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COMPLETENESS OF EIGENVECTORS IN BANACH SPACES

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ABSTRACT. We prove a general theorem on the completeness of the eigenvectors of linear operators in a Banach space. We then derive asymptotic estimates for the Green's functions of two-point boundary value problems which allow us to apply the above theorem to a wide class of such problems in the spaces $L^p(0, 1)$, $1 \leqq p < \infty$.

1. **Introduction.** Let B denote a Banach space, and let B^* denote its dual. A sequence $\{\varphi_k\}$ of elements of B is *complete* in B if the collection of all finite sums $\sum \alpha_k \varphi_k$, α_k a scalar, is dense in B. The sequence $\{\varphi_k\}$ is *closed* in B if the only element ψ of B^* for which $\psi(\varphi_k)=0$, all k, is the zero of B^*. It is easily seen that $\{\varphi_k\}$ is closed if and only if $\{\varphi_k\}$ is complete.

For the case that the scalar field is the complex field, we consider the problem of determining if a sequence $\{\varphi_k\}$ is complete in B, where the φ_k's arise as the eigenvectors and generalized eigenvectors of a linear operator $T: B \to B$. In the case that B is a Hilbert space, there are completeness results provided that the resolvent operator is a Hilbert-Schmidt operator or an operator of class C_p, and the norm of the resolvent operator obeys certain growth conditions [**1**, pp. 1042, 1089, 1115]. These results are extended to Banach spaces in [**9**], [**10**].

If $T: B \to B$ has a compact resolvent $R(\lambda, T)$ for some λ, then the spectrum of T is at most countably infinite, consisting entirely of eigenvalues λ_i which are poles of $R(\lambda, T)$ [**8**, p. 416]. The invariant subspace corresponding to an eigenvalue λ_i is of finite dimension ν_i. By the operational calculus [**7**, pp. 287, 305], the projection P_i of B onto the invariant subspace corresponding to λ_i has the form

$$P_i f = \sum_{j=1}^{\nu_i} \psi_{ij}(f)\varphi_{ij}, \qquad f \in B, \tag{1.1}$$

where $\varphi_{ij} \in B$, $\psi_{ij} \in B^*$, and

$$\psi_{ij}(\varphi_{kl}) = \delta_{ik}\delta_{jl}. \tag{1.2}$$

In §2, we shall prove the following result.

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THEOREM 1.1. *Let $T: B \to B$ be a densely defined linear operator with compact resolvent. Then the sequence $\{\varphi_{kl}\}$ is complete in B provided that for each $r>0$ sufficiently large, the annulus $2r \leqq |\lambda| \leqq 3r$ contains a circle C centered at the origin, lying entirely in the resolvent set of T, such that*

$$\|R(\lambda, T)\| \leqq K\,|\lambda|^{\mu} \tag{1.3}$$

for λ on C, where K is a constant, and μ is an integer.

2. **The completeness theorem.** Since T is densely defined in B, its adjoint $T^*: B^* \to B^*$ is well defined, and is a closed linear operator [**6**, p. 43]. Since T and T^* have the same resolvent sets we have $R(\lambda, T^*) = R^*(\lambda, T)$ [**6**, p. 56]. Thus the residue at λ_i of $R(\lambda, T^*)$ is the adjoint P_i^* of the residue of $R(\lambda, T)$ at λ_i, and P_i^* has the form

$$P_i^* g = \sum_{j=1}^{\nu_i} \varphi_{ij}(g)\psi_{ij} \tag{2.1}$$

for g in B^*. For convenience, we relabel the sequence $\{\varphi_{ij}\}$ as $\{\varphi_k\}$, and similarly for $\{\psi_k\}$. If $\{\varphi_k\}$ is not closed in B, there exists a nonzero g in B^* such that $\varphi_k(g)=0$ for all k. For such a g, $P_i^* g=0$ for all i. Using the biorthogonality (1.2), we see that the converse is true. Consequently $\{\varphi_k\}$ is closed if and only if the only element g of B^* for which $R(\lambda, T^*)g$ is entire is the zero of B^*. See also Definition 3 in [**8**, p. 443] and the resulting discussion.

LEMMA 2.1. *Let $T: B \to B$ be a linear operator, and let f be in B, $f \neq 0$. Then the equation*

$$Tu = \lambda u + \tag{2.2}$$

has no solution $u(\lambda)$ which is a polynomial in λ on an infinite set S.

