chi)e\}$ consistent. Therefore $p \cup \{(\exists\chi)e\} \in P$.
For the converse implication assume that $p \cup \{(\exists\chi)e\} \in P$ where $\chi : \Sigma \to \Sigma'$. By Lemma 6 (2) there exists a substitution $\chi' : \chi \to 1_\Sigma$ such that $p \cup \{(\exists\chi)e,\ \chi'(e)\} \in P$. Applying the induction hypothesis to $\chi'(e)$ we obtain $q \geq p \cup \{(\exists\chi)e\}$ such that $q \Vdash \chi'(e)$. Therefore, by the definition of forcing relation $q \Vdash (\exists\chi)e$.

We have the following consequence of the above proposition.

Corollary 2. *If $\mathcal{L} \subseteq \mathcal{L}_{\Sigma'}$ then for each condition $p \in P$, any generic model M for p satisfies p.*

Proof. Let $G \subseteq P$ be the generic set such that $p \in G$ and M a model for G. We prove that $M \models e$ for all $e \in p$.
Let e be an arbitrary sentence in p. Since $G \subseteq P$ is a generic set there exists $q \in G$ such that either $q \Vdash e$ or $q \Vdash \neg e$. Suppose that $q \Vdash \neg e$ then there is $r \in G$ such that $r \geq p$ and $r \geq q$. By Lemma 2 (2) $r \Vdash \neg e$. By Proposition 5 since $e \in r$ there exists $r' \geq r$ such that $r' \Vdash e$. Using Lemma 2 (2) again we get $r' \Vdash \neg e$ which is a contradiction. Therefore $q \Vdash e$ and since M is a model for G we have that $M \models e$.

Existence of generic sets. Corollary 2 does not state that for each condition there is a generic set which includes it. Therefore we need to prove that generic sets actually exists. For this we will consider only signatures that have a countable set of symbols.

Definition 16. *We say that a signature Σ is* countable *if*
- *it has a countable set of "atomic" sentences, i.e. card$(\mathbb{S}en_0(\Sigma)) \leq \omega$, and*
- *for each $\Sigma \xrightarrow{\psi} \Sigma' \in \mathcal{D}$ there exists only a countable set of substitutions $\theta : \psi \to 1_\Sigma$.*

Lemma 7. *Assume that all the signatures of I are countable and let*
- *$\chi : \Sigma \to \Sigma'$ be an extension of Σ as in Definition 15, and*
- *Γ be a countable set of Σ-sentences.*

If $\mathcal{L}$ is the least first-order fragment which contains $\chi(\Gamma)$ then every condition $p \in P$ belongs to a generic set.

Proof. Since the signature Σ is countable then the fragment $\mathcal{L}$ is countable. By Lemma 3 every condition p belongs to a generic set.

We will now enumerate the sufficient conditions for proving completeness.

Theorem 2 (First order completeness). *Consider a $\mathcal{D}$-first order institution $I = (\mathbb{S}ig,\ \mathbb{S}en, \mathbb{M}od, \models)$ over $I_0 = (\mathbb{S}ig, \mathbb{S}en_0, \mathbb{M}od, \models)$. The first order entailment system over I_0 of the institution I is complete if*
1. *all the signatures are countable and disjunctions are applied only to countable sets of sentences,*
2. *every signature Σ has a $\mathcal{D}$-extension,*
3. *every signature morphism in $\mathcal{D}$ is non-void and finitary,*
4. *the semantic entailment system $(\mathbb{S}ig, \mathbb{S}en_0, \models)$ of I_0 is compact,*

5. *every sentence of I_0 is finitary, and*
6. *for every $E \subseteq \mathbb{S}en_0(\Sigma)$ there exists a $\mathcal{D}$-reachable model M_E defining E as basic set of sentences.*

Proof. Assume that $\Gamma \nvdash_\Sigma \rho$, where Γ is a countable set of sentences and ρ is any sentence. Let $\Sigma \xrightarrow{\chi} \Sigma'$ be a $\mathcal{D}$-extension of Σ as in Definition 15. We define $\mathcal{L} \subseteq \mathbb{S}en(\Sigma')$ as the least fragment which includes $\chi(\Gamma)$, and by the first condition of the theorem $\mathcal{L}$ is countable.
Because χ is non-void we have $\chi(\Gamma) \nvdash_{\Sigma'} \chi(\rho)$. We have that $\chi(\Gamma \cup \{\neg\rho\})$ is consistent. If $\chi(\Gamma \cup \{\neg\rho\})$ is not consistent then $\Gamma \cup \{\neg\rho\}$ is not consistent which implies $\Gamma \vdash \neg\neg\rho$ and by the rule of Double negation elimination we obtain $\Gamma \vdash \rho$ which is a contradiction with our assumption. By the first hypothesis of the theorem and Lemma 7 the condition $\chi(\Gamma \cup \{\neg\rho\})$ (of the canonical forcing property $\mathbb{P} = (P, \leq, f)$) belongs to a generic set. By Theorem 1 there exists a generic $\mathcal{D}$-reachable Σ'-model M' for the condition $\chi(\Gamma \cup \{\neg\rho\})$. By Corollary 2 $M' \models \chi(\Gamma \cup \{\neg\rho\})$ and by satisfaction condition $M' \upharpoonright_\chi \models \Gamma \cup \{\neg\rho\}$ which implies $\Gamma \not\models \rho$.

7 Working Examples

In order to develop concrete sound and complete first order entailment systems we need to set the parameters of the completeness theorem for each example. In particular we need to define the institutions I_0 and give a complete finitary deduction system for its sentences in order to ensure that it is compact.
For each case we restrict the signatures to the signatures composed of countable numbers of symbols. This implies that the conditions of Lemma 7 are fulfilled.

7.1 The first order entailment system of FOL

We set the parameters of the completeness theorem for **FOL** as follows:

- the institution I is **FOL**
- the institution I_0 is **Atomic**(**FOL**)
- $\mathcal{D}$ is the class of all signature extensions with a finite number of constants with non-empty sorts,
- the system of proof rules for **Atomic**(**FOL**) is given by the following set of rules for any **FOL** signature (S, F, P):
  - $(R)\emptyset \vdash t = t$ for each term $t \in T_F$
  - $(S)t = t' \vdash t' = t$ for any terms $t, t' \in T_F$
  - $(T)\{t = t', t' = t''\} \vdash t = t''$ for any terms $t, t', t'' \in T_F$
  - $(F)\{t_i = t'_i | 1 \leq i \leq n\} \vdash \sigma(t_1, ..., t_n) = \sigma(t'_1, ..., t'_n)$ for all terms $t_i, t'_i \in T_F$ and any operation symbol $\sigma \in F$
  - $(P)\{t_i = t'_i | 1 \leq i \leq n\} \cup \{\pi(t_1, ..., t_n)\} \vdash \pi(t'_1, ..., t'_n)$ for all terms $t_i, t'_i \in T_F$ and any relation symbol $\pi \in P$

Proposition 6. **Atomic**(**FOL**) *with the above proof rules is sound and complete.*

Proof. Soundness follows by simple routine check. For proving the completeness, for any set E of atoms for a signature (S, F, P) we define

$$\equiv_E = \{(t, t') | E \vdash t = t'\}$$

By (R), (S), (T) and (F) this is a congruence on T_F. Then we define a model M_E as follows:

– the (S,F)-algebra part of M_E is defined as the quotient of the initial algebra (term algebra) T_F by $\equiv_E$, and
– for each relation symbol $\pi \in P$, we define $(M_E)_\pi = \{x/_{\equiv_E} | E \vdash \pi(x)\}$

The definition of $(M_E)_\pi$ is correct because of the rule (P). Now we note that for each (S,F,P)-atom ρ we have $E \vdash \rho$ iff $M_E \models \rho$. If $E \models \rho$ then $M_E \models \rho$ which means $E \vdash \rho$.

We are now able to formulate the following corollary of the general first order completeness theorem.

Corollary 3. *The first order entailment system of* **FOL** *and* $\mathbf{FOL}_{\omega_1,\omega}$ *is sound and complete. Moreover, this entailment system is obtained as the first order entailment system freely generated by the following system of finitary rules for a signature* (S,F,P)*.*

– $(R)\emptyset \vdash t = t$ *for each term* t
– $(S) t = t' \vdash t' = t$ *for any terms* t, t'
– $(T)\{t = t', t' = t''\} \vdash t = t''$ *for any terms* t, t', t''
– $(F)\{t_i = t'_i | 1 \le i \le n\} \vdash \sigma(t_1,...,t_n) = \sigma(t'_1,...,t'_n)$ *for any* $\sigma \in F$
– $(P)\{t_i = t'_i | 1 \le i \le n\} \cup \{\pi(t_1,...,t_n)\} \vdash \pi(t'_1,...,t'_n)$ *for any* $\pi \in P$

Remark 3. We can also consider the case when the set of relation is empty, obtaining thus a completeness result for first order equational logic, **FOEQL**.