PROOF. If we asume that $u(\lambda) = \sum_{k=0}^{m} \lambda^k u_k$, $u_m \neq 0$, is a solution of (2.2) for each λ, in S, then we easily see that each u_k is in the domain of T. Substituting this expression into (2.2), we must have $u_m = 0$, a contradiction.

If λ is in the resolvent set of T, then the unique solution to (2.2) is

$$u(\lambda) = -R(\lambda, T)f.$$

Thus any entire solution to (2.2) is an analytic continuation of $-R(\lambda, T)f$ onto the spectrum of T.

PROOF OF THEOREM 1.1. Assume $\{\varphi_k\}$ is not closed in B. Then there exists an element g in B^*, $g \neq 0$, such that $v(\lambda) = R(\lambda, T^*)g$ is entire. Let λ_0 be a fixed complex number, with $|\lambda_0|$ sufficiently large so that the annulus $2|\lambda_0|\leqq|\lambda|\leqq 3|\lambda_0|$ contains a circle C on which $\|R(\lambda, T^*)\|\leqq K|\lambda|^\mu$. Since $v(\lambda)$ is entire,

$$v(\lambda_0) = (\tfrac{1}{2}\pi i)\int_C [v(\lambda)/(\lambda - \lambda_0)]\, d\lambda.$$

Since $|\lambda-\lambda_0|\geqq|\lambda_0|$, and $|\lambda|\leqq 3|\lambda_0|$, we have

$$\|v(\lambda_0)\| \leqq (\tfrac{1}{2}\pi)\int_C [\|v(\lambda)\|/|\lambda_0|]\, |d\lambda| \leqq 3K\, |3\lambda_0|^\mu\, \|g\| = K'\, |\lambda_0|^\mu.$$

Thus $-v(\lambda)$ is a polynomial solution to $T^*v=\lambda v+g$. By Lemma 2.1, this is not possible for $g\neq 0$, so $\{\varphi_k\}$ is closed in B.

COROLLARY. *If B is reflexive, then under the assumptions of Theorem* 1.1, *the sequence $\{\psi_k\}$ is complete in B^*.*

PROOF. If $\{\psi_k\}$ is not closed in B^*, then there exists f in $B^{**}=B, f\neq 0$, such that $P_i f=0$ for all i. The remainder of the discussion is as in the previous proof.

3. **Completeness for ordinary differential operators.** Let l denote the nth order ordinary linear differential expression defined by

$$(3.1)\qquad l(u) = u^{(n)} + a_{n-1}(x)u^{(n-1)} + \cdots + a_0(x)u, \qquad 0\leqq x\leqq 1,$$

where the a_j's are bounded measurable functions, and in addition $a_{n-1}^{(n-1)}$ exists and is also a bounded measurable function. Let M, N denote two matrices of complex constants with n linearly independent columns between them. Let $\hat{u}(x)$ denote the column vector $(u(x), u^{(1)}(x), \cdots, u^{(n-1)}(x))$. Let

$$(3.2)\qquad Uu = M\hat{u}(0) + N\hat{u}(1).$$

For $1\leqq p<\infty$, let $\Delta=\Delta_p$ denote the subspace of $L^p(0, 1)$ consisting of all functions u of class $C^{n-1}[0, 1]$ such that $u^{(n-1)}$ is absolutely continuous, $u^{(n)}$ is of class $L^p(0, 1)$, and such that $Uu=0$. Let $T{:}\,L^p\rightarrow L^p$ be defined on Δ by $Tu=l(u)$. Since Δ contains all functions of class $C^n[0, 1]$, which vanish, along with their first $n-1$ derivatives, at the endpoints, we see that T is densely defined.

If λ is in the resolvent set of T, then the solution to $Tu=\lambda u+f$, f in $L^p(0, 1)$, is

$$(3.3)\qquad u(x, \lambda) = \int_0^1 G(x, t, \lambda)f(t)\, dt = -R(\lambda, T)f,$$

where G is the Green's function of T.

Since $a_{n-1}^{(n-1)}$ is in $L^\infty[0, 1]$, we can perform a substitution $u(x)=q(x)v(x)$, where

$$q(x) = \exp\left[-(1/n)\int_0^x a_{n-1}(t)\,dt\right],$$

and obtain a new differential expression and boundary conditions for v. The significant feature of the transformed problem is that the coefficient of $v^{(n-1)}$ is zero. This simplifies the discussion of the asymptotic nature of solutions to $l(u)=\lambda u$.

DEFINITION 3.1. The differential operator T is *Stone regular* if the transformed problem satisfies Definition 3.1 in [**2**, p. 487].