Corollary 4. *The first order entailment system for* **QfFOL***,* $\mathbf{QfFOL}_{\omega_1}$ *and* **EQLN** *is sound and complete.*

Proof. The only non trivial case is **EQLN**. We sketch the proof. We consider the fragment of **FOEQL** with sentences constructed by applying negation and only one round of existential quantification. More precisely the sentences are of the form $\neg \ldots \neg (\exists X) \neg \ldots \neg t = t'$. This institution satisfies the hypothesis of Theorem 2. After that we consider that $\neg\neg e = e$ and $\neg(\exists X)\neg e' = (\forall X) e'$ and conclude that the entailment system of the institution defined above is sound and complete for **EQLN** too.

7.2 The first order entailment system of POA

We set the parameters of the completeness theorem for **POA**.

– the institution I is **POA**
– the institution I_0 is **Atomic**(**POA**)
– $\mathcal{D}$ is the class of all signature extensions with a finite number of constants with non-empty sorts,
– the system of proof rules for **Atomic**(**POA**) is given by the following set of rules for any **POA** signature (S,F):
  - $(R)\emptyset \vdash t = t$ for each term $t \in T_F$
  - $(S) t = t' \vdash t' = t$ for any terms $t, t' \in T_F$
  - $(T)\{t = t', t' = t''\} \vdash t = t''$ for any terms $t, t', t'' \in T_F$
  - $(F)\{t_i = t'_i | 1 \le i \le n\} \vdash \sigma(t_1,...,t_n) = \sigma(t'_1,...,t'_n)$ for all term $t_i, t'_i \in T_F$ and any operation symbol $\sigma \in F$
  - $(R')\emptyset \vdash t \le t$ for each term $t \in T_F$
  - $(T')\{t \le t', t' \le t''\} \vdash t \le t''$ for any terms $t, t', t'' \in T_F$

- $(F')\{t_i \leq t'_i | 1 \leq i \leq n\} \vdash \sigma(t_1, ..., t_n) \leq \sigma(t'_1, ..., t'_n)$ for all terms $t_i, t'_i \in T_F$ and any operation symbol $\sigma \in F$
- $(ET)\{t_1 = t_2,\ t_2 \leq t_3,\ t_3 = t_4\} \vdash t_1 \leq t_4$ for any terms $t_1, t_2, t_3, t_4 \in T_F$

For the readers not familiar with preorder algebras we give the following definition:

Definition 17. *A (preorder) congruence relation on a (S,F)-preorder algebra M is a pair $(\equiv, \sqsubseteq)$ where $\equiv$ is a (S,F)-congruence relation and $\sqsubseteq$ is a preorder on M which*
- *preserve the preorder structure of M, i.e. $m \leq m'$ implies $m \sqsubseteq m'$ for all elements $m, m' \in M$,*
- *is compatible with operations in F, i.e. $m \leq m'$ implies $M_\sigma(m) \leq M_\sigma(m')$ for all operations $\sigma \in F_{w,s}$ and all elements $m, m' \in M_w$, and*
- *is compatible with the congruence $\equiv$, i.e. $m_1 \equiv m_2$, $m_2 \sqsubseteq m_3$ and $m_3 \equiv m_4$ implies $m_1 \sqsubseteq m_4$ for all elements $m_1, m_2, m_3, m_4 \in M$.*

Proposition 7. **Atomic**(**POA**) *with the above system of proof rules is sound and complete.*

Proof. Soundness follows by simple routine check. For proving the completeness, for any set E of (S,F)-atomic sentences we define $(\equiv_E, \sqsubseteq_E)$
- $\equiv_E = \{(t,t') \mid E \vdash t = t'\}$
- $\sqsubseteq_E = \{(t,t') \mid E \vdash t \leq t'\}$

By the above rules $(\equiv_E, \sqsubseteq_E)$ is a preorder congruence on the term algebra T_F. Then we define the preorder algebra M_E as the quotient of the term algebra by $(\equiv_E, \sqsubseteq_E)$. We note that for each equational or transitional (S,F)-atom ρ

$$E \vdash \rho \text{ iff } M_E \models \rho$$

Now if $E \models \rho$ then $M_E \models \rho$ which means $E \vdash \rho$.

7.3 The first order entailment system of OSA

We set the parameters of the completeness theorem for **OSA** as follows:
- the institution I is **OSA**,
- the institution I_0 is **Atomic**(**OSA**),
- $\mathcal{D}$ is the class of all signature extensions with a finite number of constants with non-empty sorts, and
- the system of proof rules for **Atomic**(**OSA**) is given by the following set of rules for any **OSA** signature $(S, \leq, F)$:
  - $(R)\emptyset \vdash t = t$ for each term t
  - $(S) t = t' \vdash t' = t$ for any terms t, t'
  - $(T)\{t = t', t' = t''\} \vdash t = t''$ for any terms t, t', t''
  - $(F)\{t_i = t'_i | 1 \leq i \leq n\} \vdash \sigma(t_1, ..., t_n) = \sigma(t'_1, ..., t'_n)$ for all terms t_i, t'_i and any operation symbol $\sigma \in F$

We give the definition of congruence relation on an order sorted model.

Definition 18. *A congruence relation $\equiv$ on a $(S, \leq, F)$-model M is a (S,F)-congruence relation $\equiv = (\equiv_s)_{s \in S}$ such that if $s \leq s'$ in $(S, \leq)$ and $a, a' \in M_s$ then $a \equiv_s a'$ iff $a \equiv_{s'} a'$.*

Proposition 8. **Atomic**(**OSA**) *with the above system of proof rules is sound and complete.*

Proof. Soundness follows by simple routine check. For proving the completeness, for any set E of equations for a signature $(S,\leq,F)$ we define

$$\equiv_E = \{(t,t')|E \vdash t = t'\}$$

Since the signature $(S,\leq,F)$ is regular the term algebra T_F is the initial $(S,\leq,F)$-algebra in $\mathbb{M}od(S,\leq,F)$. By (R), (S), (T) and (F) this is an F-congruence on T_F. $\equiv_E$ is also an order sorted congruence on T_F, because the definition of $\equiv_E$ does not depend upon a sort. Since the signature $(S,\leq,F)$ is locally filtered we may define a model M_E as the quotient of the initial algebra (term algebra) T_F by the order sorted congruence $\equiv_E$.
Notice that for each $(S,\leq,F)$-equation $t = t'$ we have $E \vdash t = t'$ iff $M_E \models t = t'$. Now if $E \models t = t'$ then $M_E \models t = t'$ which means $E \vdash t = t'$.

7.4 The first order entailment system of PA

We set the parameters of the completeness theorem for **PA** as follows:

- the institution I is **PA**,
- the institution I_0 is **Atomic**(PA) (the restriction of **PA** to the existence equations),
- $\mathcal{D}$ is the class of all signature extensions with a finite number of total constants, and
- the system of proof rules for **Atomic**(PA) is given by the following set of rules for any **PA** signature (S,F):
  - (S) $t \stackrel{e}{=} t' \vdash t' \stackrel{e}{=} t$ for any terms t,t'
  - (T) $\{t \stackrel{e}{=} t',\ t' \stackrel{e}{=} t''\} \vdash t \stackrel{e}{=} t''$ for any terms t,t',t''
  - (C) $\{t_i \stackrel{e}{=} t'_i,\ def(\sigma(t_1,\ldots,t_n)),\ def(\sigma(t'_1,\ldots,t'_n))\} \vdash$ $\sigma(t_1,\ldots,t_n) \stackrel{e}{=} \sigma(t'_1,\ldots,t'_n)$ for all terms t_i,t'_i and any operation symbol $\sigma \in F$
  - $(Subterm)$ $def(\sigma(t_1,\ldots,t_n)) \vdash \{def(t_i) \mid i \in \overline{1,n}\}$ for all terms t_i and any operation symbol $\sigma \in F$

We give the definition of partial congruence relation.

Definition 19. *A congruence relation $\equiv$ on a (S,F)-model M is a S-sorted equivalence relation $\equiv = (\equiv_s)_{s\in S}$ such that for every operation symbol $\sigma \in F$ and elements m, $m' \in M$ with $m \equiv m'$ if both $M_\sigma(m)$ and $M_\sigma(m')$ are defined then $M_\sigma(m) \equiv M_\sigma(m')$.*

Proposition 9. **Atomic**(**PA**) *with the above system of proof rules is sound and complete.*

Proof. Soundness follows by simple routine check. For proving the completeness, for any set E of (S,F)-atomic sentences we define

$$\equiv_E = \{(t,t')|E \vdash t \stackrel{e}{=} t'\}$$

Note that for every set $E \subseteq \mathbb{S}en(S,F)$ of existence equations we have $E \vdash def(t)$ iff $t \in T_E$, where T_E is the partial algebra having the carrier set consisting of all sub-terms appearing in E.