If $G'(x, t, \lambda)$ denotes the Green's function of the transformed problem, then as observed in [**3**, equation 2.5],

$$G(x, t, \lambda) = q(x)G'(x, t, \lambda)q^{-1}(t), \tag{3.4}$$

where in [**3**] we used the substitution $\lambda=-\rho^n$. Thus we shall dispense with the distinction between the original problem and its transformed version, and assume that $a_{n-1}\equiv 0$.

The location of the eigenvalues of T is discussed in [**2**, p. 489]. It is convenient for this purpose to refer to the ρ-plane. We will use the notation of [**2**], in particular the sectors S_i are defined on p. 483, and the constants σ and τ are defined on p. 485. Let $\delta>0$ be given. It is clear from the discussion in [**2**] that if each $\rho \in S_i$ such that $-\rho^n$ is an eigenvalue of T is centered at a disc of radius δ, then for $r>0$ sufficiently large, each region in S_i of the form $(2r)^{1/n}\leqq|\rho|\leqq(3r)^{1/n}$ contains many circular arcs centered at the origin of the ρ-plane, and not intersecting any of the discs. The image in the λ-plane of such an arc is a circle C, centered at the origin of the λ-plane, contained entirely in the resolvent set of T, and satisfying $2r\leqq|\lambda|\leqq 3r$.

THEOREM 3.1. *If the differential operator T is Stone regular, there exists an integer $m\geqq 0$ such that*

$$n\rho^{n-1}G(x, t, \rho) = \rho^m O(1) \tag{3.5}$$

as $|\rho|\to\infty$ in S_1' where the $O(1)$ term is uniform in t and x for $0\leqq t, x\leqq 1$.

PROOF. This is a direct consequence of equations (2.9) and (4.7) in [**2**, pp. 484, 492].

COROLLARY. *If $\lambda=-\rho^n$ is in the resolvent set of T, and if $|\rho-\rho_0|\geqq\delta$ for each eigenvalue $\lambda_0=-\rho_0^n$, then for $|\lambda|$ sufficiently large,*

$$|G(x, t, \lambda)| \leqq K\,|\lambda|^{(m+1-n)/n}, \qquad 0\leqq t, x\leqq 1, \tag{3.6}$$

where K is a constant.

PROOF. This is a direct consequence of equations (3.4) and (3.5).

We note at this point that there is no theoretical limit to the size of m. See [**2**, Theorem 5.3]. Let μ denote the first integer no smaller than $(m+1-n)/n$.

THEOREM 3.2. *If T is Stone regular, then for each $\lambda=-\rho^n$ in the resolvent set of T such that $|\rho-\rho_0|\geqq\delta$ for each eigenvalue $\lambda_0=-\rho_0^n$, as an operator from L^p to L^p,*

$$\|R(\lambda, T)\| \leqq K\,|\lambda|^{\mu}. \tag{3.7}$$

PROOF. By (3.6), we see that G, as a function to t, is of class $L^\infty(0, 1)$, for fixed x and λ. Thus G is in $L^q(0, 1)$ for each q, $1\leqq q\leqq\infty$. If f is in $L^p(0, 1)$, $1\leqq p<\infty$, and if $p+q=pq$, then by Hölder's inequality,

$$|u(x, \lambda)| \leqq \left[\int_0^1 |G(x, t, \lambda)|^q\,dt\right]^{1/q} \|f\|_p \leqq K\,|\lambda|^{\mu}\,\|f\|_p.$$

Thus $\|u(\cdot, \lambda)\|_p\leqq K|\lambda|^\mu\|f\|_p$.

REMARK. In particular, (3.7) holds on each circle C which is the image of a circular arc lying entirely in S_i'.

THEOREM 3.3. *If T is Stone regular, the eigenfunctions and generalized eigenfunctions of T form a sequence which is complete in $L^p(0, 1)$ for $1\leqq p<\infty$.*

COROLLARY. *The eigenfunctions and generalized eigenfunctions of T^* are complete in $L^p(0, 1)$ for $1<p<\infty$.*

REMARK. The adjoint in L^q of a two-point boundary value problem in L^p is known to be another two-point boundary value problem $(1<p<\infty)$, provided that the coefficients a_j are sufficiently differentiable [**5**], so in such a case the corollary provides no new information. If the a_j's are not sufficiently differentiable, the L^p adjoint of T is a quasi-differential operator [**4**, p. 888]. Thus in these cases the corollary provides new information.

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