Firstly we prove that $\equiv_E$ is a congruence relation on T_E. The reflexivity of $\equiv_E$ is given by the above remark. The first two rules ensure the symmetry and the transitivity of $\equiv_E$. By the rule (C) we have that $\equiv_E$ is a congruence relation on T_E.
For each existence equation $t \stackrel{e}{=} t'$ we have $E \vdash t \stackrel{e}{=} t' \iff t \equiv_E t' \iff T_E/_{\equiv_E} \models t \stackrel{e}{=} t'$. If $E \models t \stackrel{e}{=} t'$ then $T_E/_{\equiv_E} \models t \stackrel{e}{=} t'$ which implies $E \vdash t \stackrel{e}{=} t'$.

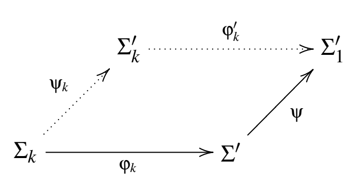
8 Conclusions and future work

We have introduced the notion of forcing parameterized by a set of sentences in institution-independent model theory. Using this we have proved the completeness of the first order entailment systems in an abstract institutional framework and then we have instantiated the result to several concrete examples of institutions. A sound and complete entailment system for **FOL**, **POA**, **OSA** and **PA** may be derived from [21], however we are not aware of similar completeness results for **EQLN**, $\mathbf{QfFOL}_{\omega_1}$, $\mathbf{POA}_{\omega_1,\omega}$, $\mathbf{OSA}_{\omega_1,\omega}$ and $\mathbf{PA}_{\omega_1,\omega}$.
The area of applications of Theorem 2 is much wider. For example we may consider variations of the institutions presented above such as first order logic with transitions or order sorted algebra with relations.
A limitation of this presentation is that the signatures morphism used for quantifications are non-void. In future work we plan to eliminate this condition and also to make the theorem applicable to signatures with uncountable symbols but with finitary conjunctions. An institutional version of the Omitting Types Theorem is also considered for future research.

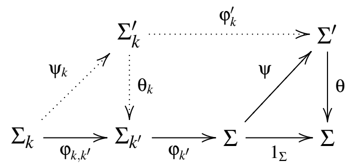
9 Exiled proofs

Proof (Lemma 4). We show $\mathbb{S}en_0(\Sigma') \subseteq \bigcup_{i\in J} \varphi_i(\mathbb{S}en_0(\Sigma_i))$. Let $e \in \mathbb{S}en_0(\Sigma')$. Since e is finitary it can be written as $v(e_f)$ where $v : \Sigma_f \to \Sigma'$ is a signature morphism such that Σ_f is finitely presented in the category $\mathbb{S}ig$. By finiteness of Σ_f there exists a signature morphism $v_i : \Sigma_f \to \Sigma_i$ such that $v_i;\varphi_i = v$. We have that $e = \varphi_i(v_i(e_f))$. Therefore $\mathbb{S}en_0(\Sigma') = \bigcup_{i\in J} \varphi_i(\mathbb{S}en_0(\Sigma_i))$.

Proof (Lemma 5). By Lemma 4 we have that $\mathbb{S}en_0(\Sigma) \subseteq \mathcal{L}_{\Sigma'}$.
The closure properties of $\mathcal{L}_{\Sigma'}$ are consequences of Remark 2 and the finiteness of signature morphisms in $\mathcal{D}$. The most interesting case is the closure of $\mathcal{L}_{\Sigma'}$ to substitutions. The remaining cases are straightforward. Let $(\exists\psi)e \in \mathcal{L}_{\Sigma'}$ (where $\psi : \Sigma' \to \Sigma'_1$) and a substitution $\theta : \psi \to 1_{\Sigma'}$. By the definition of $\mathcal{L}_{\Sigma'}$ and Remark 2 we have $(\exists\psi)\varphi'_k(e_k) = \varphi_k((\exists\psi_k)e_k)$ for some $(\exists\psi_k)e_k \in \mathbb{S}en(\Sigma_k)$, where



is a pushout of signature morphisms with $\psi_k \in \mathcal{D}$. Since ψ_k is finitary and $(\varphi_{k,i} \xrightarrow{\varphi_i} \varphi_k)_{(k \leq i) \in (J,\leq)}$ is a directed co-limit in the category $\Sigma_k/\mathbb{S}ig$, there exists $\theta_k : \psi_k \to \varphi_{k,k'}$, where $k \leq k'$ such that $\theta_k;\varphi_{k'} = \varphi'_k;\theta$.



Therefore $\theta(e) = \theta(\varphi'_k(e_k)) = \varphi_{k'}(\theta_k(e_k)) \in \mathcal{L}_{\Sigma'}$.

References

1. J. Adámek and J. Rosický. *Locally Presentable and Accessible Categories*. Number 189 in London Mathematical Society Lecture Notes. Cambridge University Press, 1994.
2. Jon Barwise. Notes on forcing and countable fragments. 1970. Mimeographed.
3. Peter Burmeister. *A Model Theoretic Oriented Approach to Partial Algebras*. Akademie-Verlag, 1986.
4. Paul J. Cohen. The independence of the continuum hypothesis. *Proceedings of the National Academy of Sciences of the United States of America*, 50(6):1143–1148, December 1963.
5. Paul J. Cohen. The independence of the continuum hypothesis. *Proceedings of the National Academy of Sciences of the United States of America*, 51(1):105–110, January 1964.
6. Răzvan Diaconescu. Institution-independent ultraproducts. *Fundam. Inf.*, 55(3-4):321–348, 2003.
7. Răzvan Diaconescu. Elementary diagrams in institutions. *Journal of Logic and Computation*, 14(5):651–674, 2004.
8. Răzvan Diaconescu. Herbrand Theorems in arbitrary institutions. *Inf. Process. Lett.*, 90:29–37, 2004.
9. Răzvan Diaconescu. An institution-independent proof of Craig interpolation theorem. *Studia Logica*, 77(1):59–79, 2004.
10. Răzvan Diaconescu. Proof systems for institutional logic. *Journal of Logic and Computation*, 16(3):339–357, 2006.
11. Răzvan Diaconescu. *Institution-independent Model Theory*. Studies in Universal Logic. Birkhäuser, 2008.
12. Răzvan Diaconescu and Petros S. Stefaneas. Ultraproducts and possible worlds semantics in institutions. *Theoretical Computer Science*, 379(1-2):210–230, 2007.
13. Joseph Goguen and Rod Burstall. Institutions: Abstract model theory for specification and programming. *Journal of the Association for Computing Machinery*, 39(1):95–146, 1992.
14. Joseph A. Goguen and Răzvan Diaconescu. An oxford survey of order sorted algebra. *Mathematical Structures in Computer Science*, 4(3):363–392, 1994.
15. Joseph A. Goguen and Jose Meseguer. Order-sorted algebra I: Equational deduction for multiple inheritance, overloading, exceptions and partial operations. *Theoretical Computer Science*, 105(2):217–273, 1992.

16. Daniel Găină and Andrei Popescu. An institution-independent proof of Robinson consistency theorem. *Studia Logica*, 85(1):41–73, 2007.
17. Leon Henkin. The completeness of the first-order functional calculus. *Journal of Symbolic Logic*, 14(3):159–166, 1949.
18. Gerard Huet and Derek C. Oppen. Equations and rewrite rules: a survey. *Formal Language Theory: Perspectives and Open Problems*, pages 349–405, 1980.
19. Jerome Keisler. Forcing and the omitting types theorem. *Studies in Model Theory*, 8:96–133, 1973.
20. J. Meseguer. General logics. In *Logic Colloquium 87*, pages 275–329. North Holland, 1989.
21. Marius Petria. An institutional version of Gödel's completeness theorem. In *CALCO*, pages 409–424, 2007.
22. Marius Petria and Răzvan Diaconescu. Abstract Beth definability in institutions. *Journal of Symbolic Logic*, 71(3):1002–1028, 2006.
23. A. Robinson. Forcing in model theory. *Symposia Mathematica*, 50:69–82, 1979.
24. Andrzej Tarlecki. Bits and pieces of the theory of institutions. In David Pitt, Samson Abramsky, Axel Poigné, and David Rydeheard, editors, *Proceedings, Summer Workshop on Category Theory and Computer Programming, Lecture Notes in Computer Science*, volume 240, pages 334–360. Springer, 1986.